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SPHERICAL DISTRIBUTIONS ON AN EXCEPTIONAL HYPERBOLIC SPACE OF TYPE F_4

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Spherical distributions on an exceptional hyperbolic space of type F_4 *)

by

M.T. Kosters **)

ABSTRACT

The spherical distributions on the pseudo Riemannian symmetric space $F_{4(-20)}/\mathrm{Spin}\,(1,8)$ are completely determined. This is done by using a construction of this space based on a Jordan algebra over the Cayley algebra of octonions. Then it is determined which of the spherical distributions are positive definite. Using some results about the Fourier-Jacobi transform a Plancherel formula for the space is obtained.

KEY WORDS & PHRASES: *pseudo Riemannian symmetric spaces, Cayley octonions, exceptional Jordan algebra's, spherical distributions*

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*) This report will be submitted for publication elsewhere.

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INTRODUCTION

In the theory of harmonic analysis on Riemannian symmetric spaces a fundamental role is played by the spherical functions. In the pseudo Riemannian case they are replaced by spherical distributions. These were completely determined in [4] for the spaces of the form G/H with $G = U(p,q,\mathbb{F})$, the group of $n\times n$- ($n=p+q$) matrices leaving invariant the hermitian form [,] on $\mathbb{F}^n$ defined by $[x,y] = \bar{y}_1x_1 + \ldots + \bar{y}_px_p - \bar{y}_{p+1}x_{p+1} - \ldots - \bar{y}_{p+q}x_{p+q}$ and with $H = U(1,\mathbb{F}) \times U(p-1,q,\mathbb{F})$.

These spaces are isotropic pseudo Riemannian spaces of rank one. Those have been classified ([10],p.379) and the classification shows that there are three spaces of this kind left, all intimately related with the algebra of Cayley octonions. One of these is Riemannian (and the spherical functions have been studied in [9]), another one is compact, and we are left with one space, called the indefinite Cayley projective plane in [10].

We shall construct a model for this space in the spirit of [9], where this is done in the Riemannian case and by using this model we shall obtain the spherical distributions. Here spherical distributions are, as in [4], defined as H-bi-invariant distributions on G which are eigendistributions of all differential operators on A which come from the center of the universal enveloping algebra. Because G/H is isotropic (i.e. H acts (via Ad) transitively on each pseudosphere in the orthoplement $\mathfrak{q}$ of $\mathfrak{h}$, the Lie algebra of H, with respect to the Killing form of the Lie algebra $\mathfrak{g}$ of G) the algebra of these differential operators is generated by the Casimir-operator and thus it suffices to look for H-bi-invariant eigendistributions of the Casimir operator.

§1. THE ALGEBRA OF OCTONIONS, THE JORDAN ALGEBRA $J_{1,2}$

This section is intended only in order to fix the notation. For a list of formulae from which the reader can verify the correctness of our future manipulations with octonions we refer to [9]p.513-514.

Let $\mathbb{H}$ be the real four dimensional algebra of quaternions. It has a basis $\{1,i,j,k\}$ with the relations $i^2 = j^2 = k^2 = 1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

We define a real linear mapping $x \mapsto \bar{x}: \mathbb{H} \to \mathbb{H}$ called conjugation by $\bar{1} = 1$, $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$.

$\mathbb{O}$, the Cayley octonion algebra is defined as follows: $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ and a multiplication in $\mathbb{O}$ is defined by:

$$(x,y)(x',y') = (xx'-\bar{y}'y, y\bar{x}'+y'x).$$

Then $\mathbb{O}$ is a real algebra of dimension 8 with basis $e_0 = (1,0)$, $e_1 = (i,0),\ldots,e_7 = (0,k)$. We define a conjugation in $\mathbb{O}$ by $\overline{(x,y)} = (\bar{x},-y)$. Then $\bar{e}_0 = e_0$, $\bar{e}_i = -e_i$ $(i = 1,\ldots,7)$. e_0 is the identity element of $\mathbb{O}$. We shall identify its multiples $\lambda e_0 (\lambda \in \mathbb{R})$ with the real numbers λ. When $x \in \mathbb{O}$ we define its real part by $\operatorname{Re} x = \frac{1}{2}(x+\bar{x}) \in \mathbb{R}$. We define an inner product $(\ ,\ )$ on O by $(x,y) = \operatorname{Re} x\bar{y}$. Then $\{e_0,\ldots,e_7\}$ is an orthonormal basis of $\mathbb{O}$. Write $(x,x)^{\frac{1}{2}} = |x|$, then the norm $|\ |$ is multiplicative: $|xy| = |x||y|\ \forall x,y \in \mathbb{O}$. It follows that $\mathbb{O}$ is a division algebra: every non-zero element x has inverse $x^{-1} = |x|^{-2}\bar{x}$.
$\mathbb{O}$ is neither commutative nor associative. $\tilde{\mathbb{O}} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ is in a natural way a complex algebra of dimension 8. Let $J_{1,2}$ be the 27-dimensional real vector-space consisting of the 3×3-matrices with coefficients in $\tilde{\mathbb{O}}$ of the form

$$X(\xi_1,\xi_2,\xi_3,u_1,u_2,u_3) = \begin{pmatrix} \xi_1 & u_3\otimes(-1)^{\frac{1}{2}} & \bar{u}_2\otimes(-1)^{\frac{1}{2}} \\ \bar{u}_3\otimes(-1)^{\frac{1}{2}} & \xi_2 & u_1 \\ u_2\otimes(-1)^{\frac{1}{2}} & \bar{u}_1 & \xi_3 \end{pmatrix}$$

with $\xi_1, \xi_2,\xi_3 \in \mathbb{R}$, $u_1,u_2,u_3 \in \mathbb{O}$. If we define a multiplication $\circ$ in $J_{1,2}$ by $X\circ Y = \frac{1}{2}(XY+YX)$ then $J_{1,2}$ is a real commutative nonassociative algebra. We use the following standard notations for elements of $J_{1,2}$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_1^{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \bar{a} & 0 \end{pmatrix} \qquad F_2^{a\otimes(-1)^{\frac{1}{2}}} = \begin{pmatrix} 0 & 0 & \bar{a}\otimes(-1)^{\frac{1}{2}} \\ 0 & 0 & 0 \\ a\otimes(-1)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

$$F_3^{a\otimes(-1)^{\frac{1}{2}}} = \begin{pmatrix} 0 & a\otimes(-1)^{\frac{1}{2}} & 0 \\ \bar{a}\otimes(-1)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (a \in \mathbb{O})$$

Finally (,) denotes the real nondegenerate bilinear form on $J_{1,2}$ defined by $(X,Y) = \operatorname{tr} X \circ Y$.

§2. THE GROUPS $G = F_{4(-20)}$ AND $H = SPIN\ (1,8)$

Let $\widetilde{G}$ be the group of automorphisms of $J_{1,2}$, G the connected component of $\widetilde{G}$ containing the identity. (Actually it is not very difficult using the results of this paper, to show that G is connected, hence $G = \widetilde{G}$, but we will not need this fact). Then G is a connected noncompact simple Lie group of type F_4. The Lie algebra $\mathfrak{g}$ of G consists of the derivations of $J_{1,2}, \mathfrak{g} = \operatorname{Der} J_{1,2}$.

We shall need from [9] the following facts about $\mathfrak{g}$. Consider O(8) as the orthogonal group of the inner product space $\mathbb{O}$ and let $\mathcal{O}(8)$ be its Lie algebra acting in $\mathbb{O}$ by antisymmetric endomorphisms. Then we have the so called principle of triality in $\mathcal{O}(8)$: if $D_2 \in \mathcal{O}(8)$ then there are unique $D_1, D_3 \in \mathcal{O}(8)$ such that $(D_1 u)v + u(D_2 v) = \overline{D_3(\overline{uv})}$ $\forall u,v \in \mathbb{O}$ (*). Thus we have a subalgebra ℓ of $\mathfrak{g}$ isomorphic with $\mathcal{O}(8)$: if we define for $D_2 \in \mathcal{O}(8)$ and D_1, D_3 as in (*) $\eta = (D_1, D_2, D_3) \in \mathfrak{g}$ by

$$\eta \begin{pmatrix} \xi_1 & u_3\otimes(-1)^{\frac{1}{2}} & \bar{u}_2\otimes(-1)^{\frac{1}{2}} \\ \bar{u}_3\otimes(-1)^{\frac{1}{2}} & \xi_2 & u_1 \\ u_2\otimes(-1)^{\frac{1}{2}} & \bar{u}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3u_3\otimes(-1)^{\frac{1}{2}} & D_2u_2\otimes(-1)^{\frac{1}{2}} \\ D_3u_3\otimes(-1)^{\frac{1}{2}} & 0 & D_1u_1 \\ D_2u_2\otimes(-1)^{\frac{1}{2}} & \overline{D_1u_1} & 0 \end{pmatrix}$$

and let ℓ consist of these η then the mappings $\phi_i : \ell \to \mathcal{O}(8)$ $(i = 1,2,3)$ defined by $\phi_i(\eta) = D_i$ are isomorphisms.

If A is an antihermitian matrix of trace 0 with coefficients in $\widetilde{\mathbb{O}}$ such that $\tilde{A}(X) = -XA + AX \in J_{1,2}$ $\forall X \in J_{1,2}$ then $\tilde{A} \in g$. So if we take for $a \in \mathbb{O}$

$$A_1^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad A_2^{a\otimes(-1)^{\frac{1}{2}}} = \begin{pmatrix} 0 & 0 & -\bar{a}\otimes(-1)^{\frac{1}{2}} \\ 0 & 0 & 0 \\ a\otimes(-1)^{\frac{1}{2}} & 0 & 0 \end{pmatrix},$$

$$A_3^{a\otimes(-1)^{\frac{1}{2}}} = \begin{pmatrix} 0 & a\otimes(-1)^{\frac{1}{2}} & 0 \\ -\bar{a}\otimes(-1)^{\frac{1}{2}} & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$$

then these define elements of g.

If we let for $i = 1,2,3$ a_i be the 8-dimensional subspace of g consisting of the $\tilde{A}_i$ then $g = \ell \oplus a_1 \oplus a_2 \oplus a_3$. Take $k = \ell \oplus a_1$, $p = a_2 \oplus a_3$ then $g = k \oplus p$ is a Cartan decomposition of g.

Now we are ready for our first purpose, the construction of an involutive automorphism σ of g which is not a Cartan involution.

First we construct a Cartan involution θ corresponding to the Cartan decomposition $g = k \oplus p$. Take

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and define for $X \in J_{1,2}$ $\tilde{J}(X) = JXJ \in J_{1,2}$ then it is easily verified that $\tilde{J} \in \mathrm{Aut}(J_{1,2})$. If $X \in g = \mathrm{Der}\, J_{1,2}$ then we define $\theta X = \tilde{J}X\tilde{J} \in g$. We see that θ is an involutive automorphism of g and that $\theta|k = \mathrm{id.}$, $\theta|p = -\mathrm{id.}$

When we apply the same construction to

$$J' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

we get an involutive automorphism σ of g which commutes with θ. An easy calculation shows that $\sigma|h = \mathrm{id.}$, $\sigma|g = -\mathrm{id.}$ where $h = \ell \oplus a_3$, $q = a_1 \oplus a_2$.

Take $L = \tilde{A}_2^{1\otimes(-1)^{\frac{1}{2}}}$ and $a = \mathbb{R}L$ then a is maximal among the abelian

subalgebras of $\mathfrak{g}$ contained in $\mathfrak{q}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$ then $\mathfrak{m} = \{\eta \in \mathfrak{l} \mid \eta F_2^{1\otimes(-1)^{\frac{1}{2}}} = 0\}$ ϕ_2 gives an isomorphism of $\mathfrak{m}$ with the subalgebra $\mathfrak{o}(7)$ of $\mathfrak{o}(8)$ consisting of those D with $De_0 = 0$.
$\mathfrak{a}$ is also contained in $\mathfrak{p}$ so it consists of semi-simple elements. The root-space decomposition which we thus have is given in [9]: for $z \in \mathbb{O}$, and $y \in \mathbb{O}$ with Re $y = 0$ take

$$Z(z) = (-A_1^z - A_3^{\bar{z}\otimes(-1)^{\frac{1}{2}}})^\sim$$

$$Z^-(z) = (-A_1^z + A_3^{\bar{z}\otimes(-1)^{\frac{1}{2}}})^\sim$$

$$Y(y) = (y(E_1-E_3) + A_2^{y\otimes(-1)^{\frac{1}{2}}})^\sim$$

$$Y^-(y) = (y(E_1-E_3)-A_2^{y\otimes(-1)^{\frac{1}{2}}})^\sim$$

then these are eigenvectors of ad L corresponding to the eigenvalues 1,−1,2 and −2 respectively so if we define $\alpha \in \mathfrak{a}^*$ by $\alpha(tL) = t$ then we have root spaces $\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}, \mathfrak{g}^{2\alpha}$ and $\mathfrak{g}^{-2\alpha}$ of dimensions 8,8,7 and 7. Take $\mathfrak{n} = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$, $\mathfrak{n}^- = \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-2\alpha}$ then $\mathfrak{n}^-$ and $\mathfrak{n}$ are nilpotent subalgebra's of $\mathfrak{g}$ of dimension 15 with center $\mathfrak{g}^{-2\alpha}$ and $\mathfrak{g}^{2\alpha}$ respectively.

Now we will describe the various subgroups of A belonging to the subalgebra's of $\mathfrak{g}$ which we have seen above. Firstly we give the group L belonging to $\mathfrak{l}$. The following is well-known:

LEMMA 1. (*principle of triality in* SO(8)).
If $\alpha_1 \in SO(8)$ *then there are* $\alpha_2, \alpha_3 \in SO(8)$ *such that*
$\alpha_1(u)\,{}_2(v) = \alpha_3(\overline{uv})\ \forall u,v \in \mathbb{O}$ (*)
If $\alpha_1(u)\alpha_2'(v) = \overline{\alpha_3'(\overline{uv})}\ \forall u,v \in \mathbb{O}$ *then either* $\alpha_2 = \alpha_2'$ *and* $\alpha_3 = \alpha_3'$ *or* $\alpha_2 = -\alpha_2'$ *and* $\alpha_3 = -\alpha_3'$.

We denote the triples in SO(8) × SO(8) × SO(8) satisfying (*) by $(\alpha_1,\alpha_2,\alpha_3)$. They form a subgroup Spin(8) of SO(8) × SO(8) × SO(8) which is a twofold covering of SO(8). Covering homomorphisms ϕ_i (i = 1,2,3) are given by $\phi_i(\alpha_1,\alpha_2,\alpha_3) = \alpha_i$.

If $\eta = (\alpha_1,\alpha_2,\alpha_3) \in$ Spin(8) then η determines an automorphism $\rho(\eta)$ of $J_{1,2}$ given by:

$$\phi(\eta)\quad \begin{pmatrix} \xi_1 & u_3\otimes(-1)^{\frac{1}{2}} & \bar{u}_2\otimes(-1)^{\frac{1}{2}} \\ \bar{u}_3\otimes(-1)^{\frac{1}{2}} & \xi_2 & u_1 \\ u_2\otimes(-1)^{\frac{1}{2}} & \bar{u}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha_3(u_3)\otimes(-1)^{\frac{1}{2}} & \overline{\alpha_2(u_2)}\otimes(-1)^{\frac{1}{2}} \\ \overline{\alpha_3(u_3)}\otimes(-1)^{\frac{1}{2}} & \xi_2 & \alpha_1(u_1) \\ \alpha_2(u_2)\ (-1)^{\frac{1}{2}} & \overline{\alpha_1(u_1)} & \xi_3 \end{pmatrix}$$

We will denote $\phi(\eta)$ also by $(\alpha_1,\alpha_2,\alpha_3)$.
ϕ is an isomorphism between Spin(8) and the subgroup L of A with Lie algebra $\mathcal{l}$ given by $L = \{\ell \in G \mid \ell E_1 = E_1,\ \ell E_2 = E_2\}$

Let M be the subgroup of L consisting of those $\ell \in L$ for which $\ell F_2^{1\otimes(-1)^{\frac{1}{2}}} = F_2^{1\otimes(-1)^{\frac{1}{2}}}$ then M has Lie algebra $\mathcal{m}$. If $\alpha \in SO(8)$ define $\kappa\alpha \in SO(8)$ by $\kappa\alpha(u) = \overline{\alpha(\bar{u})}$ then M consists of the elements $(\tilde{\alpha},\alpha,\kappa\tilde{\alpha})$ with $\alpha \in SO(7)$ (i.e. $\alpha(e_0)=e_0$). Lemma 1 says that $\tilde{\alpha}(u)\alpha(v) = \tilde{\alpha}(uv)$ for such an element . M is a twofold covering group Spin(7) of SO(7) in the obvious way.

K, the connected subgroup of A with Lie algebra $\mathcal{k}$ is isomorphic to Spin(9). It has a decomposition K = LBL where B is a subgroup of K isomorphic to the circle:

$$B = \{u_\theta = \exp \theta\tilde{A}_1^1 \mid \theta \in \mathbb{R}\} \text{ ([9], §4, Prop.1).}$$

N, the connected subgroup of A with Lie algebra $\mathcal{n}$ consists of the elements $u(y,z) = \exp(Y(y)+Z(z))$. Its center consists of the elements $u(y,0)$.

Corresponding to $\mathcal{a}$ we have the subgroup $A = \{a_t = \exp|A_2^{1\otimes(-1)^{\frac{1}{2}}} | t \in \mathbb{R}\}$.
Now we come to the important subgroup H with Lie algebra $\mathcal{h}$. If $X \in \mathcal{g}$ then we easily see $X \in \mathcal{h} \Longleftrightarrow XE_3 = 0$ so if we take $H = \{h \in G \mid hE_3 = E_3\}$ then H is a closed subgroup of A with Lie algebra $\mathcal{h}$.
We now describe H in more detail.
Let $S^0 = \{X \in J_{1,2} \mid E_3 \circ X = 0,\ \mathrm{tr}\, X = 0,\ (X,X) = 2\}$.
If $X \in S^0$, $h \in H$ then $E_3 \circ hX = hE_3 \circ hX = h(E_3 \circ X) = 0$, $\mathrm{tr}\, hX = \mathrm{tr}\, X = 0$, $(hX,hX) = (X,X) = 2$ so $hX \in S^0$, that is, S^0 is invariant under H.
$S^0 = \{\xi(E_1-E_2)+F_3^{w\otimes(-1)^{\frac{1}{2}}} = X(\xi,w) \mid \xi^2 - |w|^2 = 1\}$ a two sheeted hyperboloid in the linear subspace of $J_{1,2}$ consisting of the elements $X(\xi,w)$ ($\xi\in\mathbb{R}$, $w\in\mathbb{O}$).
Take $b_t = \exp t\tilde{A}_3^{1\otimes(-1)^{\frac{1}{2}}}$ ($t\in\mathbb{R}$) then $b_t(E_1-E_2) = X(\cosh 2t, -\sinh 2t)$. From this we see that if we let $S = \{X \in S^0 \mid (X,E_1) \geq 1\}$ then $S \subset H(E_1-E_2)$. Indeed, take $X(\xi,w) \in S$ then we show that $X(\xi,w) \in H(E_1-E_2)$. We may assume $w \neq 0$, $\xi \geq 1$, so there is $t \geq 0$ such that $\cosh 2t = \xi$. Then $\sinh 2t = |w|$. There is $\ell = (\alpha_1,\alpha_2,\alpha_3) \in L$ such that $\alpha_3(1) = -\frac{w}{|w|}$. Then

$X(\xi,w) = \ell b_t(E_1-E_2)$.

Now we want to prove that even $S = H(E_1-E_2)$. From the preceding result it is clear that we need only show that $-E_1 + E_2 \notin H(E_1-E_2)$.

We use the following result of which the easy proof can be found in [9]:

LEMMA 2. *If* $g \in G$ *then* $(gE_1,E_1) \geq 1$.

Now suppose that there is $h \in H$ such that $h(E_1-E_2) = -E_1 + E_2$. Then $h(E_1+E_2) = h((E_1-E_2)\circ(E_1-E_2)) = h(E_1-E_2)\circ h(E_1-E_2) = E_1 + E_2$ and adding the two equations yields $hE_1 = E_2$ which contradicts lemma 2.

We now know that H acts transitively on S. If $\ell \in L$ then $\ell E_1 = E_1$, $\ell E_2 = E_2$ so $\ell(E_1-E_2) = E_1 - E_2$. Conversely $\ell(E_1-E_2) = E_1 - E_2 \Rightarrow \ell(E_1+E_2) =$ $= \ell((E_1-E_2)\circ(E_1-E_2)) = \ell(E_1-E_2)\circ\ell(E_1-E_2) = (E_1-E_2)\circ(E_1-E_2) = E_1 + E_2$ and adding and subtracting these two equations yields $\ell E_1 = E_1$, $\ell E_2 = E_2$ so $\ell \in L$. Hence L is the isotropy group of $E_1 - E_2$, $S = H/L$. S and L are simply connected hence so is H. Furthermore we see that the action of H on the space $\{X(\xi,w) \mid \xi \in \mathbb{R}, w \in \mathbb{O}\}$ gives a homomorphism of H onto the connected component $SO_0(1,8)$ of $SO(1,8)$ which contains the identity (because H leaves invariant the restriction of the bilinear form (,) to this space, which is of signature (1,8)). Because dim H = dim $SO_0(1,8)$ (=36) this homomorphism is a covering, and because the fundamental group of $SO_0(1,8)$ consists of two elementes we have:

PROPOSITION 1. H *is connected and simply connected. It is a twofold covering group* Spin(1,8) of $SO_0(1,8)$. *Every element* $h \in H$ *can be written as* $h = \ell b_t \ell'$ $(\ell,\ell' \in L$, $t \geq 0$, t *uniquely determined*).

§3. THE HOMOGENEOUS SPACE X = G/H

We will construct an action of G on a certain submanifold $X \subset J_{1,2}$ admitting H as the isotropy subgroup of a certain point of X. The method by which we prove that G acts transitively on X is obviously not the simplest one but it has the advantage that it also yields the "analogue of the Iwasawa decomposition".

Take $X = \{Y \in J_{1,2} \mid Y\circ Y = Y,\ \mathrm{tr}\ Y = 1,\ (Y,E_1) \leq 0\}$ then we see (using the proof of lemma 2) that G leaves X invariant. The element E_3 of X has

has isotropy group H so it remains to prove that $GE_3 = X$. Let us first describe X in more detail. If $Y = X(\xi_1,\xi_2,\xi_3,u_1,u_2,u_3) \in X$ then it follows from $Y \circ Y = Y$ that:

(1) $\xi_1 = \xi_1^2 - |u_2|^2 - |u_3|^2$

(2) $\xi_2 = \xi_2^2 + |u_1|^2 - |u_3|^2$

(3) $\xi_3 = \xi_3^2 + |u_1|^2 - |u_2|^2$

(4) $u_3 = \bar{u}_2\bar{u}_1 + \xi_1 u_3 + \xi_2 u_3$

and from $tr(Y) = 1$ that:

(5) $\xi_1 + \xi_2 + \xi_3 = 1$

(4),(5) $\Rightarrow \xi_3 u_3 = \bar{u}_2\bar{u}_1$ so if $\xi_3 = 0$ then $u_1 = 0$ or $u_2 = 0$. Also it follows from (3) that $\xi_3 = 0 \Rightarrow |u_1| = |u_2|$ so $\xi_3 = 0 \Rightarrow u_1 = u_2 = 0$.
(1) $\Rightarrow \xi_1^2 - \xi_1 - |u_3|^2 = 0 \Rightarrow \xi_1 = \frac{1}{2}(1-(1+4|u_3|^2)^{\frac{1}{2}})$ because $\xi_1 \leq 0$. So X contains the following elements with $\xi_3 = 0$:

$$X(u) = \begin{pmatrix} \frac{1-(1+4|u|^2)^{\frac{1}{2}}}{2} & u\otimes(-1)^{\frac{1}{2}} & 0 \\ \bar{u}\otimes(-1)^{\frac{1}{2}} & \frac{1+(1+4|u|^2)^{\frac{1}{2}}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (u \in \mathbb{O})$$

Now assume $\xi_3 \neq 0$. Write $x = \frac{\bar{u}_1}{\xi_3}$, $y = \frac{u_2}{\xi_3}$ then
(3) $\Rightarrow \xi_3 = \xi_3^2(1+|x|^2-|y|^2) \Rightarrow \xi_3 = (1+|x|^2-|y|^2)^{-1}$. Substracting (2) from (1):

$$\xi_1 - \xi_2 = (\xi_1-\xi_2)(\xi_1+\xi_2) - |u_1|^2 - |u_2|^2 \Rightarrow \text{(using (5))}$$

$$(\xi_1-\xi_2)\xi_3 = -\xi_3^2(|x|^2+|y|^2) \Rightarrow \xi_1 - \xi_2 = \frac{-|x|^2-|y|^2}{1+|x|^2-|y|^2} .$$

$$\xi_1 + \xi_2 = 1 - \xi_3 = \frac{|x|^2-|y|^2}{1+|x|^2-|y|^2} \quad \text{so:}$$

$$\xi_1 = \frac{-|y|^2}{1+|x|^2-|y|^2} \quad \xi_2 = \frac{|x|^2}{1+|x|^2-|y|^2} \quad .$$

$$\xi_3\, u_3 = \bar{u}_2 \bar{u}_1 \Rightarrow u_3 = \frac{\bar{y}x}{1+|x|^2-|y|^2} \quad .$$

Thus the elements of X with $\xi_3 \neq 0$ are:

$$X(x,y) = \frac{1}{R}\begin{pmatrix} -|y|^2 & \bar{y}x\otimes(-1)^{\frac{1}{2}} & \bar{y}\otimes(-1)^{\frac{1}{2}} \\ \bar{x}y\ (-1)^{\frac{1}{2}} & |x|^2 & \bar{x} \\ y\otimes(-1)^{\frac{1}{2}} & x & 1 \end{pmatrix} \qquad (x,y\in\mathbb{O}, R=1+|x|^2-|y|^2>0)$$

Now we will prove that $GE_3 = X$.
From the definition of $u(y,z) \in N$ we calculate:

$$u(y,z)E_3 = -\left(\frac{|z|^4}{4}+|y|^2\right)E_1 + |z|^2E_2 + \left(1-|z|^2+\frac{|z|^4}{4}+|y|^2\right)E_3 + $$
$$+ F_1^{-z(1-\frac{|z|^2}{2}+y)} + F_2^{(\frac{1}{2}|z|^2(1-\frac{|z|^2}{2})-y-|y|^2)\otimes(-1)^{\frac{1}{2}}} + F_3^{-(y+\frac{|z|^2}{2})\,\bar{z}\otimes(-1)^{\frac{1}{2}}}$$

From this, together with the formulae in [9] for the action of the group A we deduce:

<u>LEMMA 3</u>. $a_t u(y,z)E_3$ *is of the form* $X(u)$ *if and only if* $y = 0$, $e^{-2t} = |z|^2-1$. $X(0) = E_2$ *is not of the form* $a_t u(y,z)E_3$. *If* $u \neq 0$ *then there are uniquely determined* t,y *and* z *such that* $a_t u(y,z)E_3 = X(u)$. *They are given by* $u = -\bar{z}(|z|^2-1)$, $e^{-2t} = |z|^2-1$, $y = 0$.

If $y \neq 0$ or $e^{-2t} \neq |z|^2-1$ then $a_t u(y,z)E_3$ is of the form $X(x_1,x_2)$. Manipulation with the above mentioned formulae yields in this case:

<u>LEMMA 4</u>. *If* $x_2 = -1$ *then* $X(x_1,x_2)$ *is not of the form* $a_t u(y,z)E_3$. *If* $x_2 \neq -1$ *then there are unique* t,y,z *such that* $X(x_1,x_2) = a_t u(y,z)E_3$. *They are given by:*

$$e^{-2t} = \frac{|1+x_2|^2}{1+|x_1|^2-|x_2|^2}$$

$$y = \tfrac{1}{2}(\bar{x}_2-x_2)/(1+|x_1|^2-|x_2|^2)$$

$$z = -\bar{x}_1(1+x_2)/(|1+x_2|(1+|x_1|^2-|x_2|^2)).$$

Now we are in a position to show that $GE_3 = X$. Because of Lemmas 3 and 4 we only have to show that $E_2 \in GE_3$ and $X(x,-1) \in GE_3$ if $1+|x|^2-1 = |x|^2 > 0$. Define for $\theta \in \mathbb{R}$ $c_\theta = \exp \theta A_1^1$ then

$$c_\theta E_3 = (\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta)E_2 + (\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta)E_3 + \tfrac{1}{2}\sin 2\theta F_1^1$$

so $c_{\pi/4}\, E_3 = E_2$.
If $0 \neq x \in \mathbb{O}$ then there is $t \in \mathbb{R}$ such that $e^t = |x|$

$$\text{and then } X(x,-1) = \begin{pmatrix} -e^{2t} & -e^{t}\bar{w}\otimes(-1)^{\frac{1}{2}} & -e^{2t}\otimes(-1)^{\frac{1}{2}} \\ -e^{t}w\otimes(-1)^{\frac{1}{2}} & 1 & e^{t}w \\ -e^{2t}\otimes(-1)^{\frac{1}{2}} & e^{t}\bar{w} & e^{2t} \end{pmatrix}$$

with $w \in \mathbb{O}$, $|w| = 1$.
Now choose $m = (\tilde{\alpha},\alpha,\kappa\tilde{\alpha}) \in M$ such that $\tilde{\alpha}(w) = 1$ (that this can be done is proved in [9], p.534 proof of Lemma 1) then $m\ X(x,-1) = a_t X(x,-1)$ so the elements $X(x,-1) \in X$ can be written in the form $X(x,-1) = ma_t X(1,-1)$ $(m\in M, t\in\mathbb{R})$ and we only have to show that $X(1,-1) \in GE_3$. Now $c_\theta X(1,-1) = X(\xi_1,\xi_2,\xi_3,u_1,u_2,u_3)$ with $\xi_2 = 1 + \sin 2\theta$, $\xi_3 = 1 - \sin 2\theta$ so if we take θ such that $0 \neq \sin 2\theta \neq 1$ then it follows from lemma 4 that $c_\theta X(1,-1) \in GE_3$ and hence $X(1,-1) \in GE_3$.
We have now proved:

PROPOSITION 2. G *acts transitively on* X *and* X *is identified with* G/H *by* $\bar{g} \leftrightarrow gE_3$ *if* $g \in G$ *is a representative for* $\bar{g} \in G/H$.

If $a_t u(y,z)E_3$ is of the form $X(x_1,x_2)$ then we have seen that

$$e^{-2t} = \frac{|1+x_2|^2}{1+|x_1|^2-|x_2|^2} = \frac{1+x_2+\bar{x}_2+|x_2|^2}{1+|x_1|^2-|x_2|^2} = (X(x_1,x_2),E_3-F_2^{1\otimes(-1)^{\frac{1}{2}}}-E_1)$$

and if $a_t\ u(y,z)E_3 = X(u)$ then $e^{-2t} = |z|^2-1 = -\tfrac{1}{2} + \tfrac{1}{2}(1+4|u|^2)^{\frac{1}{2}} =$
$= (X(u),E_3-F_2^{1\otimes(-1)^{\frac{1}{2}}}-E_1)$. Define $\xi^0 = -E_3 + F_2^{1\otimes(-1)^{\frac{1}{2}}} + E_1$ and for $Y \in X$ $P_0(Y) = -(Y,\xi^0)$, then, combining lemma's 3 and 4:

PROPOSITION 3. (*cf.* [4], Prop.4.2).
If $x \in X$ *then there are elements* $a_t \in A$ *and* $u(y,z) \in N$ *such that* $x = u(y,z)a_tE_3$ *if and only if* $P_0(x) \neq 0$. *In this case* $u(y,z)$ *and* a_t *are uniquely determined,* t *being given by* $t = -\frac{1}{2}\log P_0(x)$.

PROOF. Observe that A normalizes N

COROLLARY. *If* $g \in G$ *then* g *is of the form* $g = na_th$ $(n\in N, a_t\in A, h\in H)$ *if and only if* $P_0(gE_3) \neq 0$. *In this case* n, a_t *and* h *are unique,* t *being given by* $t = -\frac{1}{2}\log P_0(gE_3)$ *(these* g *form an open dense submanifold of* G).

This is an analogue of the Iwasawa decomposition which we have in the Riemannian case.

§4. THE CONE Ξ = G/MN AND THE POISSON KERNEL

Take $\Xi^0 = \{X \in J_{1,2} \mid \mathrm{tr}X = 0,\ X\circ X = 0\}$ then Ξ^0 is clearly invariant under G and so is $\Xi^0 - \{0\} = \{X \in \Xi^0 \mid (X,E_1) \neq 0\}$ G is connected so the connected component $\Xi = \{X \in \Xi^0 \mid (X,E_1) = 0\}$ of $\Xi^0 - \{0\}$ is also invariant under G. Analogous calculations as those which we did for the space X in the preceding section show that Ξ consists of the elements

$$X(\xi,x,y) = \xi \begin{pmatrix} 1 & y\otimes(-1)^{\frac{1}{2}} & \bar{x}\ (-1)^{\frac{1}{2}} \\ \bar{y}\otimes(-1)^{\frac{1}{2}} & -|y|^2 & -\bar{y}\bar{x} \\ x\otimes(-1)^{\frac{1}{2}} & -xy & -|x|^2 \end{pmatrix} \qquad \begin{matrix}(\xi>0, x,y\in\mathbb{O} \\ |x|^2+|y|^2=1)\end{matrix}$$

Define for $x,y \in \mathbb{O}$ with $|x|^2 + |y|^2 = 1$

$$Y(x,y) = \begin{pmatrix} 0 & y\otimes(-1)^{\frac{1}{2}} & \bar{x}\otimes(-1)^{\frac{1}{2}} \\ \bar{y}\otimes(-1)^{\frac{1}{2}} & 0 & 0 \\ x\otimes(-1)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

then it is shown in [9] p.535,536 that $S^{15} = \{Y(x,y) \mid x,y \in \mathbb{O}, |x|^2+|y|^2 = 1\}$ is invariant under K, that K acts transitively on S^{15} and that the isotropy group of $Y(1,0)$ is M.
Now $X(1,x,y) = 2E_1 + Y(x,y) + Y(x,y)\circ Y(x,y)$ and because K fixes E_1 K also

acts transitively on $\{X(1,x,y) \mid x,y \in \mathbb{O}, |x|^2 + |y|^2 = 1\} \simeq S^{15}$. $a_t X(1,1,0) = e^{2t} X(1,1,0)$ so we have:

<u>LEMMA 5</u>. G *acts transitively on* Ξ, *every element of* Ξ *can be written as* $X(\xi,x,y) = ka_t X(1,1,0)$ t *being determined uniquely by* $\xi = e^{2t}$ *and* k *being determined uniquely modulo* M *by* $kY(1,0) = Y(x,y)$.

Observe that $X(1,1,0)$ is precisely the element ξ^0 of the preceding section.
We now show that its isotropy group is equal to MN. Clearly it contains M and that it contains N follows from an easy calculation. MN has Lie algebra $\mathfrak{m}\oplus\mathfrak{n}$ of dimension 36, the isotropy group has dimension $\dim G - \dim \Xi = 52-16 = 36$ so MN is the connected component of the isotropy group containing the identity. But $\Xi \simeq S^{15} \times (0,\infty)$ is simply connected so the isotropy group is connected, hence equal to MN.

Define for $x \in X$, $\xi \in \Xi$, $P(x,\xi) = -(x,\xi)$. $P: X \times \Xi \to \mathbb{R}$ is called the Poisson kernel. It takes only nonnegative values on $X \times \Xi$ and it will play an important role in the construction of spherical distributions. It has the following obvious properties:

(i) $P(gx,g\xi) = P(x,\xi) \forall g \in G,\ x \in X,\ \xi \in \Xi$.
(ii) $P(x,\xi^0) = P_0(x) \forall x \in X$.

Define $P: \Xi \times \Xi \to \mathbb{R}$ by $P(\xi,\xi') = (\xi,\xi')$ then writing $E_3 = x^0$:

(iii) $P(g\xi^0,\xi) = 4 \lim_{t\to\infty} e^{-2t} P(ga_t x^0,\xi) \forall g \in G$.

These functions P are the analogues of Faraut's functions P ([4], p.391). An important property of the Poisson kernel is that it is for fixed ξ, considered as a function on the space X, en eigenfunction of the Casimir operator. A precise statement of this fact will be given in the next section. The proof we give there is general in the sense that it applies to Faraut's case as well. One possible method consists of extending the result of [2] to pseudo Riemannian symmetric pairs and then using the expression for the Casimir operator we then find to calculate its effect on functions left invariant under N and right invariant under H. But, as was pointed out to me by T.H. Koornwinder it is also possible to calculate directly in the universal enveloping algebra which makes things much simpler.

5. AN EXPRESSION FOR THE CASIMIR OPERATOR ON A PSEUDO RIEMANNIAN SYMMETRIC SPACE

Let G be a connected semisimple Lie group, σ an involutive automorphism of G. By abuse of language we denote the corresponding auotmorphism of the Lie algebra g of G also by σ. We have the decomposition $g = h \oplus q$ where h and q are the eigenspaces of σ for eigenvalues 1 and -1 respectively. It is well-known that there exists a Cartan involution θ of g commuting with σ. Let a be a subalgebra of g which is maximal among the abelian subalgebra's of g contained in q. Assume furthermore that a is θ-stable (it is easy to show that such a exist). Then $\{\mathrm{ad}X \mid X \in a\}$ is a collection of commuting semisimple endomorphisms of g so we have a rootspace decomposition: there is a finite set $\Phi \subset a^*$ such that $g = \sum_{\alpha\in\Phi} g^\alpha$ where

$$g^\alpha = \{X \in g \mid [H,X] = \alpha(H)X \forall H \in a\} \neq \{0\}.$$

Take $m = g^0 \cap h$ then clearly $g^0 = m \oplus a$. Choose a lexicographical ordering ">" in a^* then we have $\Phi - \{0\} = \Phi^+ \cup \Phi^-$ where

$$\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}, \quad \Phi^- = -\Phi^+.$$

Take $n = \sum_{\alpha\in\Phi^+} g^\alpha$ then n is a nilpotent subalgebra of g. One easily proves the "Iwasawa decomposition" $g = h \oplus a \oplus n$. Let H, A and N be the connected subgroups of G with Lie algebra's h,a and n, then there are open neighbourhoods H^0, A^0, N^0 and G^0 of the identities in H,A,N and G such that the mapping $(n,a,h) \mapsto nah$ from $N^0 \times A^0 \times H^0$ to G^0 is a diffeomorphism. We want to give an expression for the Casimir differential operator of G on G^0 in terms of the "Iwasawa coordinates" on G^0 given by this diffeomorphism. Let $\langle\ ,\ \rangle$ be the Killing form of g. Choose an orthonormal basis of a, i.e. a basis $\{H_1,\dots,H_r\}$ such that

$$\langle H_i,H_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ \eta_j = \pm 1 & \text{if } i = j \end{cases}$$

Let $N_1,\ldots,N_n$ be a basis of $\mathfrak{n}$ consisting of root vectors (i.e. $N_i \in \mathfrak{g}^{\alpha_i}$ for a certain $\alpha_i \in \Phi^+$ for $i = 1,\ldots,n$) such that the vectors $P_1,\ldots,P_n$ given by $P_i = -N_i + \sigma N_i$ form an orthonormal set in $\mathfrak{q}$:

$$<P_i,P_j> = \begin{cases} 0 & \text{if } i \neq j \\ -\gamma_j = \pm 1 & \text{if } i = j \end{cases}$$

then the vectors $K_i = N_i + \sigma N_i$ $(i = 1,\ldots,n)$ form an orthonormal set in $\mathfrak{h}$:

$$<K_i,K_j> = \begin{cases} 0 & \text{if } i \neq j \\ \gamma_j = \pm 1 & \text{if } i = j \end{cases}$$

Let $Y_1,\ldots,Y_q$ be a basis of $\mathfrak{m}$ such that

$$<Y_i,Y_j> = \begin{cases} 0 & \text{if } i \neq j \\ \varepsilon_j = \pm 1 & \text{if } i = j \end{cases}$$

We then can write the Casimir operator Γ as follows:

$$\begin{aligned}\Gamma &= \sum_{j=1}^{r} \eta_j H_j^2 - \sum_{j=1}^{n} \gamma_j(-N_j+\sigma N_j)^2 + \sum_{j=1}^{n} \gamma_j(N_j+\sigma N_j)^2 + \sum_{j=1}^{q} \varepsilon_j Y_j^2 \\ &= \sum_{j=1}^{r} \eta_j H_j^2 + 2\sum_{j=1}^{n} \gamma_j(N_j\ \sigma N_j + \sigma N_j\ N_j) + \sum_{j=1}^{q} \varepsilon_j Y_j^2 = \\ &= \sum_{j=1}^{r} \eta_j H_j^2 + 4\sum_{j=1}^{n} \gamma_j \sigma N_j\ N_j + 2\sum_{j=1}^{n} \gamma_j [N_j,\sigma N_j] + \sum_{j=1}^{q} \varepsilon_j Y_j^2\end{aligned}$$

where we have done our calculations inside the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ identified in the usual way with the algebra of right invariant differential operators on G. It is easy to see that $[N_j,\sigma N_j] \in \mathfrak{a}$ $(j = 1,\ldots,n)$. If $H \in \mathfrak{a}$ then $<H,[N_j,\sigma N_j]> = <[H,N_j],\sigma N_j> = \alpha_j(H)<N_j,\sigma N_j> = \frac{1}{2}\gamma_j\alpha_j(H)$.
Now it follows from the orthonormality of the basis $\{H_1,\ldots,H_r\}$ of $\mathfrak{a}$ that

$$2\sum_{j=1}^{n}\gamma_j[N_j,\sigma N_j] = \sum_{i=1}^{r}\eta_i <2\sum_{j=1}^{n}\gamma_j[N_j,\sigma N_j],H_i>H_i =$$

$$=2\sum_{i=1}^{r}\eta_i\rho(H_i)H_i$$

where we have written ρ for half the sum of the positive roots, counted with multiplicity.

Substituting the identity we have found in our expression for Γ we get:

LEMMA 6.

$$\Gamma = \sum_{j=1}^{r}\eta_j(H_j^2+2\rho(H_j)H_j) + 4\sum_{j=1}^{n}\gamma_j\sigma N_j\, N_j + \sum_{j=1}^{q}\varepsilon_j Y_j^2.$$

If $f \in C^\infty(G)$ is a function which is left invariant under N and right invariant under H it follows from the fact that the elements of m commute with those of a that

$$(\Gamma f)(\exp(s_1H_1+\ldots+s_rH_r)) = \sum_{j=1}^{r}\eta_j\left(\frac{\partial^2}{\partial s_j^2} - 2\rho(H_j)\frac{\partial}{\partial s_j}\right)f(\exp(s_1H_1+\ldots+s_rH_r))$$

COROLLARY. A. *Suppose* $\dim a = 1$, $H \in a$ *such that* $<H,H> = 1$. *Let* $a_t = \exp tH$. *If* f *is the function on* G^0 *defined by* $f(na_th) = e^{-t}$, *then, writing* $\rho = \rho(H)$: $(f^{s-\rho}) = (s^2-\rho^2)f^{s-\rho}$.

PROOF. By the lemma:

$$\Gamma(f^{s-\rho}) = (\partial^2/\partial t^2-2\rho\,\partial/\partial t)(e^{-(s-\rho)t}) =$$

$$((s-\rho)^2-2\rho(-(s-\rho)))e^{-(s-\rho)t} = (s^2-\rho^2)f^{s-\rho}.$$

COROLLARY. B. (*Notation as in* §4).

Let Ω *be the Casimir operator of* G *considered as an invariant differential operator on* X. *Let* $P : X \times \Xi \to \mathbb{R}$ *be the Poisson kernel. Write* $\rho = \rho(L)$ *with* $L = \tilde{A}_2^{1\otimes(-1)^{\frac12}}$ *as in* §2 *and consider for fixed* $\xi \in \Xi$ *and* $s \in \mathbb{C}$ *the function* $x \mapsto P(x,\xi)^{\frac12(s-\rho)}$ *on* X. *Then if* $P(x,\xi) \neq 0$ $\Omega(P(x,\xi)^{\frac12(s-\rho)}) =$
$= \frac{1}{C}\,P(x,\xi)^{\frac12(s-\rho)}$ *where* $C = <L,L>$.

PROOF. Because of the G-invariance of P and Ω we only have to consider the case $\xi = \xi^0$. But then the corollary follows from Corllary A because $P(na_t hx^0,\xi^0)^{\frac{1}{2}} = e^{-t}$ (the factor $1/C$ occurs because $a_t = \exp tL$, $\langle L,L\rangle \neq 1$).

REMARK. $\rho = \frac{1}{2}(\dim g^{\alpha} + 2 \dim g^{2\alpha}) = 11$

$$C = 2(\dim g^{\alpha} + 4 \dim g^{2\alpha}) = 72.$$

6. THE REPRESENTATIONS π_s

The subgroup of G which stabilizes the line in Ξ through ξ^0 is MAN, a parabolic subgroup of G. If $s \in \mathbb{C}$ then we define a character χ_s of MAN by $\chi_s(ma_t n) = e^{-st}$. Let π_s be the representation of G induced by χ_s. π_s can be realized on the space

$$E_s(\Xi) = \{f \in C^{\infty}(\Xi) \mid f(ga_t\xi^0) = e^{(s-\rho)t}f(g\xi^0)\forall t \in \mathbb{R},\ g \in G\}$$

on which G acts by left translation:

$$(\pi_s(g)f)(\xi) = f(g^{-1}\xi).$$

Because each $f \in E_s(\Xi)$ is determined completely by its restriction to $B = S^{15} = \{k\xi^0 \mid k \in K\}$ and because conversely every $f \in C^{\infty}(B)$ is the restriction of an element of $E_s(\Xi)$ (Lemma 5) we can identify $E_s(\Xi)$ with $C^{\infty}(B)$. In particular $E_s(\Xi)$ acquires the structure of a topological vectorspace, arising from the Schwartz topology on $C^{\infty}(B)$. Let db denote the normalized K-invariant measure on B then we have a nondegenerate bilinear form $\langle\ ,\ \rangle$ on $E_s(\Xi) \times E_{-s}(\Xi)$ defined by $\langle f,h\rangle = \int_B f(b)h(b)db$.

From the well known expression for the Haar measure of G in terms of those on K,A and N it follows easily that this form is G-invariant

$$\langle\pi_s(g)f,\pi_{-s}(g)h\rangle = \langle f,h\rangle.$$

Let $E'_s(\Xi)$ be the topological dual of $E_{-s}(\Xi)$. It can be identified as a topological vectorspace with $\mathcal{D}'(B)$ the space of distributions on B. If we let π'_s be the representation of G on $E'_s(\Xi)$ contragredient to π_{-s} then the injection $E_s(\Xi) \to E'_s(\Xi)$ which arises from < , > is G-intertwining because of the G-invariance of < , >. Whenever this is convenient we shall regard $E_s(\Xi)$ as a subspace of $E'_s(\Xi)$.

LEMMA 7 (*cf. Prop.* 5.2 *in* [4]).
$\pi_s(\Omega) = \frac{1}{C}(s^2-\rho^2)$ I, I *being the identity mapping in* $E_s(\Xi)$.

PROOF. As in [4], using Corollary B of Lemma 6.

REMARK. Corollary B of Lemma 6 also applies to Faraut's case so we find a new proof of his Proposition 5.2.

§7. CONSTRUCTION OF SPHERICAL DISTRIBUTIONS

Let P_1 be the function on Ξ defined by

$$P_1(\xi) = P(x^0,\xi).$$

If $s \in \mathbb{C}$, $\mathrm{Re}\, s > \rho$ then the function $F = P_1^{\frac{1}{2}(s-\rho)}$ determines an element (also denoted F) of $E'_s(\Xi)$ by

$$\langle F,f\rangle = \int_B P_1(b)^{\frac{1}{2}(s-\rho)} f(b)\,db.$$

We want to extend the analytic distribution-valued function $s \to F$ defined on $\{s \mid \mathrm{Re}\, s > \rho\}$ to a meromorphic distribution-valued function on $\mathbb{C}$. To this purpose we first study the behaviour of P_1 in the neighbourhood of its zeroes.

If we parametrize B by $\{(x,y) \mid x,y \in \mathbb{O}, |x|^2 + |y|^2 = 1\}$ as in §4 then we have if $b \in B$, $b \leftrightarrow (x,y)$: $P_1(b) = |x|^2$. Write

$$x = \sum_{i=0}^{7} x_i e_i, \quad y = \sum_{i=0}^{7} y_i e_i \quad (x_i, y_i \in \mathbb{R})$$

then: $P_1(b) = 0 \Rightarrow x_0 = .. = x_7 = 0$, $y_0^2 + \ldots + y_7^2 = 1$ so in a neighbourhood U of a zero of P_1 we can (renumbering the y_i if necessary) take as coordinates $x_0,\ldots,x_7, y_1,\ldots,y_6$. Assume that $f \in C^\infty(B)$ such that supp $f \subset U$, then

$$\int_B P_1(b)^{\frac{1}{2}(s-\rho)} f(b)db =$$

$$\int_{x_0^2+\ldots+y_6^2<1} f(x_0,\ldots,y_6)(x_0^2+\ldots+x_7^2)^{\frac{1}{2}(s-\rho)} g(x_0,\ldots,y_6)dx_0\ldots dy_6$$

for a certain C^∞-function g.
Integrating over the coordinates $y_0,\ldots,y_6$, expressing $x_0,\ldots,x_7$ in polar coordinates $r,\theta_1,\ldots,\theta_7$ and integrating over $\theta_1,\ldots,\theta_7$ yields:

$$\int_B P_1(f)^{\frac{1}{2}(s-\rho)} f(b)db = \int_{r=0}^{1} H(r)r^7 r^{(s-\rho)}dr \qquad (*).$$

where H is or C^∞-function on $[0,1]$ which is the restriction of an even C^∞-function on $[-1,1]$.
Using the results on integrals which have the form of the right hand side of (*) in [7] (Kapitel 1, §3) we find the following

PROPOSITION 4. *If* $f \in C^\infty(B)$ *then the function* $s \to Z(f,s) = = \int_B P_1(b)^{\frac{1}{2}(s-\rho)} f(b)db$ *which is defined for* $\mathrm{Re}\, s > \rho$ *is an analytic function on* $\{s \mid \mathrm{Re}\, s > \rho\}$ *which has a meromorphic continuation to* $\mathbb{C}$ *with at most simple poles in* $\rho-8, \rho-10, \ldots$. *When we denote the meromorphic continuation also by* $s \to Z(f,s)$ *then for fixed* $s \in \mathbb{C}$, $s \notin \{\rho-8,\rho-10,\ldots\}$ *the linear functional on* $C^\infty(B)$ *defined by* $f \to Z(f,s)$ *is a distribution on* B. *If* $s \in \{\rho-8,\rho-10,\ldots\}$ *then the linear functional on* $C^\infty(B)$ *defined by* $f \to \mathrm{Res}_s Z(f,s)$ *is a distribution on* B.

Let us write for $\mathrm{Re}\, s > \rho$ and $f \in E_{-s}(\Xi)$

$$u_s(f) = \frac{1}{\Gamma(\frac{s-\rho+8}{2})} \int_B P_1(b)^{\frac{1}{2}(s-\rho)} f(b)db \quad (*)$$

then by the above Proposition the function $s \to u_s(f)$ can be extended to an entire function and if we denote this function also by $s \to u_s(f)$ then

$$u_s \in E_s'(\Xi) \ \forall s \in \mathbb{C}.$$

For $h \in H$ we have $\pi_s'(h)u_s = u_s$: for $\mathrm{Re}\, s > \rho$ this is easily seen from (*)

but because $\pi_s'(h)u_s$ and u_s are analytic in s the result extends to all $s \in \mathbb{C}$.

Using the distributions u_s we define the distributions ζ_s on G as in [4]: if $\phi \in \mathcal{D}(G) = C_c^\infty(G)$ we define $\pi_s'(\phi)u_s \in E_s'(\Xi)$ as follows: if $f \in E_{-s}(\Xi)$ then

$$(\pi_s'(\phi)u_s)(f) = \int_G (\pi_s'(g)u_s)(f)\phi(g)dg.$$

Then we show that actually even $\pi_s'(\phi)u_s \in E_s(\Xi)$ and that the mapping $\phi \to \pi_s'(\phi)u_s$ from $\mathcal{D}(G)$ to $E_s(\Xi)$ is continuous so that for $\phi \in \mathcal{D}(G)$ $\zeta_s(\phi) = \langle\pi_s'(\phi)u_s, u_{-s}\rangle$ is well-defined and ζ_s is a distribution on G. From the H-invariance of u_s and u_{-s} and Lemma 7 respectively the two properties in the next Proposition follow:

PROPOSITION 5. ζ_s *is a spherical distribution on* G, *i.e.*

(i) ζ_s *is* H-*biinvariant*

(ii) ζ_s *is an eigendistribution of* Ω: $\Omega\zeta_s = \frac{1}{C}(s^2-\rho^2)\zeta_s$.

§8. INTERTWINING OPERATORS

If $f \in C^\infty(B)$ then we define for Re $s > \rho$

$$W_s(f) = \int_B P(\xi^0,b)^{\frac{1}{2}(s-\rho)}f(b)db.$$

We want to extend this function of s to a meromorphic function on $\mathbb{C}$ and therefore first study the behaviour of the function $b \to P(\xi^0,b)$ near the points $b \in B$ where it is zero.

If $b = X(1,x,y)$ then $P(\xi^0,b) = |1-x|^2$ so $P(\xi^0,b) = 0 \iff x = 1$. We see from the expression for the function in the coordinates x and y that in a neighbourhood U of $b \in B$ such that $P(\xi^0,b) = 0$ we can choose coordinates $x_1,\ldots,x_{15}$ on B such that in these coordinates $P(\xi^0,b) = x_1^2+\ldots+x_7^2$ $\forall b \in U$.

Using the results of [7], Kapitel 1, §3 we get the following:

LEMMA 8. *If* $f \in C^\infty(B)$ *then the function* $s \to W_s(f)$ *which is defined in the region* $\{s \in \mathbb{C} \mid \text{Re } s > \rho\}$ *and is analytic there, has a meromorphic continuation to* $\mathbb{C}$ *with at most simple poles in* $\rho-7, \rho-9, \rho-11,\ldots$

Now we calculate $W_s(1)$.

$$W_s(1) = \int_B P(\xi^0,b)^{\frac{1}{2}(s-\rho)}db \qquad (\mathrm{Re}\ s > \rho) \Rightarrow$$

$$W_s(1) = \int_K P(\xi^0,k\xi^0)^{\frac{1}{2}(s-\rho)}dk (\mathrm{Re}\ s > \rho).$$

Because the integrand is an M-biinvariant function on K we can apply Proposition 2, p.537 of [9] and get

$$W_s(1) = \frac{7}{\pi} \int_{t=0}^{\pi/2} \int_{\theta=0}^{\pi} (1-2\cos t\cos\theta + \cos^2 t)^{\frac{1}{2}(s-\rho)} \sin^7 2t \sin^6\theta d\theta dt.$$

This integral is evaluated on p.558 of [9]. It is shown there that it is equal to

$$\frac{7!\Gamma(s)}{\Gamma(\frac{11+s}{2})\Gamma(\frac{5+s}{2})} .$$

Write for $f \in E_{-s}(\Xi)$, Re $s > \rho$

$$(A_s f)(\xi) = \frac{1}{W_s(1)} \int_B P(\xi,b)^{\frac{1}{2}(s-\rho)} f(b)db$$

then we see from Lemma 8 and the expression for $W_s(1)$:

<u>LEMMA 9</u>.

a. *The function* $s \to (A_s f)(\xi)$ *is holomorphic on* $\{s \in \mathbb{C} \mid \mathrm{Re}\ s > \rho\}$. *It can be extended to a meromorphic function on* $\mathbb{C}$ *having at most poles in certain points of* $\rho + \mathbb{Z}$. *Denote this extended function also by* $s \to (A_s f)(\xi)$ *then if* $s + \rho \notin \mathbb{Z}$:

b. $A_s f \in E_s(\Xi)$ *and* $A_s : E_{-s}(\Xi) \to E_s(\Xi)$ *is continuous*

c. A_s *intertwines the actions of* G:

$$A_s \circ \pi_{-s} = \pi_s \circ A_s .$$

Diagonalization of the intertwining operators.

According to [9] the space of K-finite functions on B splits as a direct sum of mutually inequivalent K-invariant, K-irreducible subspaces $H_{p,q}$:

$$C^\infty(B) = \sum_{p,q\geq 0} H_{p-q}.$$

Moreover each $H_{p,q}$ contains a unique M-invariant function $\phi_{p,q}$ taking the value 1 at ξ^0. It is given by the following formulae (these are taken from [9], we repeat the results for convenience):
For $0 \leq \lambda \leq \pi$ ℓ_λ is the element of L given by $\ell_\lambda = (\alpha_1^\lambda, \alpha_2^\lambda, a_3^\lambda)$ where $\alpha^\lambda{}_2$ has the following matrix with respect to the basis $\{e_0, \ldots, e_7\}$ of $\mathbb{O}$:

$$\alpha_2^\lambda = \begin{pmatrix} \cos\lambda & -\sin\lambda & & & \\ \sin\lambda & \cos\lambda & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

and ℓ_λ is chosen so that it depends continuously of λ, $\ell_0 = 1$. Then every element $\ell \in L$ can be written as $\ell = m\ell_\lambda m'$ where λ is uniquely determined. Define

$$\psi_p(m\ell_\lambda m') = \frac{C_p^3(\cos\lambda)}{C_p^3(1)}$$

(C_p^3 is the Gegenbauer polynomial of index 3 and degree p). As we have already remarked in §2 every $k \in K$ can be written as $k = \ell u_\theta \ell'$ with $\ell, \ell' \in L$ and a uniquely determined $\theta \in [0, \pi/2]$. Then, writing $\phi_{p,q}(k) = \phi_{p,q}(k\xi^0)$ for $k \in K$ we have

$$\phi_{p,q}(\ell u_\theta \ell') = \psi_p(\ell\ell')\phi_{p,q}(u_\theta)$$

where

$$\phi_{p,q}(u_\theta) = \frac{q!}{4_q} \cos^p \theta P_q^{(3,p+3)}(\cos 2\theta)$$

($P_q^{(3,p+3)}$ is the Jacobi polynomial of index (3,p+3) and degree q). Now according to Schur's Lemma A_s acts as a scalar $\alpha_{p,q}(s)$ on $H_{p,q}$ which we shall presently determine. If $f \in H_{p,q}$ then on one hand we have

$$(A_s f)(\xi^0) = \frac{1}{W_s(1)} \int_B P(\xi^0, b)^{\frac{1}{2}(s-\rho)} f(b)\,db = \frac{W_s(f)}{W_s(1)}$$

and on the other hand:

$$(A_s f)(\xi^0) = \alpha_{p,q}(s) f(\xi^0).$$

Now $\phi_{p,q} \in H_{p,q}$, $\phi_{p,q}(\xi^0) = 1$ hence

$$\alpha_{p,q}(s) = \frac{W_s(\phi_{p,q})}{W_s(1)} .$$

Now the integrand in

$$W_s(\phi_{p,q}) = \int_K P(\xi^0, k\xi^0)^{\frac{1}{2}(s-\rho)} \phi_{p,q}(k\xi^0)\,dk$$

is M-biinvariant, so we can once again apply Proposition 2 on p.537 of [9]. We find

$$W_s(p,q) = \frac{7}{\pi} \int_{t=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\pi} (1-2\cos t\cos\theta + \cos^2 t)^{\frac{1}{2}(s-\rho)} \phi_{p,q}(u_t)$$

$\psi_p(\ell_\theta) \sin^7 2t \sin^6\theta \, d\theta dt$. In [9] this integral is called $W_{s,1}(p,q)$. Its value is given by formula (5) on p.558 of [9] and we get from it (remembering $\rho=11$):

$$\alpha_{p,q}(s) = \frac{(\frac{\rho-s}{2})_{p+q}(\frac{5-s}{2})_q}{(\frac{\rho+s}{2})_{p+q}(\frac{5+s}{2})_q}$$

We can also write this as:

$$\alpha_{p,q}(s) = \frac{\prod_{j=1}^{p+q}(-s+\rho+2j-2)}{\prod_{j=1}^{p+q}(s+\rho+2j-2)} \quad \frac{\prod_{k=1}^{q}(s+\rho+2k-8)}{\prod_{k=1}^{q}(s+\rho+2k-8)}$$

which the reader should compare to [4], Theorème 6.3. If $s \in \mathbb{Z}$ then $\alpha_{p,q}(s)\alpha_{p,q}(-s) = 1$ $\forall p,q$ and because of the density of the space of K-finite vectors in $C^\infty(B)$ and the continuity of A_s and A_{-s} it follows:

PROPOSITION 6. *If* $s \notin \mathbb{Z}$ *then* $A_s \circ A_{-s} = I$, *so then* A_s *is an isomorphism.*

9. THE FOURIER TRANSFORM, EXPRESSION FOR $\zeta_s(\phi)$ WHEN ϕ IS A K-FINITE FUNCTION

LEMMA 10. *Every element* $x \in X$ *can be written as* $x = ka_t x^0$ *with* $t \geq 0$, $k \in K$. *Then* t *is uniquely determined,* k *is determined uniquely modulo* M. *If* dx *is a suitably normalized* G-*invariant measure on* X *then*

$$\int_X \phi(x)dx = \int_K \int_0^\infty \phi(ka_t x^0) \sinh^7 t \cosh^{15} t \, dk dt \;\; \forall \phi \in C_c(X).$$

We leave the proof of this Lemma to the reader.

DEFINITION. If $\phi \in \mathcal{D}(X)$ then the Fouriertransform $\hat{\phi}$ of ϕ is the function on $\Xi \times \mathbb{C}$ given by

$$\hat{\phi}(\xi,s) = \frac{1}{\Gamma(\frac{s-\rho+8}{2})} \int_X P(x,\xi)^{\frac{1}{2}(s-\rho)} \phi(x)dx \quad \text{for Re } s > \rho$$

and by analytic continuation of this expression for all other $s \in \mathbb{C}$. Then $\hat{\phi}$ is well-defined (the proof is like that of Proposition 4) and $\hat{\phi}$ has the following properties (cf.[4], Proposition 7.1):

(i) for fixed ξ, $\phi(\xi,s)$ is an entire function of s

(ii) for fixed s, $\phi(\xi,s)$ is a C^∞ function of ξ which is even contained in $E_s(\Xi)$:

$$\hat{\phi}(ga_t\xi^0,s) = e^{(s-\rho)t}\hat{\phi}(g\xi^0,s) \forall g \in G, \; t \in \mathbb{R}, \; s \in \mathbb{C}$$

(iii) The Fourier transform commutes with the action of G: if we denote for $g \in G$, $\phi \in \mathcal{D}(X)$ the function $x \mapsto \phi(g^{-1}x)$ which is also contained in $\mathcal{D}(X)$ by $\tau_g\phi$ then $\widehat{\tau_g\phi}(\xi,s) = \pi_s(g)\hat{\phi}(\xi,s)$

(iv) $(\widehat{\Omega\phi})(\xi,s) = 1/C(s^2-\rho^2)\hat{\phi}(\xi,s)$

The reason why we study this Fourier transform is its connection with the spherical distributions ζ_s: for

$$\text{Re } s < -\rho + 8 \quad \zeta_s(\rho) = \frac{1}{\Gamma(\frac{-s-\rho+8}{2})} \int_B \hat{\phi}(b,s)P_1(f)^{-s-\rho} \, db$$

(when we consider ζ_s as a distribution on $X = G/H$). We will now make a quite detailed study of the Fourier transform and apply the results to get information about ζ_s, as in [4].

We have seen in Lemma 10 that we can parametrize X by $B \times [0,\infty)$ as follows: if $b \in B$, $b = k\xi^0$ and $t \in [0,\infty)$ then we assign to (b,t) the point $ka_t x^0 \in X$. Remark that $\{kx^0 \mid k \in K\} \simeq B$. We shall identify the two sets by $kx^0 \leftrightarrow k\xi^0$. If $Y \in C^\infty(B)$, $F \in \mathcal{D}$ even $(\mathbb{R})$ we can make from these functions a function $\phi \in \mathcal{D}(X)$ via the parametrization. We shall permit ourselves to write $\phi(x) = F(t)Y(b)$. Now let ϕ be of the form $\phi(x) = F(t)Y(b)$ with $Y \in H_{p,q}$, then ϕ is a K-finite function on X, and all K-finite functions on X are finite linear combinations of such functions.

Then:

$$\hat{\phi}(X(\xi,x,y),s) = \frac{(\xi)^{\frac{1}{2}(s-\rho)}}{\Gamma(\frac{s-\rho+8}{2})} \int_0^\infty \int_K |(X(1,x,y),ka_t x^0)|^{\frac{1}{2}(s-\rho)} F(t) \cdot$$

$$\cdot\, Y(kx^0)A(t)dtdk =$$

$$\frac{(\xi)^{\frac{1}{2}(s-\rho)}}{\Gamma(\frac{s-\rho+8}{2})} \int_0^\infty F(t)A(t)\Big(\int_K |(X(1,x,y),ka_t x^0)|^{\frac{1}{2}(s-\rho)} Y(kx^0)dk\Big)dt$$

where we have written $A(t) = \sinh^7 t \cosh^{15} t$ (cf. Lemma 10). Following the reasoning of [4] p.404 we see that the integral over K in the last expression is a multiple $\Phi^*_{p,q}(t,s)Y(X(1,x,y))$ of $Y(X(1,x,y))$ where $\Phi^*_{p,q}(t,s)$ does not depend on $X(1,x,y) \in B$ and on $Y \in H_{p,q}$. This observation allows us to calculate $\Phi^*_{p,q}(t,s)$ by taking $x = 1$, $y = 0$ (i.e. $X(1,x,y) = \xi^0$) and $Y = \phi_{p,q}$. We get:

$$\Phi^*_{p,q}(t,s) = \int_K |(\xi^0,ka_t x^0)|^{\frac{1}{2}(s-\rho)} \phi_{p,q}(k)dk.$$

The integrand being M-biinvariant we can again apply the integration formula (7) of [9], p.551. We find, with notations as in §8:

$$\Phi^*_{p,q}(t,s) = \frac{7}{\pi} \int_{\theta=0}^{\pi/2} \int_{\lambda=0}^{\pi} |(\xi^0,u_\theta \ell_\lambda a_t x^0)|^{\frac{1}{2}(s-\rho)} \phi_{p,q}(u_\theta \ell_\lambda x^0) \sin^7 2\theta \sin^6 \lambda d\theta d\lambda$$

$$= \frac{7}{\pi} \int_{\theta=0}^{\pi/2} \int_{\lambda=0}^{\pi} |\tfrac{1}{2}-\tfrac{1}{2}\cosh 2t-(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta)(\tfrac{1}{2}+\tfrac{1}{2}\cosh 2t) + \sinh 2t \cos\lambda \cos\theta|^{\frac{1}{2}(s-\rho)}$$

$$\frac{C_p^3(\cos\lambda)}{C_p^3(1)} \frac{q!}{4_q} \cos^p\theta P_q^{(3,p+3)}(\cos 2\theta)\sin^7 2\theta \sin^6 \lambda d\theta d\lambda.$$

Use formula 22.5.27 of [1] to write the Gegenbauer polynomial as a Jacobi polynomial with equal upper indices, use that $C_p^3(1) = \frac{6_p}{p!}$, write $r = \cos\theta$, $\rho = \lambda$, then, as was pointed out kindly to me by T.H. Koornwinder this integral can be calculated using the methods of [8].

This is done in the appendix to this paper. The result is (with $x = -\cosh 2t$)

$$\Phi^*_{p,q}(t,s) = \frac{7}{\sqrt{\pi}}\,\frac{\Gamma(7/2)\Gamma(4)(-1)^{p+q}}{2^{p+q-6}\Gamma(p+4)}\;\frac{\Gamma(\frac{1}{2}(s-\rho)+1)\Gamma(\frac{1}{2}(s-\rho)+4)}{\Gamma(\frac{1}{2}(s-\rho)+q+8)\Gamma(\frac{1}{2}(s-\rho)-p-q+1)}$$

$$(-x+1)^{\frac{p}{2}+q}(-x-1)^{\frac{p}{2}}\,{}_2F_1(-\tfrac{1}{2}(s-\rho)+p+q,\tfrac{1}{2}(s-\rho)+p+q+11;\ p+4;\ \tfrac{1+x}{2})$$

and we have the expression for $\hat{\phi}$ (cf. [4], ch.7)

$$\hat{\phi}(X(\xi,x,y),s) = (\xi)^{\frac{1}{2}(s-\rho)}Y(X(1,x,y))\int_0^\infty \Phi_{p,q}(t,s)F(t)A(t)dt$$

if

$$\phi(x) = Y(b)F(t) \quad (F \in C^\infty_{even}(\mathbb{R}),\ Y \in H_{p,q})$$

where

$$\Phi_{p,q}(t,s) = \frac{1}{\Gamma(\frac{s-\rho+8}{2})}\,\Phi^*_{p,q}(t,s) = \beta_{p,q}(s)\Psi_{p,q}(t,s)$$

with

$$\beta_{p,q}(s) = b_{p,q}\,\frac{(\frac{1}{2}(s-\rho)-p-q+1)\ldots(\frac{1}{2}(s-\rho))}{\Gamma(\frac{1}{2}(s-\rho)+q+8))}$$

$$b_{p,q} = \frac{7}{\sqrt{\pi}}\,\frac{\Gamma(4)\Gamma(7/2)2^6}{\Gamma(p+4)}$$

and $\Psi_{p,q}(t,s) = (\cosh t)^{p+2q}(\sinh t)^p.$

$${}_2F_1(-\tfrac{1}{2}(s-\rho)+p+q,\tfrac{1}{2}(s-\rho)+p+q+11;\ p+4;\ -\sinh^2 t).$$

Define $F_s\colon \mathcal{D}(X) \to E_s(\Xi)$ by

$$F_s\colon \phi \to \hat{\phi}(\cdot,s).$$

Let $E(s) = \{(p,q) \mid \beta_{p,q}(s) \neq 0\}$ and

$$I_s = \sum_{(p,q)\in E(s)} H_{p,q}$$

then

$$I_s \subset F_s(\mathcal{D}(X)) \subset \bar{I}_s.$$

Inspection of the numbers $\beta_{p,q}(s)$ learns us that:

(i) if $s \neq \rho + 2r \ \forall r \in \mathbb{N} = \{0,1,2,\ldots\}$ and $s \neq \rho - 16 - 2h \ \forall h \in \mathbb{N}$ then $(p,q) \in E(s) \ \forall p \geq 0,\ q \geq 0$

(ii) if $r \in \mathbb{N}$ then $E(\rho+2r) = \{(p,q) \mid p + q \leq r\}$

(iii) if $h \in \mathbb{N}$ then $E(\rho-16-2h) = \{(p,q) \mid q \geq h + 1\}$.

We conclude the analogue of [4] Prop.7.2:

PROPOSITION 7. $s + \rho \notin \mathbb{Z} \Rightarrow \{\pi'_s(\psi)u_s \mid \psi \in \mathcal{D}(G)\}$ *is a dense linear subspace of* $E_s(\Xi)$.

PROOF. This follows from the above because if $\phi \in \mathcal{D}(G)$ then $\pi'_s(\phi)u_s = F_s(\phi^0)$ where $\phi^0 \in \mathcal{D}(X)$ is defined by $\phi^0(gx^0) = \int_H \phi(gh)dh$, the mapping $\phi \to \phi^0$ being a mapping of $\mathcal{D}(G)$ onto $\mathcal{D}(X)$.

Application to the distributions ζ_s

For Re $s < -\rho + 8$:

$$\zeta_s(\phi) = \frac{1}{\Gamma(\frac{-s-\rho+8}{2})} \int_B \hat{\phi}(b,s)P_1(b)^{-\frac{1}{2}(s+\rho)}db =$$

$$\beta_{p,q}(s) \int_0^\infty \Psi_{p,q}(t,s)F(t)A(t)dt \ \frac{1}{\Gamma(\frac{-s-\rho+8}{2})} \int_B Y(b)P_1(b)^{-\frac{1}{2}(s+\rho)}db$$

when $\phi \in \mathcal{D}(X)$ is a K-finite function of the form

$$\phi(x) = F(t)Y(b) \quad (Y\in H_{p,q}).$$

We now consider the factor

$$\frac{1}{\Gamma(\frac{-s-\rho+8}{2})} \int_B Y(b)P_1(b)^{-\frac{1}{2}(s+\rho)}db =$$

$$\frac{1}{\Gamma(\frac{-s-\rho+8}{2})} \int_K Y(k\xi^0)P(k\xi^0,x^0)^{-\frac{1}{2}(s+\rho)}dk.$$

Because the function $k \to |P(k\xi^0,x^0)|^{-\frac{1}{2}(s+\rho)}$ is M-biinvariant this is equal to

$$\frac{1}{\Gamma(\frac{-s-\rho+8}{2})} \int_K c_{p,q}(s)Y(k)\phi_{p,q}(k)dk$$

for a certain constant which we will now determine by taking $Y = \phi_{p,q}$. We see that

$$c_{p,q}(s) = \begin{cases} 0 & \text{if} \quad p \neq 0 \\ c_q \quad \beta_{0,q}(-s) & \text{if } p = 0 \end{cases}$$

where we have put $c_q^{-1} = \int_K (\phi_{p,q}(k))^2 dk > 0$.
Thus we have proved:

PROPOSITION 8. *If* $\phi(x) = F(t)Y(b)$ *with* $Y \in H_{p,q}$ *then*

$$\zeta_s(\phi) = c_q \beta_{p,q}(s)\beta_{0,q}(-s) \int_0^\infty \Psi_{p,q}(t,s)F(t)A(t)dt \int_B Y(b)\phi_{0,q}(b)db.$$

§10. DETERMINATION OF THE SPHERICAL DISTRIBUTIONS

We consider the following function Q on the space X which parametrizes the H-orbits on X: if $Y \in X$ then $Q(Y) = (Y,E_3)$. Q has the following properties:

a. Q is H-invariant: if $Y \in X$, $h \in H$ then

$$Q(hY) = (hY,E_3) = (Y,h^{-1}E_3) = (Y,E_3) = Q(Y)$$

b. E_3 is a nondegenerate critical point for Q, the Hessian of Q in this point having signature (8,8). This follows from the parametrization $(x,y) \to X(x,y)$ which holds in a neighbourhood of E_3. In these coordinates

$$Q(X(x,y)) = \frac{1}{1+|x|^2-|y|^2}$$

from which the result follows.

c. Q is a real analytic function on X

d. $Q(Y) \geq 0 \ \forall Y \in X$

e. If $Q(Y) = 0$ then Y is a degenerate critical point of Q.

The Hessian of Q in Y then has signature (8,0).

PROOF. As in §3 one shows that a neighbourhood of $Q^{-1}(\{0\})$ in X can be parametrized by an open subset of $\mathbb{O} \oplus \mathbb{O}$ when we take

$$x = \frac{\bar{u}_1}{\xi_2}, \quad u = \frac{\bar{u}_3}{\xi_2}.$$

Then $|y| < 1$ and $\xi_3 = |x|^2/(1+|x|^2-|y|^2)$. Thus when $Q(Y) = 0$ then Y has parameters $(0,y_0)$ $(y_0 \in \mathbb{O}, |y_0|<1)$ and in a neighbourhood of Y Q is given in terms of the parameters x,y by

$$Q = |x|^2/1+|x|^2-|y|^2 = \frac{1}{1-|y_0|^2}|x|^2 + \text{higher order terms.}$$

f. If $Q(Y) \neq 0$ or 1 then Y is not a critical point of Q. To see this consider the curves $t \to a_t Y$ and $\theta \to u_\theta Y$ through Y respectively, then if $Y = X(\xi_1,\xi_2,\xi_3,u_1,u_2,u_3)$ the derivative of Q along these curves in Y is equal to $-2\text{Re } u_2$ or $-2\text{Re } u_1$ respectively. If $\ell = (\alpha_1,\alpha_2,\alpha_3) \in L$ then it follows that the derivative of Q along the curves $t \to Q(\ell^{-1}a_t\ell Y)$ and $\theta \to Q(\ell^{-1}u_\theta \ell Y)$ in Y are equal to $-\text{Re } \alpha_2(u_2)$ and $-\text{Re } \alpha_1(u_1)$ respectively. By Lemma 1 we see that if $u_1 \neq 0$ or $u_2 \neq 0$ then Y is not a stationary point of Q. But if $u_1 = u_2 = 0$ it follows from §3 that $Y = E_3$ or $Q(Y) = 0$.

In order to avoid having to write the constant C (see Corollary B of Lemma 6) too often we write $\square = C\Omega$. $\square$ is the Laplace Beltrami operator on X belonging to the G-invariant pseudo Riemannian metric on X which comes

from the G-invariant bilinear form (,) on $J_{1,2}$.
We have the following Lemma on $\Box$:

LEMMA 11. *If* F *is a* C^2*-function on* $\mathbb{R}$ *then* $\Box(F\circ Q) = (LF)\circ Q$ *where* L *is the second order differential operator on* $\mathbb{R}$ *given by* $L = a(t)d^2/dt^2 + b(t)d/dt$, $a(t) = 4t(t-1)$, $b(t) = 16(3t-1)$.

REMARK. This is the analogue for our case of a result in [4], ch.3. However, since we have no direct construction of $\Box$ as in [4] our method of proof has to be different.

PROOF *of Lemma* 11. If $Y \in X$ then we define the tangent sectors D_i^j $(i = 0,\dots,7,\ j = 1,2)$ to X at Y as follows: if f is a C^∞-function on X defined in a neighbourhood of Y then

$$(D_i^j f)(Y) = \begin{cases} \dfrac{d}{dt} f(\exp t\tilde{A}_1^{e_i} Y)\big|_{t=0} & \text{if } j = 1 \\ \dfrac{d}{dt} f(\exp t\tilde{A}_2^{e_i\otimes(-1)^{\frac{1}{2}}} Y)\big|_{t=0} & \text{if } j = 2 \end{cases}$$

then the D_i^j are first order differential operators on X and

$$\Box = \sum_{i=0}^{7} \sum_{j=1,2} (-1)^j (D_i^j)^2 .$$

By the chain-rule, applied twice:

$$\Box(F\circ Q) = (F''\circ Q)\sum_{i,j}(-1)^j(D_i^jQ)^2 + (F'\circ Q)\sum_{i,j}(-1)^j(D_i^j)^2Q$$

(where we have written $F'(t) = \dfrac{dF}{dt}$, $F''(t) = \dfrac{d^2F}{dt^2}$).

By definition of $D_i^1, (D_i^1Q)(Y)$ is the coefficient of E_3 in $\tilde{A}_1^{e_i}Y$. From this we see that if $Y = X(\xi_1,\xi_2,\xi_3,u_1,u_2,u_3)$ then $(D_i^1Q)(Y) = -2(u_1,e_i)$.
In the same way: $(D_i^2Q)(Y) = -2(u_2,e_i)$. Hence $a(Q(Y)) =$

$$\sum_{i,j} (-1)^j((D_i^jQ)(Y))^2 = 4(u_2,e_0)^2+\dots+4(u_2,e_7)^2-4(u_2,e_0)^2 -$$

$$-\dots-4(u_1,e_7)^2 = -4|u_1|^2+4|u_2|^2 .$$

By (3) of §3 this is equal to $4\xi_3(\xi_3-1) = 4\,Q(Y)(Q(Y)-1$ and we see: $a(t) = 4t(t-1)$. The same method works for finding b: $((D_1^1)^2Q)(Y) = D_1^1(D_1^1Q)(Y)$ and we have seen $(D_1^1Q)(Y) = -2\,\mathrm{Re}\,u_1$ so $((D_1^1)^2Q)(Y)$ is -2 times the real part of the coefficient of F_1 in $\tilde{A}_1^1$ which is $\xi_3 - \xi_2$. Proceeding analogously for other i and j we find:

$$((D_i^1)^2Q)(Y) = -2(\xi_3-\xi_2)$$

$$((D_i^2)^2Q)(Y) = 2(\xi_3-\xi_1)$$

so

$$b(Q(Y)) = 16\xi_3 - 16\xi_2 + 16\xi_3 - 16\xi_1 =$$

$$48\xi_3 - 16(\xi_1+\xi_2+\xi_3) = 48\xi_3 - 16 = 16(3\xi_3-1) = 16(3Q(Y)-1)$$

as we had to prove.

From the properties of Q mentioned in the beginning of this section we deduce following the reasoning of [4], ch.3 the existence of a linear mapping M which assigns to every $f \in \mathcal{D}(X)$ a continuous function Mf on $[0,\infty)$ such that

$$\int_X F(Q(x))\, f(x)dx = \int_0^\infty F(t)(Mf)(t)dt \qquad \forall F \in C_c^\infty([0,\infty)).$$

((Mf)(t) can be thought of as the mean of the values of f on the submanifold $\{Y \in X \mid Q(Y) = t\}$ of X) the space $M\mathcal{D}(X) = \mathcal{H}$ consisting of all functions ϕ of the form

$$\phi(t) = t^3(\phi_0(t)+\eta(t)\phi_1(t))(\phi_1,\phi_2\in C_c^\infty([0,\infty)))$$

where $\eta(t) = (t-1)^7Y(t-1)$ (Y being Heaviside's function.) As in [4], $\mathcal{H}$ is given the structure of a locally convex vectorspace such that the following holds: (cf. [4], Theorème 3.1)

a. $M:\mathcal{D}(X) \to \mathcal{H}$ is continuous.

b. The transpose mapping $M': \mathcal{H}' \to \mathcal{D}'(X)$ between the dual topological vectorspaces (which is injective because M is surjective) has the space of

H-invariant distributions on X as image.

c. $\Box \circ M' = M' \circ L$

c. is proved using Lemma 11.

b. and c show how the biinvariance of the spherical distributions is used to reduce the differential equation for these distributions to an ordinary differential equation which has to be solved in a certain space of distributions (namely the space $\mathcal{H}'$). What we need from the results we get by solving the differential equation is the following (analogue of Theorème 3.2 of [4], the proof is, after our preparations, also analogous):

PROPOSITION 9. *Let* $\mathcal{D}'_{\lambda,H}(X)$ *be the space of* H*-invariant distributions* T *on* X *satisfying* $\Box T = \lambda T$ *then*:

a. *If* $\lambda \neq 4r(r+\rho)$ *for* $r = 0,1,2,\ldots$ *then* $\dim \mathcal{D}'_{\lambda,H}(X) = 1$.

b. *If* $r \in \{0,1,2,\ldots\}$ *and* $\lambda = 4r(r+\rho)$ *then* $\dim \mathcal{D}'_{\lambda,H}(X) = 2$.

11. EXCEPTIONAL SERIES OF REPRESENTATIONS AND CORRESPONDING SPHERICAL DISTRIBUTIONS

From Proposition 5 and Proposition 8 and by inspection of the numbers $\beta_{p,q}(s)$ we see that in case *a* of Proposition 9 $\mathcal{D}'_{\lambda,H}(X)$ is spanned by ζ_s when $s^2 - \rho^2 = \lambda$. In case *b* however when $s^2 - \rho^2 = \lambda$, i.e. $s = \pm(\rho+2r)$, $\zeta_s = 0$. We will treat this case in this section. For each $r \in \mathbb{N}$ we shall construct a finite dimensional subrepresentation of $\pi_{\rho+2r}$. This representation and the representation on the quotient space give rise to spherical distributions η_r and θ_r and very nicely η_r and θ_r span $\mathcal{D}'_{\lambda,H}(X)$. Let P_r be the space of functions on Ξ which are the restrictions to Ξ of polynomials on $J_{1,2}$ homogeneous of degree r.

Because $a_t\xi^0 = e^{2t}\xi^0$, if $P \in P_r$:

$$P(ga_t\xi^0) = e^{2rt}P(g\xi^0) \quad \forall g \in G, \ t \in \mathbb{R}.$$

Hence $P_r \subset E_{\rho+2r}(\Xi)$. Clearly P_r is a nonzero finite dimensional invariant subspace of $E_{\rho+2r}(\Xi)$.

Under the action of K P_r splits as a finite direct sum (by abuse of terminology we identify an element of $E_{\rho+2r}(\Xi)$ with its restriction to B) of

$H_{p,q}$'s, the following Lemma saying which:

LEMMA 12. $P_r = \sum_{p+q\leq r} H_{p,q}$.

PROOF. We consider the elements of both spaces of functions in the Lemma as functions on $B = \{X(1,x,y) \mid x,y \in \mathbb{O}, |x|^2+|y|^2 = 1\}$ because they are completely determined by their restrictions to B. Write for each p and q $H^M_{p,q}$ for the space of M-invariants in $H_{p,q}$. It is spanned by $\Phi^s_{p,q}$ (with $s = \rho + 2r$). Write P^M_r for the space of M-invariants in P_r then because each $H_{p,q}$ is M-stable we have to show that $P^M_r = \sum_{p+q\leq r} H^M_{p,q}$, i.e. that P^M_r is spanned by the $\Phi^s_{p,q}$ with $p + q \leq r$. Because for fixed indices the Jacobi-polynomials of degree $\leq n$ (and for fixed index the Gegenbauer polynomials of degree $\leq n$) span the space of all polynomials of degree $\leq n$ and because a Gegenbauer polynomial is odd or even (according to whether its degree is odd or even) we see from the expression for the functions $\Phi^s_{p,q}$ that this is equivalent to showing that P^M_r is spanned by the monomials of the form $X(1,x,y) \to (|x|^2)^{r_1}(\mathrm{Re}x)^{r_2}$ with $r_1 + r_2 \leq r$. It is easy to show that these are indeed contained in P^M_r. As to the converse, remark that each M-orbit on B contains an element of the form $X(1,\cos\theta(\cos\lambda+\sin\lambda e_1),\sin\theta)$ $(0\leq\theta\leq\pi/2,\ 0\leq\lambda\leq\pi)$, $X(1,x,y)$ being contained in this orbit if and only if $|x| = \cos\theta$, $\mathrm{Re}\, x = \cos\theta \cos\lambda$ (this follows from Lemma 5 and [9] Lemma 3 on p.536). In particular then $X(1,x,y)$ is also in the M-orbit of

$$X(1,\cos\theta(\cos\lambda+\varepsilon_1\sin\lambda e_1),\ \varepsilon_2\sin\theta) \quad \text{for} \quad \varepsilon_1 = \pm 1,\ \varepsilon_2 = \pm 1.$$

Now if $P \in P^M_r$ and $X(1,x,y)$ is in the M-orbit of

$$X(1,\cos\theta(\cos\lambda+\sin\lambda e_1),\sin\theta)$$

then

$$P(X(1,x,y)) = P(X(1,\cos\theta(\cos\lambda+\sin\lambda e_1),\sin\theta) =$$

$$= P_0(\cos\theta\cos\lambda,\cos\theta\sin\lambda,\sin\theta)$$

for a certain polynomial P_0. Also

$$P(X(1,x,y)) = \frac{1}{4} \sum_{\varepsilon_1,\varepsilon_2=\pm 1} P_0(\cos\theta\cos\lambda, \varepsilon_1\cos\theta\sin\lambda, \varepsilon_2\sin\theta)$$

so we may assume that P_0 contains only even powers of its second and third variable, i.e. it is a polynomial in $\cos\theta\cos\lambda$, $\cos^2\theta\sin^2\lambda$ and $\sin^2\theta$. Because $\cos^2\theta\sin^2\lambda = 1 - \sin^2\theta - (\cos\theta\cos\lambda)^2$ it is a polynomial in $\cos\theta\cos\lambda$ and $\sin^2\theta$, say $P(X(1,x,y)) = P_1(\cos\theta\cos\lambda, \sin^2\theta)$. Thus $P(X(1,x,y)) = P_1(\mathrm{Re}\, x, 1-|x|^2)$. From the construction it is clear that P_1 is of degree $\leq r$ and the proof is finished.

If $P,Q \in P_r$ then the function $s \to \langle A_s P,Q\rangle$ is holomorphic in $s = -\rho - 2r$. Indeed, if $p + q \leq r$ then the denominator of

$$\alpha_{p,q}(s) = \frac{\prod_{j=1}^{p+q}(-s+\rho+2j-2)\prod_{k=1}^{q}(-s+\rho+2k-8)}{\prod_{j=1}^{p+q}(s+\rho+2j-2)\prod_{k=1}^{q}(s+\rho+2k-8)}$$

in $s = -\rho - 2r$ is a product of strictly negative numbers so it is nonzero. So if $P,Q \in P_r$ then $\langle P,Q\rangle_r = \langle A_s P,Q\rangle_{s=-\rho-2r}$ is well defined. From the intertwing property of A_s for s in a punctured neighbourhood of $-\rho - 2r$ it follows that the bilinear form $\langle\ ,\ \rangle_r$ on P_r is G-invariant. P_r contains the H-invariant element U_r defined by

$$U_r(\xi) = \frac{1}{\Gamma(r+4)} (P_1(\xi))^r \quad (\xi\in\Xi).$$

Write

$$\gamma(s) = \frac{\beta_{00}(-s)}{\beta_{00}(s)} = \frac{\Gamma(\frac{1}{2}(s-\rho)+8)}{\Gamma(\frac{1}{2}(-s-\rho)+8)} .$$

Define for $\phi \in \mathcal{D}(G)$

$$\eta_r(\phi) = \frac{d\gamma}{ds}(\rho+2r)\langle\pi_{\rho+2r}(\phi)U_r, U_r\rangle_r$$

then η_r is a spherical distribution belonging to the eigenvalue $\lambda = 4r(r+\rho)$ of $\square$.

Now we construct in the same way a spherical distribution corresponding to the representation of G on $E_{\rho+2r}(\Xi)/P_r$. Take

$$V_r(\xi) = \frac{p_1(\xi)^r}{\Gamma(r+4)} \log(p_1(\xi))$$

and define for $\phi \in \mathcal{D}(G)$

$$\theta_r(\phi) = \int_B V_r(b)(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r})(b)db$$

then θ_r is an eigendistribution of $\square$ for the eigenvalue $\lambda = 4r(r+\rho)$ and because $u_{-\rho-2r}$ is H-invariant θ_r is right H-invariant.
If $t \in \mathbb{R}$ then the function on Ξ defined by

$$g\xi^0 \to V_r(ga_t\xi^0) - e^{2rt}V_r(g\xi^0)$$

is in P_r. Because of this we can prove in the same way as we proved the G-invariance of the bilinear form $<\ ,\ >$ on $E_s(\Xi)xE_{-s}(\Xi)$ that if $f \in E_{-\rho-2r}(\Xi)$ is such that

$$\int_B f(b)p(b)db = 0 \quad \forall p \in P_r$$

then

$$\int_B V_r(gb)f(gb)db = \int_B V_r(b)f(b)db \quad \forall g \in G.$$

Now if $f \in P_r$ then $<u_{-\rho-2r},f> = 0$: we may assume $f \in H_{p,q}$ with $p + q \le r$ and then it follows from the results of §9. So if $f \in P_r$, $\phi \in \mathcal{D}(G)$ then

$$\int_B f(b)(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r})(b)db = 0.$$

From these observations we see that

$$\int_B V_r(gb)(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r})(gb)db =$$

$$\int_B V_r(b)(\pi'_{-\rho-2r}(\phi)u_{-\rho-2r})(b)db \quad \forall g \in G,\ \phi \in \mathcal{D}(G)$$

and this, together with the H-invariance of the function V_r shows that the distribution θ_r is also left H-invariant, hence is a spherical distribution.

<u>LEMMA 13</u>. η_r *and* θ_r *are linearly independent.*

<u>PROOF</u>. As in [4] we see that if $\phi \in \mathcal{D}(X)$ is K-finite, of the form $\phi(x) = F(t)Y(b)$ $(Y \in H_{p,q})$ as in §9, then

$$\theta_r(\phi) = c_q\left(\frac{d}{ds}\,\beta_{p,q}(\rho+2r)\right)\beta_{0,q}(-\rho-2r)\int_0^\infty \Psi_{p,q}(t,\rho+2r)F(t))A(t)dt \cdot$$

$$\cdot \int_B Y(b)\phi_{0,q}(b)db$$

$$\eta_r(\phi) = c_q\beta_{p,q}(\rho+2r)\left(\frac{d}{ds}\,\beta_{0,q}(-\rho-2r)\right)\int_0^\infty \Psi_{p,q}(t,\rho+2r)F(t)A(t)dt \,.$$

$$\cdot \int_B Y(b)\phi_{0,q}(b)db.$$

From our expression for the numbers $\beta_{p,q}(s)$ we see that

$$\eta_r \neq 0, \quad \theta_r \neq 0, \quad \eta_r(\phi) = 0 \text{ if } p+q > r, \quad \theta_r(\phi) = 0 \text{ if } q \leq r$$

from which the assertion follows.

Collecting the results contained in Propositions 8 and 9 and Lemma 13 we obtain:

<u>THEOREM 1</u>. *If*

$$\lambda \neq 4r(r+\rho)\forall r \in \mathbb{N}$$

then $\mathcal{D}'_{\lambda,H}(X)$ *is spanned by* ζ_s *if* $\lambda = s^2 - \rho^r$, $\zeta_s \neq 0$.
If $r \in \mathbb{N}$ *and* $\lambda = 4r(r+\rho)$ *then* $\mathcal{D}'_{\lambda,H}(X)$ *is spanned by the linearly independent elements* η_r *and* θ_r.

We are now in a position to state and prove the analogues of Theorems

7.3 and 7.4 of [4]. This is left to the reader.

§12. SPHERICAL DISTRIBUTIONS OF POSITIVE TYPE

Because it follows from our preceding results along exactly the same line of reasoning as that in [4], ch.9 we leave the proof of the following to the reader:

THEOREM 2. *The following and only the following spherical distributions are of positive type*:

a. $c\zeta_s$ ($c > 0$, $s \in i\mathbb{R}$, *the principal series*)

b. $c\zeta_s$ ($c > 0$, $s \in \mathbb{R}$, $-5 \leq s \leq 5$, $s \neq 0$, *the complementary series*)

c. $c\zeta_s$ ($c > 0$, $s = \rho - 2r$ *for a certain* $r \in \mathbb{N}$, $s > 5$)

d. $c_1\eta_0 - c_2\theta_0$ ($c_1, c_2 > 0$, η_0 *is a constant distribution*)

e. $c.(-1)^{r+1}\theta_r$ ($c > 0$, $r > 0$).

Statement and proof of the analogue of [4], Theorème 9.5 are again left to the reader.

§13. THE PLANCHEREL FORMULA

We first introduce some notations. If f is a meromorphic function, $s_0 \in \mathbb{C}$ then we denote the coefficient of $(s-s_0)^{-2}$ in the Laurent series for f near s_0 by $c_{-2}(f(s), s_0)$.
With $W_s(1)$ as in §8 we write

$$c(s) = \frac{2^{\rho-s}}{\Gamma(\frac{s-\rho+8}{2})} W_s(1).$$

We write

$$C = \frac{7!2^{\rho}}{b_{p,q}\Gamma(p+4)} .$$

From the expression for $b_{p,q}$ one sees that C does not depend on p and q.
We denote the spherical kernels on $\mathcal{D}(X)$ corresponding to the spherical

distributions ζ_s and θ_r by Z_s and Θ_r respectively.
We have the following expressions for these kernels (cf. [4], ch.9):
when $s \in i\,\mathbb{R}$:

$$Z_s(\phi,\phi) = \int_B |\hat{\phi}(b,s)|^2 db.$$

When $s \in \mathbb{R}$, $s - \rho \in \mathbb{Z}$ and ϕ is a K-finite function of the form $\phi(x) = F(t)Y(b) \quad (Y \in H_{p,q})$ then

$$Z_s(\phi,\phi) = \gamma(-s)\alpha_{p,q}(s)(\beta_{p,q}(-s))^2 \Big| \int_0^\infty F(t)\Psi_{p,q}(t,s)A(t)dt\Big|^2 \cdot \int_B |Y(b)|^2\, db,$$

and when

$$s - \rho = k \in \mathbb{Z}\,,\ \phi(x) = F(t)Y(b)(Y \in H_{p,q})$$

$$Z_s(\phi,\phi) = \lim_{s\to\rho+k} \gamma(-s)\alpha_{p,q}(s)(\beta_{p,q}(-s))^2 \Big| \int_0^\infty F(t)\Psi_{p,q}(t,s)A(t)dt\Big|^2 \cdot \int_B |Y(b)|^2\, db.$$

We have written

$$\gamma(s) = \frac{\beta_{00}(-s)}{\beta_{00}(s)}\ .$$

When $r \in \mathbb{N}$, $\phi(x) = F(t)Y(b)$, $(Y \in H_{p,q})$

$$\Theta_r(\phi,\phi) = \lim_{s\to\rho+2r} \frac{1}{s-\rho-2r}\, \gamma(-s)\alpha_{pq}(s)\beta_{pq}(-s)\beta_{pq}(-\rho-2r) \cdot \int_0^\infty F(t)\Psi_{p,q}(t,s)A(t)dt \int_0^\infty \overline{F(t)}\Psi_{p,q}(t,\rho+2r)A(t)dt \int_B |Y(b)|^2\, db.$$

We now prove the Plancherel formula:

THEOREM 3. *If* $\phi \in \mathcal{D}(X)$ *then*

$$\int_X |\phi(x)|^2 dx = C^2 \left\{ \frac{1}{2\pi} \int_0^\infty Z_{i\nu}(\phi,\phi) \frac{d\nu}{|c(i\nu)|^2} \right.$$

$$+ \sum_{0<\rho+2r<\rho} Z_{\rho+2r}(\phi,\phi) \operatorname{Res}\left(\frac{1}{c(s)c(-s)}, \rho+2r\right) +$$

$$\left. \sum_{\rho+2r>\rho} \Theta_r(\phi,\phi) c_{-2} \left(\frac{1}{c(s)c(-s)}, \rho+2r\right)\right\}.$$

REMARK. This is the analogue of Theorème 10.1 in [4]. In that theorem no factor C^2 occurs. This has the following reason: Faraut uses two different measures dx and $\underline{dx}$ on X which are proportional to each other. The measure $\underline{dx}$ is used in the left hand side of his Plancherel formula and the measure dx in the right hand side, via the definitions of the distributions ζ and θ. We only use one measure dx in both sides of the formula. It is the analogue of Faraut's $\underline{dx}$.

PROOF OF THEOREM 3. Let ϕ be a K-finite function of the form

$$\phi(ka_t x^0) = F(t)(\sinh t)^p (\cosh t)^{p+2q} Y(k) (Y \in H_{p,q}, F \in \mathcal{D}_{\text{even}}(\mathbb{R})) .$$

Then we shall prove the Plancherel formula for this ϕ. Because such ϕ span a dense linear subspace of $\mathcal{D}(X)$ this will prove the theorem. Now

$$\int_X |\phi(x)|^2 dx = \int_B |Y(b)|^2 db \int_0^\infty |F(t)|^2 (\sinh t)^{2p+7} (\cosh t)^{4p+4q+15} dt.$$

Write

$$\alpha = p + 3,\ \beta = p + 2q + 7,\ \Delta_{\alpha,\beta}(t) = 2^{2(\alpha+\beta+1)} (\sinh t)^{2\alpha+1} (\cosh t)^{2\beta+1}$$

then

$$\int_X |\phi(x)|^2 dx = \int_B |Y(b)|^2 |db\ 2^{-2(\alpha+\beta+1)} \int_0^\infty |F(t)|^2 \Delta_{\alpha,\beta}(t)\, dt$$

The integral over t can be rewritten using the Plancherel formula for the Fourier-Jacobi transform (see [6], Appendix 1 and proof of Theorem 2.5). Define for $\nu \in \mathbb{C}$ the Jacobi function $\phi_\nu^{(\alpha,\beta)}$ as in [6], (A.3), and

$$\hat{F}(\nu) = (2\pi)^{-\frac{1}{2}} \int_0^\infty F(t)\phi_\nu^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt$$

$$C(\nu) = \frac{2^{\alpha+\beta+1-i\nu}\Gamma(i\nu)\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}(\alpha+\beta+1+i\nu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\nu))}$$

then we have

$$\int_0^\infty |F(t)|^2\Delta_{\alpha,\beta}(t)dt = \frac{1}{2\pi}\int_0^\infty |\hat{F}(\nu)|^2 \frac{d\nu}{|C(\nu)|^2} +$$

$$- 2\pi i \sum_{\substack{\nu\in i\mathbb{R} \\ C(-\nu)=0}} |F(\nu)|^2 \operatorname{Res} \frac{1}{C(\nu)C(-\nu)} \qquad (*)$$

We now show that the expression for $\int_X |\phi(x)|^2 dx$ which this yields is equal to the right hand side in our Plancherel formula. Because

$$\Psi_{p,q}(t,i\nu) = \sinh^p t \cosh^{p+2q} t \; \phi_\nu^{(\alpha,\beta)}(t):$$

$$Z_{i\nu}(\phi,\phi) = \int_B |\phi(b,i\nu)|^2 db =$$

$$\int_B |Y(b)|^2 db |\beta_{p,q}(i\nu)|^2 | \int_0^\infty F(t) \sinh t^{p+7} \cosh t^{p+2q+15} \cdot$$

$$\cdot \sinh t^p \cosh t^{p+2q} \phi_\nu^{(\alpha,\beta)}(t)dt|^2 =$$

$$\int_B |Y(b)|^2 |\beta_{p,q}(i\nu)|^2 2\pi . 2^{-4(\alpha+\beta+1)} |\hat{F}(\nu)|^2.$$

But

$$2^{-2(\alpha+\beta+1)} \frac{1}{|C(\nu)|^2} \frac{|c(i\nu)|^2}{2\pi \; 2^{-4(\alpha+\beta+1)} |\beta_{p,q}(i\nu)|^2} =$$

$$c^2\left|\frac{\Gamma(\frac{1}{2}(i\nu-\rho)+q+8)\Gamma(\frac{1}{2}(i\nu-\rho)-p-q+1)}{\Gamma(\frac{1}{2}(i\nu-\rho)+1)}\ \frac{\Gamma(\frac{1}{2}(2p+2q+11+i\nu))\Gamma(\frac{1}{2}(-2q-3+i\nu))}{\Gamma(i\nu)}\ \cdot\right.$$

$$\left.\cdot\ \frac{\Gamma(i\nu)}{\Gamma(\frac{1}{2}(i\nu+\rho))\Gamma(\frac{1}{2}(i\nu-\rho)+8)\Gamma(\frac{1}{2}(i\nu-\rho)\ +4)}\right|^2 =$$

$$c^2\left|\frac{\Gamma(\frac{1}{2}(i\nu-\rho)+1)}{\Gamma(\frac{1}{2}(i\nu-\rho)+q+8)}\ \frac{\Gamma(\frac{1}{2}(i\nu+\rho))}{\Gamma(\frac{1}{2}(i\nu+\rho)-q-7)}\ \frac{\Gamma(\frac{1}{2}(i\nu+\rho)-3)}{\Gamma(\frac{1}{2}(i\nu+\rho)+p+q)}\ \frac{\Gamma(\frac{1}{2}(i\nu-\rho)+4)}{\Gamma(\frac{1}{2}(i\nu-\rho)-p-q+1)}\right|^2 =$$

$$c^2\left|\frac{(\frac{1}{2}(i\nu+\rho)-q-7)_{q+7}(\frac{1}{2}(i\nu-\rho)-p-q+1)_{p+q+3}}{(\frac{1}{2}(i\nu-\rho)+1)_{q+7}(\frac{1}{2}(i\nu+\rho))_{p+q+3}}\right|^2 = c^2.$$

The last equality holds because the numerator of the fraction is the complex conjugate of the denominator, as is easily seen (remember that ν is real). In an analogous fashion the reader may show that the other terms in (*) correspond to the other terms in the right hand side of the Plancherel formula.

REMARK. Essentially our proof of the Plancherel formula is the same as Faraut's "seconde démonstration de la formule de Plancherel". Faraut proves for a discrete set of values of α and β the Plancherel formula for the Fourier Jacobi transform using essentially the same method as is used in [5] (using spectral theory of Sturm-Liouville operators).

REMARK. From the Plancherel formula we see that we have a "relative discrete series" (cf. [4], p.431-432) consisting of the distributions listed under c. and e. in Theorem 2.

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APPENDIX

CALCULATION OF A CERTAIN DOUBLE INTEGRAL

Define as in [8] for $\alpha,\beta \geq -\frac{1}{2}$ and integers n,k with $n \geq k \geq 0$ the function $P_{n,k}^{(\alpha,\beta)}$ of two variables by

$$P_{n,k}^{(\alpha,\beta)}(u,v) = P_k^{(\alpha,\beta+n-k+\frac{1}{2})}(2v-1)v^{(n-k)\frac{1}{2}}P_{n-k}^{(\beta,\beta)}(v^{-\frac{1}{2}}u)$$

where for $\alpha,\beta \geq -\frac{1}{2}$ $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n with indices α and β. Write

$$P_{k,\ell}^{(\alpha,\beta)}(r,\phi) = P_{k,\ell}^{(\alpha-\beta-1,\beta-\frac{1}{2})}(r\cos\phi,r^2)$$

and

$$dm_{\alpha,\beta}(r,\phi) = \mu_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}dr\,d\phi$$

with

$$\mu_{\alpha,\beta} = \frac{2\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha,\beta)\Gamma(\beta+\frac{1}{2})}\,.$$

For $t \geq 0$ we want to compute the integral

$$\int_0^1\int_0^\pi (\sinh^2 t+r^2\cosh^2 t-2\sinh t\cosh t\; r\cos\phi)^{\frac{1}{2}(s-\alpha-\beta-1)}\cdot$$

$$\cdot\, P_{k,\ell}^{(\alpha,\beta)}(r,\phi)dm_{\alpha,\beta}(r,\phi) \qquad (*)$$

for $s \in \mathbb{C}$, Re $s > \alpha + \beta + 1$.

First we consider the case $\frac{1}{2}(s-\alpha-\beta-1) = n = 0,1,2,\ldots$. Take $x = -\cosh 2t$ and apply formula (4.9) of [8] (both sides of this formula being defined so that they are single valued analytic functions on the complex plane with a cut along the interval $[-1,1]$) then we see that our integral is equal to

$$\frac{(-1)^n n!(\alpha-\beta)_\ell(n-\ell+\beta+1)_\ell(\beta+\frac{1}{2})_{k-\ell}}{2^k \ell!(k-\ell)!(\alpha+1)_{n+\ell}}(-2)^k \cdot$$

$$\cdot\,(\cosh t)^{k+\ell}(\sinh t)^{k-\ell} P_{n-k}^{(\alpha+k+\ell,\beta+k-\ell)}(-\cosh 2t) =$$

$$\frac{n!(\alpha-\beta)_\ell(n-\ell+\beta+1)_\ell(\beta+\frac{1}{2})_{k-\ell}(\beta+k-\ell+1)_{n-k}}{\ell!(k-\ell)!(\alpha+1)_{n+\ell}(n-k)!}\,\cdot$$

$$\cdot\,(\sinh t)^{k-\ell}(\cosh t)^{k+\ell}\,{}_2F_1(-n+k,n+\alpha+\beta+1;\beta+k-\ell+1;-\sinh^2 t)$$

where in the last step we have used one of the well-known expressions for Jacobi polynomials as hypergeometric functions (see e.g. [3] 10.8 formula (16), the second expression).

Let for $\lambda \in \mathbb{C}$ the Jacobi function $\phi_\lambda^{(\alpha,\beta)}$ be defined by

$$\phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1-i\lambda),\tfrac{1}{2}(\alpha+\beta+1+i\lambda);\ \alpha+1;\ -\sinh^2 t)$$

then we have now proved for $s = \alpha + \beta + 1 + 2n$, $n = 0,1,2,\ldots$ that our integral (*) is equal to

$$(-1)^k(\tfrac{1}{2}(\alpha+\beta+1-s))_k \frac{\Gamma(\frac{1}{2}(s-\alpha+\beta+1))}{\Gamma(\frac{1}{2}(s+2\ell+\alpha-\beta+1))}\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}\frac{(\alpha-\beta)_\ell(\beta+\frac{1}{2})_{k-\ell}}{\ell!(k-\ell)!(\beta+1)_{k-\ell}}\,\cdot$$

$$\cdot\,(\sinh t)^{k-\ell}(\cosh t)^{k+\ell}\,\phi_{i(s-k)}^{(\beta+k-\ell,\alpha+\ell)}(t)\,. \qquad (**)$$

Now both sides of this identity are analytic in s for Re $s > \alpha + \beta + 1$ and this is still true if we divide both by $e^{(s-\alpha-\beta-1)t}$. But then, as a function of s the left member is smaller in absolute value than

$$\int_0^1\int_0^\pi |P_{-k,\ell}^{(\alpha,\beta)}|\,dm_{\alpha,\beta}(r,\phi)$$

which is independent of s and the absolute value of the right hand side is smaller than const.$|s|^{k-2\ell-2(\alpha-\beta)}$ where the constant does not depend on s (cf. T.H. KOORNWINDER, *A new proof of a Paley-Wiener type theorem for the Jacobi transform,* Ark. Mat. 13 (1975), p.150 Lemma 2.3) Now it follows from

Carlson's Theorem (cf. E.C. TITCHMARSH, *The theory of functions*, Oxford University Press, second edition (1939), on p.186) that for all s with Re $s > \alpha + \beta + 1$ our integral (*) is equal to (**).
The formula for the function $\Phi^{*}_{p,q}(t,s)$ which is mentioned in §9 of this paper is a particular case of this identity (with $k = p+q$, $\ell = q$, $\alpha = 7$, $\beta = 3$).