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CLEBSCH-GORDAN COEFFICIENTS FOR SU(2) AND HAHN POLYNOMIALS

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Clebsch-Gordan coefficients for SU(2) and Hahn polynomials*)

by

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ABSTRACT

The Clebsch-Gordan coefficients for SU(2) are introduced in an algebraic way, in the context of polynomials in four variables, homogeneous of certain degrees in the first and second pair of variables, respectively. Next it is shown that the Clebsch-Gordan coefficients can be expressed in terms of Hahn polynomials and that the Clebsch-Gordan coefficients and Hahn polynomials can be identified with each other as orthogonal systems.

KEY WORDS & PHRASES: *canonical matrix elements of irreducible representations of* SU(2)*; decomposition of tensor products of irreducible representations of* SU(2)*; Clebsch-Gordan coefficients; Wigner coefficients;* 3-j *symbols; symmetries for Clebsch-Gordan coefficients; Hahn polynomials; hypergeometric functions*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

WILSON [18] proved that there is a class of discrete orthogonal polynomials expressible in terms of ${}_4F_3$ hypergeometric functions of unit argument such that the 6-j symbols (or Racah coefficients) are expressible in terms of these polynomials and the orthogonality relations for the 6-j symbols and the orthogonal polynomials coincide. With knowledge of this result it is a natural question to ask for a similar, but more simple result concerning 3-j symbols (or Wigner coefficients or Clebsch-Gordan coefficients). In this paper we will prove that the orthogonality relations for the 3-j symbols coicide with the orthogonality relations for Hahn polynomials (or their duals). Of course, this is no surprise, since both 3-j symbols and Hahn polynomials have explicit expressions in terms of ${}_3F_2$ hypergeometric functions of unit argument. In fact, the result is well-known in a small circle of people interested in the relationship between special functions and group theory, but, as far as I know, no proof or even statement of the result appeared in the literature until now.

Before proving this result in Section 4 we give a rather self-contained introduction to Clebsch-Gordan (= CG) coefficients in Section 3, while some properties of the canonical matrix elements of irreducible representations of SU(2) are recapitulated in Section 2. Section 5 contains some historical notes. We have tried to give a possibly original approach to CG coefficients. The key formulas (3.12) together with (3.2) are derived in a very elegant algebraic way, with hardly any calculations. However, we have to make our hands dirty in deriving (2.10), (3.5) and (4.2). We could not resist to derive, in passing, Regge's beautiful formula (3.19), which implies a symmetry group of order 72 for CG coefficients.

This paper is related to some other work by the author. In [7] we proved that Krawtchouk polynomials (discrete orthogonal polynomials expressible as ${}_2F_1$'s) are related to matrix elements for irreducible representations of SU(2). In a forthcoming paper we will give an unification of the present group theoretic interpretation of Hahn polynomials as CG coefficients and another interpretation as spherical functions on symmetric groups. A similar identification in the Krawtchouk polynomial case was made in [7].

2. THE CANONICAL MATRIX ELEMENTS OF THE IRREDUCIBLE UNITARY REPRESENTATIONS OF SU(2)

Let $\ell \in \frac{1}{2}\mathbb{Z}_+ := \{0,\frac{1}{2},1,\frac{3}{2},2,\ldots\}$. Let H_ℓ be the space of homogeneous polynomials of degree 2ℓ in two complex variables. Choose an inner product on H_ℓ such that an orthonormal basis of H_ℓ is given by the functions ψ_n^ℓ $(n = -\ell,-\ell+1,\ldots,\ell)$:

$$\psi_n^\ell(x,y) := \binom{2\ell}{\ell-n}^{1/2} x^{\ell-n} y^{\ell+n}. \tag{2.1}$$

Let T^ℓ be the representation of $GL(2,\mathbb{C})$ on H_ℓ defined by

$$\left(T^\ell\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} f\right)(x,y) := f(\alpha x+\gamma y, \beta x+\delta y), \qquad f \in H_\ell. \tag{2.2}$$

It is well-known that T^ℓ is irreducible as a representation of $GL(2,\mathbb{C})$, $U(2)$ or $SU(2)$, unitary as a representation of $U(2)$ or $SU(2)$ and that each irreducible unitary representation of $SU(2)$ is equivalent to some T^ℓ $(\ell \in \frac{1}{2}\mathbb{Z}_+)$, cf. for instance HEWITT & ROOS [5, Theorems (29.20), (29.27)].

The canonical matrix elements t_{mn}^ℓ of T^ℓ are given by

$$T^\ell(g)\psi_n^\ell = \sum_{m=-\ell}^{\ell} t_{mn}^\ell(g)\psi_m^\ell, \qquad g \in GL(2,\mathbb{C}). \tag{2.3}$$

The unitariness of $T^\ell|_{U(2)}$ implies

$$t_{mn}^\ell(g) = \overline{t_{nm}^\ell(g^{-1})}, \qquad g \in U(2). \tag{2.4}$$

It follows from (2.1), (2.2) and (2.3) that

$$\binom{2\ell}{\ell-n}^{\frac{1}{2}}(\alpha x+\gamma y)^{\ell-n}(\beta x+\delta y)^{\ell+n} = \sum_{m=-\ell}^{\ell} t_{mn}^\ell\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\binom{2\ell}{\ell-m}^{\frac{1}{2}} x^{\ell-m}y^{\ell+m}. \tag{2.5}$$

Hence

$$\overline{t_{mn}^\ell\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}} = t_{mn}^\ell\begin{pmatrix}\bar\alpha & \bar\beta\\ \bar\gamma & \bar\delta\end{pmatrix}. \tag{2.6}$$

It follows from (2.4) and (2.6) that

$$(2.7)\qquad t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t^{\ell}_{nm}\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$$

for $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in U(2)$ and, by analytic continuation also for $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in GL(2,\mathbb{C})$. Formulas (2.5) and (2.7) imply:

LEMMA 2.1.
(a) $t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ *is a homogeneous polynomial of degree* 2ℓ *in* α, β, γ, δ *with real coefficients.*
(b) t^{ℓ}_{mn} *is homogeneous of degree* $\ell-m$ *in* α, β *and homogeneous of degree* $\ell+m$ *in* γ, δ.
(c) $t^{\ell}_{m,n}$ *is homogeneous of degree* $\ell-n$ *in* α, γ *and homogeneous of degree* $\ell+n$ *in* β, δ.

It follows from (2.5) that

$$(2.8)\qquad t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t^{\ell}_{-m,-n}\begin{pmatrix}\delta & \gamma\\ \beta & \alpha\end{pmatrix}.$$

Combination with (2.7) yields:

$$(2.9)\qquad t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t^{\ell}_{-n,-m}\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}.$$

For fixed ℓ the set $\{(m,n) \mid m,n \in \{-\ell,-\ell+1,\ldots,\ell\}\}$ is the union of the four subsets

$$\{(m,n) \mid m+n \geq 0,\ m-n \geq 0\},$$

$$\{(m,n) \mid m+n \geq 0,\ m-n \leq 0\},$$

$$\{(m,n) \mid m+n \leq 0,\ m-n \geq 0\},$$

$$\{(m,n) \mid m+n \leq 0,\ m-n \leq 0\}.$$

Because of the symmetries (2.7), (2.8), (2.9), $t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is completely known if it is known for (m,n) restricted to one of these four subsets.

It can be derived from (2.5) that, for $m+n \geq 0$, $m-n \geq 0$, we have:

$$t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} =$$

$$= \frac{1}{(m+n)!}\left(\frac{(\ell+m)!(\ell+n)!}{(\ell-m)!(\ell-n)!}\right)^{\frac{1}{2}}\beta^{\ell-m}\gamma^{\ell-n}\delta^{m+n}\ {}_2F_1(-\ell+m,-\ell+n;\ m+n+1;\ \frac{\alpha\delta}{\beta\gamma})$$

(2.10) $$= \frac{1}{(m+n)!}\left(\frac{(\ell+m)!(\ell+n)!}{(\ell-m)!(\ell-n)!}\right)^{\frac{1}{2}}\gamma^{m-n}\delta^{m+n}(\beta\gamma-\alpha\delta)^{\ell-m}\ \cdot$$

$$\cdot\ {}_2F_1(-\ell+m,\ell+m+1;\ m+n+1;\ \frac{\alpha\delta}{\alpha\delta-\beta\gamma})$$

$$= \left(\frac{(\ell+m)!(\ell-m)!}{(\ell+n)!(\ell-n)!}\right)^{\frac{1}{2}}\gamma^{m-n}\delta^{m+n}(\beta\gamma-\alpha\delta)^{\ell-m}\ P^{(m+n,m-n)}_{\ell-m}\left(\frac{\beta\gamma+\alpha\delta}{\beta\gamma-\alpha\delta}\right),$$

where [3, 2.1(22) and 10.8(16)] are used and $P^{(\alpha,\beta)}_n(x)$ denotes a Jacobi polynomial (cf. VILENKIN [14, Ch.3, §3] or KOORNWINDER [7, §2]). Then Schur's orthogonality relations on SU(2) for the matrix elements t^{ℓ}_{mn} imply the orthogonality relations for Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$ $(\alpha,\beta \in \mathbb{Z}_+)$.

3. THE DEFINITION OF THE CLEBSCH-GORDAN COEFFICIENTS FOR SU(2)

From now on, a condition on indices like $|j| \le \ell$ will mean that j runs over the integers or half integers $\ell,\ell-1,\ldots,-\ell+1,-\ell$.

The tensor product $H_{\ell_1} \otimes H_{\ell_2}$ can be identified with the space of polynomials in four complex variables x, y, u, v, homogeneous of degree $2\ell_1$ in x, y and homogeneous of degree $2\ell_2$ in u,v. An orthonormal basis of $H_{\ell_1} \otimes H_{\ell_2}$ is given by the polynomials

(3.1) $$(\psi^{\ell_1}_{j_1} \otimes \psi^{\ell_2}_{j_2})(x,y,u,v) = \binom{2\ell_1}{\ell_1-j_1}^{\frac{1}{2}}\binom{2\ell_2}{\ell_2-j_2}^{\frac{1}{2}} x^{\ell_1-j_1}y^{\ell_1+j_1}u^{\ell_2-j_2}v^{\ell_2+j_2},$$

$$|j_1| \le \ell_1,\ |j_2| \le \ell_2.$$

$T^{\ell_1} \otimes T^{\ell_2}$ is an irreducible unitary representation of SU(2) × SU(2) on $H_{\ell_1} \otimes H_{\ell_2}$. The restriction of this representation to $SU(2)^* :=$ diag(SU(2) × SU(2)) is unitary but in general not irreducible.

<u>THEOREM 3.1</u>.

(a) $$T^{\ell_1} \otimes T^{\ell_2}\big|_{SU(2)^*} \simeq \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} T^{\ell}.$$

Denote the subspace of $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$ *corresponding to* T^{ℓ} *by* $\mathcal{H}_{\ell_1,\ell_2,\ell}$.

(b) *For suitable complex constants* $a_{\ell_1,\ell_2,\ell} \neq 0$ *the functions* $\phi_j^{\ell_1,\ell_2,\ell}$ $(|j| \le \ell)$ *defined by*

$$\phi_j^{\ell_1,\ell_2,\ell}(x,y,u,v) := a_{\ell_1,\ell_2,\ell}(xv-yu)^{\ell_1+\ell_2-\ell}\, t^{\ell}_{\ell_2-\ell_1,j}\begin{pmatrix} x & y \\ u & v \end{pmatrix} \tag{3.2}$$

form an orthonormal basis for $\mathcal{H}_{\ell_1,\ell_2,\ell}$.

(c) *The matrix elements of the representation* $T^{\ell_1} \otimes T^{\ell_2}|_{SU(2)^*}$ *on* $\mathcal{H}_{\ell_1,\ell_2,\ell}$ *with respect to the basis* $\{\phi_j^{\ell_1,\ell_2,\ell}\}$ *are equal to* t^{ℓ}_{mn}, *so*

$$\begin{aligned} &\phi_n^{\ell_1,\ell_2,\ell}(\alpha x-\bar\beta y, \beta x+\bar\alpha y, \alpha u-\bar\beta v, \beta u+\bar\alpha v) \\ &= \sum_{m=-\ell}^{\ell} t^{\ell}_{mn}\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \phi_m^{\ell_1,\ell_2,\ell}(x,y,u,v), \qquad \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \in SU(2). \end{aligned} \tag{3.3}$$

PROOF. By (3.2) and Lemma 2.1(b) the function $\phi_j^{\ell_1,\ell_2,\ell}$ is a polynomial, homogeneous of degree $2\ell_1$ in its first two variables and homogeneous of degree $2\ell_2$ in its last two variables, so it belongs to $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$. Formula (3.3) is clear from (3.2) and the homomorphism property of T^{ℓ}. Since the representations T^{ℓ} are irreducible and mutually inequivalent, part (b) of the theorem is implied by (3.3). Finally, the completeness of the orthonormal system $\{\phi_j^{\ell_1,\ell_2,\ell}\}$ in $\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$ follows from the equality of dimensions:

$$\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} (2\ell+1) = (2\ell_1+1)(2\ell_2+1). \qquad \square$$

THEOREM 3.2. *The constant* $a_{\ell_1,\ell_2,\ell}$ *is uniquely determined by the condition*

$$(\phi_{\ell}^{\ell_1,\ell_2,\ell}, \psi_{\ell_1}^{\ell_1} \otimes \psi_{\ell-\ell_1}^{\ell_2}) > 0. \tag{3.4}$$

Then

$$a_{\ell_1,\ell_2,\ell} = (-1)^{\ell_1+\ell_2-\ell}\left(\frac{(2\ell+1)(2\ell_1)!(2\ell_2)!}{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right)^{\frac{1}{2}}. \tag{3.5}$$

PROOF. It follows from (2.5) that

$$(3.6)\qquad t^{\ell}_{m\ell}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \binom{2\ell}{\ell-m}^{\frac{1}{2}}\beta^{\ell-m}\delta^{\ell+m},$$

so combination with (3.2) yields

$$\phi_{\ell}^{\ell_1,\ell_2,\ell}(x,y,u,v)$$

$$= a_{\ell_1,\ell_2,\ell}\binom{2\ell}{\ell_1-\ell_2+\ell}^{\frac{1}{2}} y^{\ell_1-\ell_2+\ell}\, v^{-\ell_1+\ell_2+\ell}\,(xv-yu)^{\ell_1+\ell_2-\ell}$$

$$= a_{\ell_1,\ell_2,\ell}\binom{2\ell}{\ell_1-\ell_2+\ell}^{\frac{1}{2}} \sum_{k=0}^{\ell_1+\ell_2-\ell} (-1)^{\ell_1+\ell_2-\ell-k}\,\cdot$$

$$\cdot\binom{\ell_1+\ell_2-\ell}{k}\binom{2\ell_1}{k}^{-\frac{1}{2}}\binom{2\ell_2}{\ell_1+\ell_2-\ell-k}^{-\frac{1}{2}} \psi^{\ell_1}_{\ell_1-k}(x,y)\,\psi^{\ell_2}_{-\ell_1+\ell+k}(u,v),$$

by the binomial formula and (2.1). Thus (3.4) implies that $(-1)^{\ell_1+\ell_2+\ell} a_{\ell_1,\ell_2,\ell} > 0$. By taking squared L^2-norms in the first and last member of the above identities we obtain

$$1 = |a_{\ell_1,\ell_2,\ell}|^2\binom{2\ell}{\ell_1-\ell_2+\ell}\sum_{k=0}^{\ell_1+\ell_2-\ell}\binom{\ell_1+\ell_2-\ell}{k}^2\binom{2\ell_1}{k}^{-1}\binom{2\ell_2}{\ell_1-\ell_2-\ell-k}^{-1}$$

$$= |a_{\ell_1,\ell_2,\ell}|^2\,\frac{(2\ell)!\,(\ell_1+\ell_2-\ell)!}{(2\ell_2)!(\ell_1-\ell_2+\ell)!}\;{}_2F_1(-\ell_1-\ell_2+\ell,-\ell_1+\ell_2+\ell+1;\,-2\ell_1;\,1)$$

$$= |a_{\ell_1,\ell_2,\ell}|^2\,\frac{(\ell_1+\ell_2-\ell)!\,(\ell_1+\ell_2+\ell+1)!}{(2\ell+1)(2\ell_1)!(2\ell_2)!}\;.$$

Here we used the Chu-Vandermonde sum

$$(3.7)\qquad {}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n},$$

$n \in \mathbb{Z}_+$; $c-b, c \neq 0,-1,\ldots,-n+1$, cf. SLATER [11, (1.7.7)]. □

Note that (3.5), (3.2) and Lemma 2.1(a) imply that $\phi_j^{\ell_1,\ell_2,\ell}$ is a polynomial in x, y, u, v with real coefficients.

The symmetries (2.7), (2.8), (2.9) applied on (3.2) yield corresponding symmetries for $\phi_j^{\ell_1,\ell_2,\ell}(x,y,u,v)$:

$$(3.8)\quad \begin{aligned} &((2\ell_1)!(2\ell_2)!)^{-\frac{1}{2}}\phi_j^{\ell_1,\ell_2,\ell}(x,y,u,v) \\ &= ((\ell_1+\ell_2-j)!(\ell_1+\ell_2+j)!)^{-\frac{1}{2}}\phi_{\ell_2-\ell_1}^{\frac{1}{2}(\ell_1+\ell_2-j),\frac{1}{2}(\ell_1+\ell_2+j),\ell}(x,u,y,v) \\ &= ((2\ell_2)!(2\ell_1)!)^{-\frac{1}{2}}\phi_{-j}^{\ell_2,\ell_1,\ell}(v,u,y,x) \\ &= ((\ell_1+\ell_2+j)!(\ell_1+\ell_2-j)!)^{-\frac{1}{2}}\phi_{\ell_1-\ell_2}^{\frac{1}{2}(\ell_1+\ell_2+j),\frac{1}{2}(\ell_1+\ell_2-j),\ell}(v,y,u,x). \end{aligned}$$

We have introduced two canonical orthonormal bases for $H_{\ell_1} \otimes H_{\ell_2}$: one basis consisting of the polynomials $\psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2}$ ($|j_1| \le \ell_1$, $|j_2| \le \ell_2$), which behaves nicely with respect to the subgroup $S(U(1) \times U(1)) \times S(U(1) \times U(1))$ of $SU(2) \times SU(2)$, and another basis consisting of the polynomials $\phi_j^{\ell_1,\ell_2,\ell}$ ($|\ell_1-\ell_2| \le \ell \le \ell_1+\ell_2$, $|j| \le \ell$), which behaves nicely with respect to the subgroup $SU(2)^*$. The matrix elements of the unitary matrix which maps the one basis onto the other basis are called *Clebsch-Gordan coefficients* $C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}$:

$$(3.9)\qquad \phi_j^{\ell_1,\ell_2,\ell} = \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}\, \psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2}$$

Now observe that

$$t_{mn}^{\ell}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix} = e^{-2im\phi}\delta_{m,n}$$

and apply

$$T^{\ell_1}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix} \otimes T^{\ell_2}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix}$$

on both sides of (3.9). It follows that

(3.10) $$C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} = 0 \quad \text{if } j_1 + j_2 \neq j.$$

Thus we only need to consider CG coefficients with parameters satisfying

(3.11) $$|\ell_1-\ell_2| \le \ell \le \ell_1+\ell_2, \quad |j| \le \ell,\ |j_1| \le \ell_1,\ |j_2| \le \ell_2,\ j = j_1+j_2.$$

It follows from (3.9) and (2.1) that

(3.12) $$((2\ell_1)!(2\ell_2)!)^{-\frac{1}{2}}\phi_j^{\ell_1,\ell_2,\ell}(x,y,u,v) = \sum_{j_1=-\ell_1}^{\ell_1}\sum_{\substack{j_2=-\ell_2 \\ j_1+j_2=j}}^{\ell_2} \frac{x^{\ell_1-j_1}y^{\ell_1+j_1}u^{\ell_2-j_2}v^{\ell_2+j_2}}{((\ell_1-j_1)!(\ell_1+j_1)!(\ell_2-j_2)!(\ell_2+j_2)!)^{1/2}}\, C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j}\,.$$

Note that, by(3.12), the coefficients $C^{\ell_1,\ell_2,\ell}_{j1,j2,j}$ are real-valued. By combination of (3.8) and (3.12) we obtain symmetries for these coefficients:

(3.13) $$C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} = C^{\frac{1}{2}(\ell_1+\ell_2-j),\frac{1}{2}(\ell_1+\ell_2+j),\ell}_{\frac{1}{2}(-\ell_1+\ell_2+j_1-j_2),\frac{1}{2}(-\ell_1+\ell_2-j_1+j_2),\ell_2-\ell_1} = C^{\ell_2,\ell_1,\ell}_{-j_2,-j_1,-j}$$
$$= C^{\frac{1}{2}(\ell_1+\ell_2+j),\frac{1}{2}(\ell_1+\ell_2-j),\ell}_{\frac{1}{2}(\ell_1-\ell_2+j_1-j_2),\frac{1}{2}(\ell_1-\ell_2-j_1+j_2),\ell_1-\ell_2}$$

The set of points in $(\ell_1,\ell_2,\ell,j_1,j_2,j)$-space satisfying the inequalities (3.11) is the union of four subsets determined, respectively, by the inequalities:

(3.14) $$\begin{cases} \text{(i)} & \ell_1-\ell_2 \le j \le \ell_2-\ell_1 \le \ell \le \ell_1+\ell_2;\ -\ell_1 \le j_1 \le \ell_1;\ j_1+j_2=j. \\ \text{(ii)} & \ell_2-\ell_1 \le j \le \ell_1-\ell_2 \le \ell \le \ell_1+\ell_2;\ -\ell_2 \le j_2 \le \ell_2;\ j_1+j_2=j. \\ \text{(iii)} & j \le \ell_1-\ell_2 \le -j \le \ell \le \ell_1+\ell_2;\ -\ell_1 \le j_1;\ -\ell_2 \le j_2;\ j_1+j_2 = j. \\ \text{(iv)} & -j \le \ell_1-\ell_2 \le j \le \ell \le \ell_1+\ell_2;\ j_1 \le \ell_1;\ j_2 \le \ell_2;\ j_1+j_2 = j. \end{cases}$$

Those four subsets are mapped onto each other by the symmetries (3.13). Thus it is sufficient to know the CG coefficients with parameters restricted to one of the four above subsets.

By combination of (3.12), (3.2), (3.5), (2.7) and (2.5) we obtain the formula

$$\frac{(-xv+yu)^{\ell_1+\ell_2-\ell}(sx+ty)^{\ell_1-\ell_2+\ell}(su+tv)^{-\ell_1+\ell_2+\ell}}{((\ell_1+\ell_2-\ell)!(\ell_1-\ell_2+\ell)!(-\ell_1+\ell_2+\ell)!)^{1/2}}$$

$$= \left(\frac{(\ell_1+\ell_2+\ell+1)!}{2\ell+1}\right)^{\frac{1}{2}} \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} \sum_{\substack{j=-\ell \\ j=j_1+j_2}}^{\ell} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} \cdot$$

$$\cdot \frac{x^{\ell_1-j_1}y^{\ell_1+j_1}u^{\ell_2-j_2}v^{\ell_2+j_2}s^{\ell-j}t^{\ell+j}}{((\ell_1-j_1)!(\ell_1+j_1)!(\ell_2-j_2)!(\ell_2+j_2)!(\ell-j)!(\ell+j)!)^{1/2}} .$$

This can be considered as a generating function for the CG coefficients. Note that, by putting v = 0 in (3.15), one concludes

$$(3.16) \qquad C^{\ell_1,\ell_2,\ell}_{\ell_1,-\ell_2,\ell_1-\ell_2} > 0.$$

This corresponds to VILENKIN's [14, Ch.3, §8.2 (9)] normalization.

For reasons of symmetry, two other notations for CG coefficients are used:

$$(3.17) \qquad \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ j_1 & j_2 & -j \end{pmatrix} := (-1)^{\ell_1-\ell_2+j}(2\ell+1)^{-\frac{1}{2}} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j}$$

(Wigner's 3-j symbol) and

$$(3.18) \qquad \begin{array}{|ccc|} \ell_2+\ell-\ell_1 & \ell+\ell_1-\ell_2 & \ell_1+\ell_2-\ell \\ \ell_1-j & \ell_2-j_2 & \ell+j \\ \ell_1+j_1 & \ell_2+j_2 & \ell-j \end{array} := (-1)^{\ell_1-\ell_2+j}(2\ell+1)^{-\frac{1}{2}} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j}$$

(Regge's 3×3 array). Note that the points of $(\ell_1,\ell_2,\ell,j_1,j_2,j)$-space satisfying the inequalities (3.11) and also $\ell_1+\ell_2+\ell = L$ for some $L \in \mathbb{Z}_+$ are in one-to-one correspondence by (3.18) with 3×3 arrays having nonnegative integer entries such that for each row and each column the sum of the entries equals L.

Replace in (3.15) x, y, u, v, s, t by x_{21}, x_{31}, x_{22}, x_{32}, $-x_{33}$, x_{23}, respectively, multiply both sides by

$$\frac{(-1)^{\ell_1-\ell_2+\ell}\, x_{11}^{\ell_2+\ell-\ell_1}\, x_{12}^{\ell+\ell_1-\ell_2}\, x_{13}^{\ell_1+\ell_2-\ell}}{((\ell_2+\ell-\ell_1)!\,(\ell+\ell_1-\ell_2)!\,(\ell_1+\ell_2-\ell)!)^{1/2}}$$

and sum upon all ℓ_1,ℓ_2,ℓ such that $|\ell_1-\ell_2| \le \ell \le \ell_1+\ell_2$. Then, by using (3.18), we obtain the beautiful formula

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}^L = (-1)^L L!((L+1)!)^{\frac{1}{2}} \;.$$

$$\cdot \sum_{\substack{a_{ij} \\ a_{11}+\ldots+a_{33}=3L}} \boxed{\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}} \prod_{i,j=1}^{3} \frac{x_{ij}^{a_{ij}}}{(a_{ij}!)^{1/2}} \;.$$

This formula is due to REGGE [9] (apart from a minor error in the coefficient in front of the summation sign). Regge pointed out in [9] that the symmetries of the determinant under row or column permutation and transposition, together with (3.19), yield a symmetry group of order 72 for the CG coefficients.

4. EXPRESSION OF CLEBSCH-GORDAN COEFFICIENTS IN TERMS OF HAHN POLYNOMIALS

In the generating function (3.15) the left hand side is elementary but the right hand side involves a double summation. We now derive another generating function from (3.12), which only involves a single summation, but which has not an elementary left hand side. It follows from (3.12), (3.2), (3.5) and the second ${}_2F_1$ in (2.10) that

$$(4.1)\qquad \frac{1}{(-\ell_1+\ell_2+j)!}\left(\frac{(\ell+j)!(-\ell_1+\ell_2+\ell)!(2\ell+1)}{(\ell-j)!(\ell_1-\ell_2+\ell)!(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right)^{\frac{1}{2}}\cdot$$

$$\cdot\, u^{-\ell_1+\ell_2-j}\, v^{-\ell_1+\ell_2+j}\,(-xv+yu)^{2\ell_1}\cdot$$

$$\cdot\, {}_2F_1(-\ell_1+\ell_2-\ell,-\ell_1+\ell_2+\ell+1;\; -\ell_1+\ell_2+j+1;\; \frac{xv}{xv-yu})$$

$$= \sum_{\substack{j_1=-\ell_1\\ j_1+j_2=j}}^{\ell_1} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j}\,\frac{x^{\ell_1-j_1}\,y^{\ell_1+j_1}\,u^{\ell_2-j_2}\,v^{\ell_2+j_2}}{((\ell_1-j_1)!(\ell_1+j_1)!(\ell_2-j_2)!(\ell_2+j_2)!)^{1/2}}$$

where $\ell_1-\ell_2 \le j \le \ell_2-\ell_1 \le \ell \le \ell_1+\ell_2$. Expand the left hand side as a power series (double summation) and compare coefficients:

$$(4.2)\qquad C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} = \frac{(-1)^{\ell_1-j_1}(2\ell_1)!}{(-\ell_1+\ell_2+j)!}\cdot$$

$$\cdot\left(\frac{(2\ell+1)(\ell_2-j_2)!(\ell_2+j_2)!(\ell+j)!(-\ell_1+\ell_2+\ell)!}{(\ell_1-j_1)!(\ell_1+j_1)!(\ell-j)!(\ell_1-\ell_2+\ell)!(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right)^{\frac{1}{2}}$$

$$\cdot\, {}_3F_2\left(\begin{matrix}-\ell_1+\ell_2-\ell,\; -\ell_1+\ell_2+\ell+1,\; -\ell_1+j_1\\ -\ell_1+\ell_2+j+1,\; -2\ell_1\end{matrix}\;\middle|\; 1\right),$$

where the parameters satisfy (3.14)(i) and the ${}_3F_2$ stands for the series

$$\sum_{k=0}^{\ell_1-\ell_2+\ell}\frac{(-\ell_1+\ell_2-\ell)_k(-\ell_1+\ell_2+\ell+1)_k(-\ell_1+j_1)_k}{(-\ell_1+\ell_2+j+1)_k(-2\ell_1)_k k!}$$

Let $N \in \mathbb{Z}_+$ and $\alpha,\beta > -1$ or $\alpha,\beta < -N$. *Hahn polynomials* $Q_n(x;\alpha,\beta,N)$ ($n = 0,1,2,\ldots,N$) are orthogonal polynomials in x on the set $\{0,1,2,\ldots,N\}$ with respect to the weights

$$\frac{(\alpha+1)_x}{x!}\,\frac{(\beta+1)_{N-x}}{(N-x)!},$$

cf. KARLIN & McGREGOR [6], we use the slightly different notation from ASKEY

[1, (2.36)]. Hahn polynomials can be expressed in terms of ${}_3F_2$'s:

$$(4.3)\qquad Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n,-x,n+\alpha+\beta+1\\ -N,\alpha+1\end{matrix}\,\middle|\,1\right) := \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(n+\alpha+\beta+1)_k}{(-N)_k(\alpha+1)_k k!}\ .$$

The precise form of the orthogonality relations is:

$$(4.4)\qquad \begin{aligned}&\sum_{x=0}^{N} Q_n(x;\alpha,\beta,N)Q_{n'}(x;\alpha,\beta,N)\,\frac{(\alpha+1)_x}{x!}\,\frac{(\beta+1)_{N-x}}{(N-x)!}\,\frac{N!}{(\alpha+\beta+2)_N} =\\ &= \left(\frac{(2n+\alpha+\beta+1)(\alpha+1)_n(\alpha+\beta+2)_n N!}{(n+\alpha+\beta+1)(\beta+1)_n(N+\alpha+\beta+2)_n n!(N-n)!}\right)^{-1}\delta_{n,n'}\ .\end{aligned}$$

Now return to (4.2). Let

$$\begin{cases} x = \ell_1-\ell_2, \quad n = \ell_1-\ell_2+\ell, \quad N = 2\ell_1,\\ \alpha = -\ell_1+\ell_2+j, \quad \beta = -\ell_1+\ell_2-j.\end{cases}$$

Then (3.14)(i) is equivalent to the condition that x, n, N, α, β are integers and

$$0 \le x \le N, \qquad 0 \le n \le N, \qquad \alpha \ge 0, \qquad \beta \ge 0.$$

We derive from (4.2) and (4.3):

$$(4.5)\qquad \begin{aligned}&C^{\frac{1}{2}N,\frac{1}{2}(N+\alpha+\beta),n+\frac{1}{2}(\alpha+\beta)}_{\frac{1}{2}N-x,\frac{1}{2}(\alpha-\beta-N)+x,\frac{1}{2}(\alpha-\beta)} =\\ &= \frac{(-1)^x N!}{\alpha!}\left(\frac{(2n+\alpha+\beta+1)(N-x+\beta)!(x+\alpha)!(n+\alpha)!(n+\alpha+\beta)!}{x!(N-x)!(n+\beta)!n!(N-n)!(N+n+\alpha+\beta+1)!}\right)^{\frac{1}{2}} Q_n(x;\alpha,\beta,N).\end{aligned}$$

By (3.9) we have the orthogonality relations

$$(4.6)\qquad \sum_{x=0}^{N} C^{\frac{1}{2}N,\frac{1}{2}(N+\alpha+\beta),n+\frac{1}{2}(\alpha+\beta)}_{\frac{1}{2}N-x,\frac{1}{2}(\alpha-\beta-N)+x,\frac{1}{2}(\alpha-\beta)}\; C^{\frac{1}{2}N,\frac{1}{2}(N+\alpha+\beta),n'+\frac{1}{2}(\alpha+\beta)}_{\frac{1}{2}N-x,\frac{1}{2}(\alpha-\beta-N)+x,\frac{1}{2}(\alpha-\beta)} = \delta_{n,n'}\ .$$

In view of (4.5) there are precisely the orthogonality relations (4.4).

Of course there are also orthogonality relations dual to (4.6):

$$(4.7)\qquad \sum_{n=0}^{N} C^{\frac{1}{2}N,\frac{1}{2}(N+\alpha+\beta),n+\frac{1}{2}(\alpha+\beta)}_{\frac{1}{2}N-x,\frac{1}{2}(\alpha-\beta-N)+x,\frac{1}{2}(\alpha-\beta)}\, C^{\frac{1}{2}N,\frac{1}{2}(N+\alpha+\beta),n+\frac{1}{2}(\alpha+\beta)}_{\frac{1}{2}N-x',\frac{1}{2}(\alpha-\beta-N)+x',\frac{1}{2}(\alpha-\beta)} = \delta_{x,x'}.$$

Let

$$(4.8)\qquad R_x(n(n+\alpha+\beta+1);\alpha,\beta,N) := Q_n(x;\alpha,\beta,N).$$

for $n, x \in \{0,1,\ldots,N\}$. Then, by (4.3), the function $y \to R_x(y;\alpha,\beta,N)$ extends to a polynomial of degree x in y: the so-called *dual Hahn polynomial* (cf. KARLIN & McGREGOR [6]). They satisfy orthogonality relations dual to (4.4):

$$(4.9)\qquad \sum_{n=0}^{N} R_x(n(n+\alpha+\beta+1);\alpha,\beta,N)R_{x'}(n(n+\alpha+\beta+1);\alpha,\beta,N)\cdot$$

$$\cdot\frac{(2n+\alpha+\beta+1)(\alpha+1)_n(\alpha+\beta+2)_n N!}{(n+\alpha+\beta+1)(\beta+1)_n(N+\alpha+\beta+2)_n n!(N-n)!}$$

$$= \left(\frac{(\alpha+1)_x}{x!}\frac{(\beta+1)_{N-x}}{(N-x)!}\frac{N!}{(\alpha+\beta+2)_N}\right)^{-1}\delta_{x,x'}\quad .$$

In view of (4.7) and (4.5) these are precisely the orthogonality relations (4.7).

Now we claim that

$$(4.10)\qquad R_x(n(n+\alpha+\beta+1);\alpha,\beta,N) = \frac{(-\beta-N)_x}{(\alpha+1)_x}\cdot$$

$$\cdot\, R_x(n(n+\alpha+\beta+1)-N(N+\alpha+\beta+1);-N-\beta-1,-N-\alpha-1,N).$$

Indeed, for $x = 0,1,\ldots,N$ the functions $y \to R_x(y;\alpha,\beta,N)$ and

$$y \to R_x(y-N(N+\alpha+\beta+1);-N-\beta-1,-N-\alpha-1,N)$$

are both polynomials of degree x which are, because of (4.9), orthogonal at the points $y = n(n+\alpha+\beta+1)$ $(n = 0,1,\dots,N)$ with respect to the same weights. Thus the two functions are equal up to a factor not depending on n. This factor can be determined by putting $n = N$. By (4.8), (4.3) and the Chu-Vandermoude sum we obtain:

$$R_x(N(N+\alpha+\beta+1);\alpha,\beta,N) = Q_N(x;\alpha,\beta,N)$$

$$= {}_2F_1(-x,N+\alpha+\beta+1;\alpha+1;1) = \frac{(-\beta-N)_x}{(\alpha+1)_x} .$$

It follows from (4.10) and (4.8) that

$$Q_n(x;\alpha,\beta,N) = \frac{(-\beta-N)_x}{(\alpha+1)_x} Q_{N-n}(x;-N-\beta-1,-N-\alpha-1,N). \tag{4.11}$$

Combination with (4.5) yields

$$C^{\frac{1}{2}N,\frac{1}{2}(N+\alpha+\beta),N-n+\frac{1}{2}(\alpha+\beta)}_{\frac{1}{2}N-x,\frac{1}{2}(\beta-\alpha-N)+x,\frac{1}{2}(\beta-\alpha)} = (\alpha+N)!N! \cdot$$

$$\cdot \left(\frac{(2N-2n+\alpha+\beta+1)(n+\beta)!(n+\alpha+\beta)!}{x!(N-x)!(\alpha+N-x)!(\beta+x)!n!(N-n)!(N-n+\beta)!(2N-n+\alpha+\beta+1)!}\right)^{\frac{1}{2}} \cdot \tag{4.12}$$

$$\cdot\ Q_n(x;-N-\alpha-1,-N-\beta-1,N),$$

where x, n, N, α, β are integers and $0 \le x \le N$, $0 \le n \le N$, $\alpha \ge 0$, $\beta \ge 0$. In view of (4.12), the orthogonality relations (4.4), with α, β replaced by $-N-\alpha-1$, $-N-\beta-1$, are the same as the orthogonality relations for the CG coefficients in the left hand side of (4.12) as a function of x.

5. NOTES

5.1. See SPRINGER [13, p.68,69] for a short description of the 19^{th} century concept of "development in a Clebsch-Gordan series".

5.2. CG coefficients in their present meaning (also called Wigner coefficients) were first introduced in VON NEUMANN & WIGNER [8, Anhang]. WIGNER [16, (17.27)] first derived a ${}_3F_2(1)$ type summation formula for these coefficients. The notation using 3-j symbols was introduced in WIGNER [17], where he also discussed 3n-j symbols for more general n.

5.3. Formula (3.15) occurs in SCHWINGER [10, (3.39)]; see also VILENKIN [14, Ch. 3, § 8.9] (where a factor $(-1)^{\ell_1+\ell_2-\ell}$ is missing). A variant of (3.19) (an expansion for $\exp(\det(x_{ij}))$) is also derived by SCHWINGER [10, (3.42)]. Formula (3.19) and the resulting symmetries are given by REGGE [9].

5.4. GELFAND, MINLOS & SHAPIRO [4, Supplement III] (see also VILENKIN [14, Ch. 3, § 8.7]) point out that CG coefficients are, in a certain sense, analogous to Jacobi polynomials, since they can be expressed by means of a Rodrigues type formula involving repeated differences. In view of our expressions of CG coefficients in terms of Hahn polynomials (cf. § 4) and the Rodrigues type formula for Hahn polynomials (cf. WEBER & ERDÉLYI [15]), this is no surprise.

5.5. The theory of CG coefficients was highly motivated by the quantum theory of angular momentum, cf. the introduction and further papers in BIEDENHARN & VAN DAM [2]. A useful survey is also given by SMORODINSKIĬ & SHELEPIN [12].

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