stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 166/81 OKTOBER
(DEPARTMENT OF PURE MATHEMATICS)

J. DE VRIES

ON THE G-COMPACTIFICATION OF PRODUCTS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 54H15, 54D35

On the G-compactification of products *)

by

Jan de Vries

ABSTRACT

Let $\beta_G X$ denote the maximal equivariant compactification (G-compactification) of the G-space X (i.e. a topological space X, completely regular and Hausdorff, on which the topological group G acts as a continuous transformation group). If G is locally compact and locally connected, then we show that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ if and only if $X \times Y$ is what we call G-pseudocompact, provided X and Y satisfy a certain non-triviality condition. This result generalizes Glicksberg's well-known result about Stone-Čech compactifications of products to the case of topological transformation groups.

KEY WORDS & PHRASES: *G-space, topological transformation group, G-compactification, G-pseudocompact, Stone-Čech compactification, Glicksberg's theorem*

*) This report will be submitted fot publication elsewhere

1. INTRODUCTION

In this paper we prove a generalization to the case of topological transformation groups of Glicksberg's well-known result about Stone-Čech compactifications of products. Recall, that a topological space X is *pseudocompact*, whenever $C(X) = C^*(X)$, i.e. every continuous real-valued function on X is bounded. A convenient characterization of pseudocompactness of a completely regular Hausdorff space X is, that X contains no infinite sequence of non-empty open subsets which is locally finite. Cf. [3] and, for more about pseudocompactness, [4]. Glicksberg's theorem says that, if X and Y are infinite completely regular spaces, then $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact. See [5]; also [3] and [9] for short proofs. Adopting the techniques of [3] and [9], we were able to prove (terminology will be explained in 1.1 and 2.1 below):

<u>THEOREM</u>. *Let G be a locally compact, locally connected topological Hausdorff group, and let X and Y be two G-infinite, completely regular Hausdorff G-spaces. Then* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ *if and only if* $X \times Y$ *is G-pseudocompact.*

Before explaining the terminology we wish to point out one shortcoming of our theorem. It is clear why Glicksberg's theorem has to contain the condition that X and Y are infinite: if either X or Y is finite, then always $\beta(X \times Y) = \beta X \times \beta Y$ without any further condition on $X \times Y$. However, compared with this situation, our "non-triviality condition" in the theorem above is too strong: if either X or Y is *not* G-infinite, then it is not true that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ without additional conditions. See Section 5 below.

The organization of the paper is as follows. In the remainder of this section we present the necessary definitions and preliminary results. In Section 2 we shall deal with the concept of G-pseudocompactness. In particular, we give some necessary and some sufficient conditions. In Section 3, the "if" part of our theorem is proven, and in Section 4 the "only if" part. Finally, in Section 5 we discuss some open questions and present some additional material. In particular, we prove that $\beta_G X = \beta X$ if X is pseudocompact and G is a topological group such that, as a topological space, G is a k-space. This slightly generalizes a result by SMIRNOV [8].

1.1. In this paper, except in 5.5 and 5.7, G will always denote a locally compact Hausdorff topological group with unit element e. The neighbourhood filter of e in G will be denoted by $\mathcal{V}_e$. (In general, $\mathcal{V}_x$ will denote the neighbourhoodfilter of x in a given topological space.) A G-*space* (or: a topological transformation group with acting group G) is a pair $\langle X,\pi\rangle$ consisting of a topological space X and an *action* π. This means, that π is a continuous mapping from $G \times X$ into (in fact, onto) X such that the following conditions are fulfilled:

(i) $\forall x \in X$: $\pi(e,x) = x$;

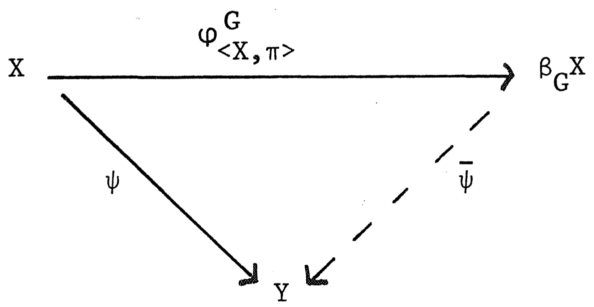
(ii) $\forall x \in X$, $\forall (s,t) \in G \times G$: $\pi(s,\pi(t,x)) = \pi(st,x)$.

Then for every $t \in G$ the mapping $\pi^t : x \mapsto \pi(t,x) : X \to X$ is a homeomorphism, and for every $x \in X$ the mapping $\pi_x : t \mapsto \pi(t,x) : G \to X$ is continuous. For brevity, we shall write in most cases tx for $\pi(t,x)$, tA for $\pi^t[A]$, Ux for $\pi_x[U]$ and, in general, UA for $\pi[U \times A]$. Also, we shall often write "the G-space X" instead of "the G-space $\langle X,\pi\rangle$". The G-space $\langle X,\pi\rangle$ will be called compact, Hausdorff, etc. whenever X is.

If $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, then a mapping $\varphi: X \to Y$ is called *equivariant* whenever $\varphi\pi^t = \sigma^t\varphi$ for all $t \in G$ (i.e. $\varphi(tx) = t\varphi(x)$ for all $t \in G$, $x \in X$). A *morphism of* G-*spaces* is a continuous, equivariant mapping $\varphi:\langle X,\pi\rangle \to \langle Y,\sigma\rangle$. A G-*compactification* of a G-space $\langle X,\pi\rangle$ is a morphism of G-spaces $\varphi:\langle X,\pi\rangle \to \langle Y,\sigma\rangle$ such that Y is a compact Hausdorff space and $\varphi[X]$ is dense in Y. If, in addition φ is an embedding of X into Y then φ is called a *proper* G-*compactification*. A necessary condition for the existence of a proper G-compactification of $\langle X,\pi\rangle$ is, that X is a Tychonov space. Because of the fact that G is assumed to be locally compact, this condition is also sufficient (cf. [11]). Every G-space $\langle X,\pi\rangle$ has an essentially unique *maximal* G-*compactification*, denoted by

$$\varphi^G_{\langle X,\pi\rangle} : \langle X,\pi\rangle \to \beta_G\langle X,\pi\rangle.$$

For convenience, the underlying topological space of $\beta_G\langle X,\pi\rangle$ will be denoted by $\beta_G X$. The maximal G-compactification of $\langle X,\pi\rangle$ is defined by the property that for every G-compactification $\psi:\langle X,\pi\rangle \to \langle Y,\sigma\rangle$ there exists a unique morphism of G-spaces $\bar{\psi}$: $\beta_G\langle X,\pi\rangle \to \langle Y,\sigma\rangle$ such that $\psi = \bar{\psi} \circ \varphi^G_{\langle X,\pi\rangle}$.



If in, this situation, ψ happens to be a proper G-compactification, then so is $\varphi^G_{<X,\pi>}$. So from our remarks above it follows, that every Tychonov G-space $<X,\pi>$ has a *proper* maximal G-compactification. *From now on, we shall assume that all* G-*spaces* $<X,\pi>$, $<Y,\sigma>$, *etc. are Tychonov spaces*. Moreover, if $<X,\pi>$ is such a G-space, then we shall identify X with its image under $\varphi^G_{<X,\pi>}$ in $\beta_G X$. Thus, X is an invariant subset of $\beta_G X$.

1.2. If $G = \{e\}$, then every mapping between G-spaces is equivariant, and the category of all G-spaces and continuous equivariant mappings is identical with the category of all topological spaces and continuous mappings. In particular, for every G-space X we have $\beta_G X = \beta X$, the ordinary G-compactification of X. From completeness, we mention three other cases where $\beta_G X = \beta X$:

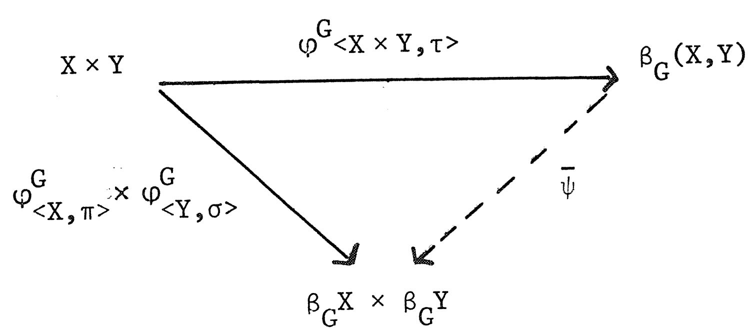
(i) G is a discrete group (cf. [10], 7.3.10(ii));

(ii) the action of G on X is trivial, i.e. $tx = x$ for all $t \in G$, $x \in X$;

(iii) G is a k-space and X is pseudocompact (cf. Section 5 below).

In a future paper, we hope to study this problem in more detail.

1.3. Let $<X,\pi>$ and $<Y,\sigma>$ be two G-spaces, and let τ denote the action of G on $X \times Y$, defined by $\tau^t(x,y) := (\pi^T x, \sigma^t y)$ (or briefly: $t(x,y) = (tx,ty)$ for $t \in G$ and $(x,y) \in X \times Y$). Then we have the following commutative diagram:



If in this diagram $\bar{\psi}:\beta_G(X \times Y) \to \beta_G X \times \beta_G Y$ is a homeomorphism, then we shall say that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$. Notice, that it follows from 1.2 (ii) above that Glicksberg's theorem gives a necessary and sufficient condition for the equality $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ to occur for the special case that the actions π and σ (hence τ) are both trivial. Taking into account that "G-infinite" means in this special situation just "infinite" (see below), it is clear that our theorem above contains Glicksberg's result as a special case.

1.4. Let $<X,\pi>$ be a G-space. A real-valued function f on X will be called *π-uniformly continuous* (cf. [8], [11]) whenever the following conditions are fulfilled

1^o. f is continuous

2^o. the set $\{f \circ \pi_x\}_{x \in X}$ is equicontinuous at e.

The second condition can also be formulated as follows:

$$\forall \varepsilon > 0 \quad \exists U \in \mathcal{V}_e : |f(tx) - f(x)| < \varepsilon \text{ for all } (t,x) \in U \times X.$$

The set of all π-uniformly continuous functions on X will be denoted by $UC<X,\pi>$, and the set of all *bounded* π-uniformly continuous functions by $UC^*<X,\pi>$ (in [11], the notation $\pi UC(X)$ was used). In [11] it was shown that $UC^*<X,\pi>$ is a closed subalgebra of $C^*(X)$ (the bounded real-valued continuous function on X), containing the constant functions, and that for every G-compactification $\varphi:<X,\pi> \to <Y,\sigma>$ we have $\{g \circ \varphi : g \in C(Y)\} \subseteq UC^*<X,\pi>$. In particular, the maximal G-compactification $\varphi^G_{<X,\pi>} : X \to \beta_G X$ is, up to isomorphism of G-spaces, completely characterized by the formula

$$UC^*<X,\pi> = \{g \circ \varphi^G_{<X,\pi>} : g \in C(\beta_G X)\}.$$

The following remark is included in order to clarify the relationship between $UC^*<X,\pi>$ and ordinary uniform continuity. If $(X,\mathcal{U})$ is a uniform space, then $UC^*(X,\mathcal{U})$ will denote the set of all $\mathcal{U}$-uniform continuous, bounded real-valued functions on X, and $\mathcal{U}^*$ will denote the weakest uniformity on X such that $UC^*(X,\mathcal{U}^*) = UC^*(X,\mathcal{U})$. If $(X,\mathcal{U})$ is a uniform space and, in addition, π is a continuous action of G on X (the topology on X, of course, being induced by $\mathcal{U}$) then π is called *$\mathcal{U}$-bounded* (cf. [10], [11]; in the literature on topological dynamics one also calls π *motion-equicontinuous*) whenever $\{\pi_x\}_{x\in X}$ is equicontinuous w.r.t. $\mathcal{U}$ at e, that is,

$$\forall\alpha \in \mathcal{U} \;\; \exists U \in \mathcal{V}_e : (x,tx) \in \alpha \text{ for all } (t,x) \in U \times X.$$

Now it is easy to show that the following two statements are equivalent for an arbitrary G-space $<X,\pi>$ and a uniformity $\mathcal{U}$, compatible with the topology of X:

(i) the action π is U^*-bounded;
(ii) $UC^*(X,\mathcal{U}) \subseteq UC^*<X,\pi>$.

1.5. Next we wish to point out the relationship between $UC^*<X,\pi>$ and the algebra $E(X,C_c^*(G))$ of [1]. Let $C_c^*(G)$ denote the space of all bounded real-valued functions on G, endowed with the compact-open topology. Then $<C_c^*(G),\rho>$ is a G-space, where $\rho^t f(s) := f(st)$ for all $f \in C_c^*(G)$, $s \in G$ and $t \in G$ (cf. [10], 2.1.3). Let $Mor_u^G(X,C_c^*(G))$ denote the set of all morphisms of G-spaces from a given G-space $<X,\pi>$ to $<C_c^*(G),\rho>$, endowed with the uniform structure and the corresponding topology of uniform convergence on X. If $f \in C^*(X)$, then the mapping

$$Tf : x \mapsto f \circ \pi_x : X \to C_c^*(G)$$

is continuous and equivariant (cf. [10], 8.1.12), i.e. $Tf \in Mor_u^G(X,C_c^*(G))$. Conversely, if $g \in Mor_u^G(X,C_c^*(G))$, then

$$Sg: x \mapsto g(x)(e) : X \to \mathbb{R}$$

is an element of $C^*(X)$. It is easily verified that $T: C^*(X) \to Mor_u^G(X,C_c^*(G))$ and $S: Mor_u^G(X,C_c^*(G)) \to C^*(X)$ are mutually inverse isomorphisms of algebras. Moreover, if we endow $C^*(X)$ with the topology of uniform convergence on X, then it is standard to show, that T and S are both continuous. So $C_u^*(X)$ and $Mor_u^G(X,C_c^*(G))$ are isomorphic as topological algebra's (consequently, the latter algebra is metrizable, though G is *not* supposed to be compact or even sigma-compact!) Under this correspondence, $E(X,C_c^*(G)) := T[UC^*<X,\pi>]$

is easily seen to be the set of all those elements $g \in Mor_u^G(X, C_c^*(G))$ for which $g[X]$ is equicontinuous in $C_c^*(G)$, that is, for which $g[X]$ has compact closure in $C_c^*(G)$. Using this relationship between $UC^*\langle X,\pi\rangle$ and $E(X,C_c^*(G))$, the correspondence between $\beta_G X$ and $UC^*\langle X,\pi\rangle$ can be reformulated as follows: for every element $g \in E(X,C_c^*(G))$ there exists a unique morphism of G-spaces $\bar{g}:\beta_G X \to C_c^*(G)$ such that $g = \bar{g} \circ \varphi^G_{\langle X,\pi\rangle}$; moreover, the embedding of X into $\beta_G X$ is completely characterized by this property (up to isomorphism of G-spaces).

2. G-PSEUDOCOMPACTNESS AND G-INFINITENESS

2.1. A collection $\mathcal{B}$ of subsets in a G-space $\langle X,\pi\rangle$ will be called *internally linked* whenever there exists $U \in \mathcal{V}_e$ and there are points $x_B \in B$ $(B \in \mathcal{B})$ such that $Ux_B \subseteq B$ for every $B \in \mathcal{B}$.

A finite (infinite) sequence of mutually disjoint, non-empty open sets which is internally linked will be called a *finite* (*infinite*) G-*dispersion*; if the sequence of sets is locally finite, then the G-dispersion will be called *locally finite*. Modifying the characterizations of infiniteness and pseudocompactness of ordinary Tychonov spaces, we obtain the following crucial (at least, for this paper) definitions. The G-space $\langle X,\pi\rangle$ will be called

- G-*infinite*, whenever it contains an infinite G-dispersion
- G-*pseudocompact*, whenever every locally finite G-dispersion in X is finite.

Clearly, if $\langle X,\pi\rangle$ is not G-infinite or if X is pseudocompact (in the usual sense) then X is G-pseudocompact. As to the converse, cf. Section 5 below.

2.2. <u>REMARKS</u>. 1^o. If G is a *discrete* group, then every family of non-empty subsets of X is internally linked, because $\{e\} \in \mathcal{V}_e$. It follows that in this case X is G-infinite if and only if X is infinite. Similarly, X is G-pseudocompact if and only if X is pseudocompact. (These statements are also valid if the action of G on X is trivial.)

2^o. Suppose that the *orbit space* X/G (= space of equivalence classes of the form Gx, $x \in X$, having the quotient topology) contains an infinite sequence of mutually disjoint, non-empty open subsets (e.g. because the Hausdorff modification of X/G is infinite; in particular, this happens if X/G is itself an infinite Hausdorff space. Recall, that X/G is usually not Hausdorff, but it is if G is compact, or the action of G on X is proper). Taking inverse images under the canonical projection $X \to X/G$ one obtains an infinite G-dispersion (the elements of which are even invariant under all of G). Hence X is G-infinite. Similarly, if X/G is not pseudocompact, then X/G contains an infinite sequence of non-empty open sets which is locally finite (for this statement, complete regularity of X/G is not required, nor its being Hausdorff), hence X contains an infinite G-dispersion which is locally finite, i e. X is not G-pseudocompact. *Thus, if* X *is* G-*pseudocompact, then* X/G *is pseudocompact.*

3°. Suppose X/G consists of one point and for some (hence for every) point x in X the mappint $\pi_x: t \to tx: G \to X$ is open (thus, $X \simeq G/H$, where $H := \{t \in G : tx = x\}$. *In this case,* X *is* G-*infinite if and only if* X *is not compact.* (Suppose X is not compact. Let $U \in \mathcal{V}_e$ be compact. Construct by induction a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that, for every $n \in \mathbb{N}$, $x_{n+1} \notin \bigcup_{i=1}^{n} Ux_i$. Let $V \in \mathcal{V}_e$ be open, $V^{-1}V \subseteq U$; then Vx_i is open in X, hence $\{Vx_i\}_{i\in\mathbb{N}}$ is an infinite G-dispersion. Conversely, suppose that X is compact and that $\{B_n\}_{n\in\mathbb{N}}$ is an infinite G-dispersion in X. We may assume that, for every $n \in \mathbb{N}$, $B_n = Uy_n$ with $y_n \in X$ and $U \in \mathcal{V}_e$, U open and $U^{-1} = U$. The sequence $\{y_n\}_{n\in\mathbb{N}}$ has an accumulation point $z \in X$. Then $y_n \in Uz$ for infinitely many values of n, contradicting the disjointness of the sequence $\{Uy_n\}_{n\in\mathbb{N}}$.) *Similarly, in this case* X *is* G-*pseudocompact if and only if* X *is compact.* (In the above proof, replace V by open $W \in \mathcal{V}_e$ such that $W^{-1} = W$ and $W^2 \subseteq V$.)

Observe, that this example shows that the converse of the final remark in 2° above is not generally true (X/G is pseudocompact, but one can have X not compact, e.g. X = G).

4°. According to the definition, a G-space $\langle X,\pi\rangle$ is G-pseudocompact whenever every sequence of *mutually disjoint* open sets which is internally linked and locally finite is finite. In this definition, disjointness can be omitted.

Indeed, let $\{B_n\}_{n\in\mathbb{N}}$ be an infinite sequence of non-empty open sets, internally linked and locally finite. Then there exists $U \in \mathcal{V}_e$, U compact, and for every $n \in \mathbb{N}$ there is $x_n \in B_n$ such that $Ux_n \subseteq B_n$. As Ux_n is compact and $\{B_i\}_{i\in\mathbb{N}}$ is locally finite, there exists an open neighbourhood B_n' of

Ux_n such that $B'_n \subseteq B_n$, and B'_n meets only finitely many of the sets B_i. Selecting from the sequence $\{B'_n\}_{n\in\mathbb{N}}$ a *disjoint* subsequence, one obtains an infinite, locally finite G-dispersion. Thus, $\langle X,\pi\rangle$ *is* G-*pseudocompact if and only if every sequence of open sets which is internally linked and locally finite, is finite.*

2.3. Before stating a (simple, yet crucial) result about the connection between π-uniformly continuous functions on a G-space $\langle X,\pi\rangle$ and G-pseudocompactness of $\langle X,\pi\rangle$, we recall from [11] a method of transforming elements of $C^*(X)$ into elements of $UC^*\langle X,\pi\rangle$. Let $f \in C^*(X)$, $f \geq 0$ and let $\|f\| := \sup\{f(x) : x \in X\}$. Let $U \in \mathcal{V}_e$ be compact and select a left-uniformly continuous function $\varphi: G \to [0,\|f\|]$ such that $\varphi(e) = 0$ and $\varphi(t) = \|f\|$ for all $t \in G\backslash U$. If we put

$$f^U(x) := \inf_{t\in G} \{\varphi(t) + f(tx)\}, \qquad x \in X,$$

then it turns out, that $f^U \in UC^*\langle X,\pi\rangle$. Moreover, $0 \leq f^U \leq f$ on X and, in addition we have for all $x \in X$.

$$f^U(x) = \inf_{t\in U} \{\varphi(t) + f(tx)\}$$

In particular, if $x \in X$ *is such that* $f(tx) = f(x)$ *for every* $t \in U$, *then clearly* $f^U(x) = f(x)$.

2.4. <u>PROPOSITION</u>. *Let* $\{B_n\}_{n\in\mathbb{N}}$ *be an infinite, locally finite* G-*dispersion in* X, *and let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers in the interval* $[0,1]$. *Then there exists* $f \in UC^*\langle X,\pi\rangle$ *such that* $f \geq 0$, $f[B_n] \subseteq [0,a_n]$ *and* $f^{\leftarrow}[a_n] \cap B_n \neq \emptyset$ *for every* $n \in \mathbb{N}$, *whereas* $f(x) = 0$ *for all* $x \in X\backslash\bigcup_{n=1}^{\infty} B_n$.

<u>PROOF</u>. There exist $U \in \mathcal{V}_e$, U compact, and $x_n \in B_n$ $(n \in \mathbb{N})$ such that $Ux_n \subseteq B_n$. For every $n \in \mathbb{N}$, Ux_n is a compact subset of the Tychonov space X, so there exists $g_n \in C^*(X)$ such that $g_n[X] \subseteq [0,a_n]$, $g_n(x) = a_n$ for all $x \in Ux_n$ and $g_n(x) = 0$ for all $x \in X\backslash B_n$. As $\{B_n\}_{n\in\mathbb{N}}$ is locally finite, $g := \sum_{n=1}^{\infty} g_n$ is a bounded, continuous function. Choosing φ according to the specification of 2.3 above, we can form the function g^U, which belongs to $UC^*\langle X,\pi\rangle$. Using the properties of this construction, mentioned in 2.3,

it is easy to verify that g^U satisfies the conditions specified in our Proposition. □

In our next Proposition we relate the property of being G-pseudocompact with boundedness properties of π-uniformly continuous functions on a G-space $\langle X,\pi\rangle$. For the problem, whether of (ii) $\Rightarrow$ (i) or not, we refer to Section 5.

2.5. PROPOSITION. *Consider the following properties for a* G-*space* $\langle X,\pi\rangle$.

(i) *Every* $f \in UC^*\langle X,\pi\rangle$ *has a maximum and a minimum on* X, *i.e.* $\sup f[X] \in f[X]$ *and* $\inf f[X] \in f[X]$;

(ii) X *is* G-*pseudocompact*;

(iii) X *is totally bounded* (= *precompact*) *in every uniformity* $\mathcal{U}$ *which has the property that the action* π *is* $\mathcal{U}$-*bounded;*

(iv) $UC\langle X,\pi\rangle = UC^*\langle X,\pi\rangle$, *that is, every* π-*uniformly continuous function on* X *is bounded.*

Then (i) $\Rightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Rightarrow$ (iv) *and* (iv) $\not\Rightarrow$ (iii).

PROOF. (i) $\Rightarrow$ (ii): Suppose X is not G-pseudocompact. Then we can apply Proposition 2.4 with $a_n = 1 - 1/n$ in order to obtain $f \in UC^*\langle X,\pi\rangle$ which has no maximum on X.

(ii) $\Rightarrow$ (iii): Suppose $\mathcal{U}$ is a uniformity for X such that the action π is $\mathcal{U}$-bounded, but X is not totally bounded w.r.t. $\mathcal{U}$. So there exists $\alpha \in U$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that, for all $n \in \mathbb{N}$, $x_{n+1} \notin \bigcup_{i=1}^n \alpha[x_i]$. Let $\beta \in U$, $\beta^4 \subseteq \alpha$ and $\beta^{-1} = \beta$, and let $U \in \mathcal{V}_e$ be such, that $(x,tx) \in \beta$ for all $(t,x) \in U \times X$, i.e. $Ux \subseteq \beta[x]$ for all $x \in X$. Then $\{\beta[x_n]\}_{n\in\mathbb{N}}$ is a locally finite G-dispersion, and therefore, X is not G-pseudocompact.

(iii) $\Rightarrow$ (ii): Suppose X is not G-pseudocompact, and let $\{B_n\}_{n\in\mathbb{N}}$ be a locally finite G-dispersion. Let $U \in \mathcal{V}_e$ be such that for every $n \in \mathbb{N}$ there exists $x_n \in B_n$ with $Ux_n \subseteq B_n$. Let $V \in \mathcal{V}_e$ and $W \in \mathcal{V}_e$ be such, that $V^2 \subseteq U$, $W^2 \subseteq V$, $W^{-1} = W$, and W compact, put $D := X\backslash\bigcup_{n=1}^\infty Wx_n$ and $\alpha := \bigcup_{n=1}^\infty (B_n \times B_n) \cup (D \times D)$. Local finiteness of $\{Wx_n\}_{n\in\mathbb{N}}$ implies that D is open in X. Hence, if $\mathcal{U}$ is a uniformity for X, then the uniformity $\mathcal{U}'$, generated by $\mathcal{U} \cup \{\alpha\}$ is also a uniformity for X. Also, if π is $\mathcal{U}$-bounded, then π is also $\mathcal{U}'$-bounded (indeed, if $x \in Vx_n$, then $Wx \subseteq V^2x_n \subseteq Ux_n \subseteq B_n$, hence $Wx \subseteq \alpha[x]$;

if $x \notin \bigcup_{n=1}^{\infty} Vx_n$, then $Wx \cap Wx_n = \emptyset$ for all n, i.e. $Wx \subseteq D$, hence $Wx \subseteq \alpha[x]$). Since $B_n = \alpha[x_n]$, X is not totally bounded w.r.t. $\mathcal{U}'$. Thus, starting with a uniformity $\mathcal{U}$ for X such that π is $\mathcal{U}$-bounded, we end up with a uniformity $\mathcal{U}'$ for X such that π is $\mathcal{U}'$-bounded, but X is not totally bounded w.r.t. $\mathcal{U}'$.
(iii) ⇒ (iv): If $\mathcal{U}$ is the weakest uniformity in X making every member of $UC\langle X,\pi\rangle$ uniformly continuous, then $\mathcal{U}$ generates the topology of X ($UC\langle X,\pi\rangle$ separates points and closed subsets of X because $UC^*\langle X,\pi\rangle$ does: cf. 1.4). Moreover, it is easily checked, that π is $\mathcal{U}$-bounded. Since every uniformly continuous function on a precompact uniform space is bounded, the result follows.
(iv) ⇏ (iii): Consider the following example. Let X be the orbit of a given point in the irrational flow on the torus. Then X is dense in the torus, but not pseudocompact. We show, that X is not $\mathbb{R}$-pseudocompact ($\mathbb{R}$ is the acting group!). In the following way one can construct an infinite, locally finite $\mathbb{R}$-dispersion in X. Representing the torus by $(\mathbb{R}/\mathbb{Z})^2$, construct a disjoint sequence of rectangular open sets in the torus, each with one side of a given length (say, 1/10) parallel to the direction of the chosen orbit X in the torus, and converging to a segment in the torus which does *not* belong to X. Since X is dense in the torus, the trace of this sequence in X is an infinite sequence of non-empty open sets in X which is clearly a locally finite $\mathbb{R}$-dispersion in X. So $\langle X,\pi\rangle$ is not $\mathbb{R}$-pseudocompact.

However, let $f \in UC\langle X,\pi\rangle$. We show, that f is bounded. Let $x_0 \in X$. Since $\langle X,\pi\rangle$ is almost periodic, there exists a relatively dense subset P in $\mathbb{R}$ such that

(1) $$|f(x_0+t) - f(x_0)| < 1$$

for all $t \in P$. (Here we view X as the set $\mathbb{R}$ with a topology, which differs from the usual one, the action of $\mathbb{R}$ on X being given by $\pi(t,x) := x + t$ for $x \in X$, $t \in \mathbb{R}$). That P is relatively dense in $\mathbb{R}$ means, that there exists a number $\ell > 0$ such that $\mathbb{R} = P + [0,\ell]$. Since $f \in UC\langle X,\pi\rangle$, there is $\delta > 0$ such that

(2) $$|f(x+s) - f(x)| < 1 \quad \text{for all } x \in X, \quad s \in \mathbb{R}, \quad |s| < \delta.$$

For every $u \in [0,\ell]$ there is a sequence $0 = u_0 < u_1 < \dots < u_k = u$, where $k \le [\frac{2\ell}{\delta}] + 1 =: k_0$, and $|u_{i+1}-u_i| < \delta$ for $i = 0,1,\dots,k-1$. Consequently, (2) implies that

$$(3) \qquad |f(x+u) - f(x)| \le \sum_{i=0}^{k-1} |f(x+u_{i+1}) - f(x+u_i)| < k \le k_0$$

for every $x \in X$ and $u \in [0,\ell]$. However, for every $s \in \mathbb{R}$ there are $t \in P$ and $u \in [0,\ell]$ with $s = t + u$, hence by (1) and (3):

$$|f(x_0+s) - f(x_0)| \le |f(x_0+t+u) - f(x_0+t)| + |f(x_0+t) - f(x_0)| \le$$

$$\le k_0 + 1.$$

This implies, that f is bounded on $X = \{x_0+s: s \in \mathbb{R}\}$. □

2.6. PROPOSITION. *If* $\varphi: \langle X,\pi\rangle \to \langle Y,\sigma\rangle$ *is a morphism of* G-*spaces and* X *is* G-*pseudocompact, then so is* Y.

PROOF. Obvious. □

2.7. PROPOSITION. *If* $\langle X,\pi\rangle$ *and* $\langle Y,\sigma\rangle$ *are* G-*spaces*, X *is* G-*pseudocompact and* Y *is compact, then* $\langle X \times Y,\tau\rangle$ *is* G-*pseudocompact* (τ as in 1.3).

PROOF. Using 2.5 (i) ⇒ (ii) and the lemma below, the proof can easily be given along the lines of [3], 3.4. □

2.8. LEMMA. *Let* $\langle X,\pi\rangle$ *be an arbitrary* G-*space and let* $\langle Y,\sigma\rangle$ *be a compact* G-*space. Define for* $f \in UC^*\langle X \times Y,\tau\rangle$

$$F(x) := \inf_{y \in Y} f(x,y), \qquad x \in X.$$

Then $F \in UC^*\langle X,\pi\rangle$.

PROOF. It is standard to show, that $F \in C^*(X)$ (cf. for instance Lemma 1.1 in [3]), and it is straightforward to verify, that $F \in UC^*\langle X,\pi\rangle$. □

3. PROOF OF NECESSITY IN THE MAIN THEOREM

In this section we suppose G to be a *locally connected* locally compact Hausdorff topological group. In addition, $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, and $\langle X \times Y,\tau\rangle$ is their product. We shall prove in this section:

3.1. THEOREM. *If* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ *then either one of the* G*-spaces* X *or* Y *is not* G*-infinite, or* $X \times Y$ *is* G*-pseudocompact.*

The proof is basically the same as the proof of necessity in Glicksberg's theorem as given by FROLÍK in [3], additional complications being caused by the fact that we need sequences of open sets which are *internally linked,* whereas in [3] the open sets are only required to be non-empty. We start with the following lemma.

3.2. LEMMA. *Suppose* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$. *If* $f \in UC^*\langle X \times Y,\tau\rangle$ *then for every* $\varepsilon > 0$ *there exists* $V \in \mathcal{V}_e$ *such that*

$$|f(tx,sy) - f(x,y)| < \varepsilon \text{ for all } (x,y) \in X \times Y \text{ and } (t,s) \in V \times V.$$

REMARK. The definition of τ-uniform continuity includes only the above inequality with $s = t$.

PROOF. According to 1.4 the assumption implies that f has a continuous extension $\bar{f}$ to $\beta_G X \times \beta_G Y$. Then each point $(x,y) \in \beta_G X \times \beta_G Y$ has a neighbourhood $W_1 \times W_2$ such that $|\bar{f}(x',y') - \bar{f}(x,y)| < \varepsilon/2$ for $(x',y') \in W_1 \times W_2$. Moreover, there are $V \in \mathcal{V}_e$ and neighbourhoods W_1' of x and W_2' of y such that $VW_1' \subseteq W_1$ and $VW_2' \subseteq W_2$. In particular,

$$|\bar{f}(tx',sy') - \bar{f}(x',y')| \leq |\bar{f}(tx',sy') - \bar{f}(x,y)| + |\bar{f}(x',y') - \bar{f}(x,y)| < 2\varepsilon$$

for $(x',y') \in W_1' \times W_2'$ and $(t,s) \in V \times V$. Now a compactness argument completes the proof. □

3.3. LEMMA. *Suppose* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$, *and let* $\{W_n\}_{n\in\mathbb{N}}$ *be a* G*-dispersion in* $X \times Y$ *which is locally finite. Then there exists* $U \in \mathcal{V}_e$, U *compact, and* *for every* $n \in \mathbb{N}$ *there exist a point* $(a_n,b_n) \in W_n$ *and open sets* A_n *in* X, B_n *in* Y *such that*

$$U(a_n,b_n) \subseteq Ua_n \times Ub_n \subseteq A_n \times B_n \subseteq W_n.$$

PROOF. It is sufficient to find compact $U \in \mathcal{V}_e$ and points $(a_n,b_n) \in W_n (n \in \mathbb{N})$ such that $Ua_n \times Ub_n \subseteq W_n$: compactness then guarantees the existence of open sets A_n and B_n such that $Ua_n \times Ub_n \subseteq A_n \times B_n \subseteq W_n$.

According to Proposition 2.4 there exists $f \in UC^*\langle X \times Y,\tau\rangle$ such that $f(z) = 0$ for all $z \in X \times Y \setminus \bigcup_{n=1}^{\infty} W_n$ and such, that for every $n \in \mathbb{N}$ there is a point $(a_n,b_n) \in W_n$ with $f(a_n,b_n) = 1$. In view of Lemma 3.2 there is $U \in \mathcal{V}_e$, U compact and connected, such that $f(ta_n,sb_n) > 1/2$ for all $n \in \mathbb{N}$ and $(t,s) \in U \times U$. This implies, that for every $n \in \mathbb{N}$,

$$Ua_n \times Ub_n \subseteq \bigcup_{k=1}^{\infty} W_k.$$

However, the sets W_k are mutually disjoint and open, $Ua_n \times Ub_n \cap W_n \neq 0$, and U, hence $Ua_n \times Ub_n$, is connected. Therefore, $Ua_n \times Ub_n \subseteq W_n$ for every $n \in \mathbb{N}$. □

3.4. LEMMA (cf. [3];1.2). *Suppose that* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$, *that* $X \times Y$ *is not* G-*pseudocompact, and that, in addition, the spaces* X *and* Y *are both* G-*infinite. Then there exists a locally finite* G-*dispersion* $\{P_n \times Q_n\}_{n\in\mathbb{N}}$ *in* $X \times Y$ *such that the sequences* $\{P_n\}_{n\in\mathbb{N}}$ *and* $\{Q_n\}_{n\in\mathbb{N}}$ *are disjoint (hence* G-*dispersions in* X *and* Y, *respectively).*

PROOF. We consider two cases. First, assume that one of the G-spaces, say $\langle X,\pi\rangle$, is not G-pseudocompact. Then in X there exists a locally finite G-dispersion $\{P_n\}_{n\in\mathbb{N}}$. By assumption, Y is G-infinite, so in Y there exists a G-dispersion $\{Q_n\}_{n\in\mathbb{N}}$. Then $\{P_n \times Q_n\}_{n\in\mathbb{N}}$ is easily seen to be a G-dispersion in $X \times Y$ which is locally finite. Next, suppose that both X and Y are G-pseudocompact. Since $X \times Y$ is not G-pseudocompact, there exists a locally finite G-dispersion $\{W_n\}_{n\in\mathbb{N}}$ in $X \times Y$. Choose $U \in \mathcal{V}_e$, $(a_n,b_n) \in W_n$ and $A_n \subseteq X$, $B_n \subseteq Y$ according to Lemma 3.3. In particular, we have for every $n \in \mathbb{N}$

(1) $U(a_n,b_n) \subseteq A_n \times B_n \subseteq W_n$.

The sequence $\{A_n \times B_n\}_{n\in\mathbb{N}}$ is locally finite as well, hence every compact subset K of $X \times Y$ has an open neighbourhood 0 such that

(2) $0 \cap (A_n \times B_n) = \emptyset$ for almost all $n \in \mathbb{N}$.

Now we claim the following: for every sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$ and for every $x \in X$ there exists a neighbourhood W of Ux in X such that $W \cap A_{n_i} = \emptyset$ for infinitely many values of $i \in \mathbb{N}$. For assume the contrary. Then there are a sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$ and a point $x \in X$ such that every neighbourhood of Ux meets A_{n_i} for almost all $i \in \mathbb{N}$. By formula (1), the sequence $\{B_{n_i}\}_{i\in\mathbb{N}}$ is internally linked. Hence by 2.2(4^o), as Y is G-pseudocompact, there exists $y \in Y$ such that every neighbourhood of y meets infinitely many of the sets B_{n_i}. Consequently, every neighbourhood of the compact set $Ux \times \{y\}$ in $X \times Y$ meets infinitely many of the sets $A_{n_i} \times B_{n_i}$, contradicting formula (2). This proves our claim.

By induction one can show now, using our claim, that there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$ and mutually disjoint open sets P_i such that

$$Ua_{n_i} \subseteq P_i \subseteq A_{n_i} \qquad (i \in \mathbb{N})$$

A similar reasoning shows the existence of a subsequence $\{k_j\}_{j\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that there are mutually disjoint open sets Q_j with

$$Ub_{k_j} \subseteq Q_j \subseteq B_{k_j} \qquad (j \in \mathbb{N}).$$

Now it is clear, that the sequence $\{P_{k_j} \times Q_j\}_{j\in\mathbb{N}}$ meets the requirements of our lemma. □

3.5. PROOF OF THEOREM 3.1. This proof can now be given completely similar to the proof of the implication (3) ⇒ (1) in Theorem 2.1 of [3]. For completeness, we repeat it here, adapted to the present situation. Suppose that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ and that $X \times Y$ is not pseudocompact. Then one of the spaces X or Y is not G-infinite. For if they are both G-infinite, then there exists a locally finite G-dispersion $\{P_n \times Q_n\}_{n\in\mathbb{N}}$ according to Lemma 3.4. By Proposition 2.4 there exists $f \in UC^*\langle X \times Y,\tau\rangle$ such that $f(x,y) = 0$ for $(x,y) \in X \times Y\setminus\bigcup_{n=1}^{\infty} P_n \times Q_n$, and for every $n \in \mathbb{N}$ there is $(p_n,q_n) \in P_n \times Q_n$ with $f(p_n,q_n) = 1$. Then f has a continuous extension $\bar{f}$ to $\beta_G X \times \beta_G Y$, and for $\varepsilon = 1/2$ there is a finite covering of $\beta_G X \times \beta_G Y$ with open rectangles, on each of which $\bar{f}$ varies less than ε. Hence there is such an open rectangle, say $A \times B$, which contains infinitely many of the points (p_n,q_n). However, if $(p_n,q_n) \in A \times B$ and $(p_k,q_k) \in A \times B$ with $n \neq k$, then also $(p_n,q_k) \in A \times B$, hence

$$f(p_n,q_k) > f(p_n,q_n) - \varepsilon = \frac{1}{2}.$$

However, since the sets $\{P_i\}_{i\in\mathbb{N}}$ are mutually disjoint, as are the sets $\{Q_i\}_{i\in\mathbb{N}}$, we have $(p_n,q_k) \notin \bigcup_{i=1}^{\infty} P_i \times Q_i$, which implies that $f(p_n,q_k) = 0$. This contradiction concludes the proof. □

3.6. The following examples show that some additional condition (e.g. that X and Y are both G-infinite) is needed in order to be sure that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ implies that $X \times Y$ is G-pseudocompact.

1^o. If G is *discrete*, then $\beta_G Z = \beta Z$ for all Tychonov G-spaces Z. If X is not G-infinite, then X is finite, and then for *every* Tychonov G-space Y we have

$$\beta_G(X \times Y) = \beta(X \times Y) = \beta X \times \beta Y = \beta_G X \times \beta_G Y.$$

In particular, if Y is not pseudocompact, then $X \times Y$ is not pseudocompact, hence not G-pseudocompact.

2^o. Let G be *compact*, Y an arbitrary Tychonov space which is *not* pseudocompact, and consider the G-spaces $\langle G,\mu\rangle$ and $\langle Y,\sigma\rangle$, where $\mu^t s := ts$ and $\sigma^t y := y$ for $t \in G$, $s \in G$ and $y \in Y$. Then it can be shown, that $\beta_G(G \times Y) = G \times \beta Y$ (cf. [10], 4.4.13 (iv)), and consequently, that $\beta_G(G \times Y) = \beta_G G \times \beta_G Y$. However, $G \times Y$ is not pseudocompact and since the action of G on Y is trivial, it follows that $G \times Y$ is not G-pseudocompact. This is in accordance with the fact, that $\langle G,\mu\rangle$ is in this case not G-infinite (cf. 2.2(3^o) with $X = G$).

More about this additional condition can be found in Section 5 below.

4. PROOF OF SUFFICIENCY IN THE MAIN THEOREM

In this section G is a locally compact Hausdorff topological group, *not* necessarily locally connected. Again, $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces and $\langle X \times Y,\tau\rangle$ is their product. In this section, we shall prove:

4.1. THEOREM. *If* $X \times Y$ *is* G*-pseudocompact, then* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$.

Again, the proof was inspired by [3] and [9]. However, a serious obstruction to a straightforward application of the methods used there was caused by the fact that in general for $f \in UC^*\langle X \times Y,\tau\rangle$ it is not true that for every $y \in Y$ the function $x \mapsto f(x,y)$ belongs to $UC^*\langle X,\pi\rangle$ (for an example, cf. 5.2 below); compare this with Lemma 3.2 above. We avoid this difficulty, or rather, we prove it (in an implicit way) for the case that $X \times Y$ is G-pseudocompact, by means of the trick, introduced in 4.3 below.

First, we need a modification of Lemma 1.3 of [3]; cf. also Lemma in [5]. Due to a possibly weaker hypothesis (cf. Section 5 below) we have to consider τ-uniformly continuous functions instead of functions which are just continuous. The proof is basically the same as in [3], but we have to be careful in connection with internal connectedness of sequences of open sets.

4.2. LEMMA. *Let* $X \times Y$ *be* G*-pseudocompact and let* $f \in UC^*\langle X \times Y,\tau\rangle$. *Then the family of all functions* $x \mapsto f(x,y): X \to \mathbb{R}$ *with* $y \in Y$ *is equicontinuous on* X, *that is,*

$$\forall x_0 \in X \quad \forall \varepsilon > 0 \quad \exists W \in \mathcal{V}_{x_0} : |f(x,y) - f(x_0,y)| < \varepsilon \text{ for all } (x,y) \in W \times Y.$$

PROOF. Suppose the contrary. Then there exists $x_0 \in X$ such that for some $\varepsilon > 0$ we have

$$\forall\, W \in \mathcal{V}_{x_0} \quad \exists (x,y) \in W \times Y : |f(x,y) - f(x_0,y)| > 5\varepsilon.$$

Now by induction it follows that there exist points $(x_n,y_n) \in X \times Y$ and open neighbourhoods $W_n \times V_n$ of (x_n,y_n), $W_n' \times V_n$ of (x_0,y_n) in $X \times Y$ such that

(1)
$$|f(x',y') - f(x_n,y_n)| < \tfrac{1}{2}\varepsilon \text{ for } (x',y') \in W_n \times V_n;$$
$$|f(x'',y'') - f(x_0,y_n)| < \tfrac{1}{2}\varepsilon \text{ for } (x'',y'') \in W_n' \times V_n;$$

(2) $$W_n \subseteq W'_{n-1} \text{ and } W_n' \subseteq W'_{n-1};$$

(3) $$|f(x_n,y_n) - f(x_0,y_n)| > 5\varepsilon$$

(compare with the proof of Lemma 1.3 in [3]). Since $f \in UC^*\langle X \times Y,\tau\rangle$ there exists $U_0 \in \mathcal{V}_e$ such that U_0 is compact, $U_0^{-1} = U_0$ and

$$|f(tx,ty) - f(x,y)| < \tfrac{1}{2}\varepsilon \text{ for all } t \in U_0,\ (x,y) \in X \times Y.$$

This implies, together with (1), that for every $n \in \mathbb{N}$:

(1)*
$$|f(x',y') - f(x_n,y_n)| < \varepsilon \text{ for } (x',y') \in U_0(W_n \times V_n)$$
$$|f(x'',y'') - f(x_0,y_n)| < \varepsilon \text{ for } (x'',y'') \in U_0(W_n' \times V_n).$$

The sequence $\{U_0(W_n \times V_n)\}_{n\in\mathbb{N}}$ is clearly internally linked and consists of non-empty open sets, so in view of 2.2(4^o) it is not locally finite. Hence there exists a point $(\bar{x},\bar{y})$ in $X \times Y$ such that

(4) $$\forall O \in \mathcal{V}_{(\bar{x},\bar{y})}: O \cap U_0(W_n \times V_n) \neq \emptyset \text{ for infinitely many values of } n \in \mathbb{N}.$$

As the mapping $(s,t,x,y) \mapsto f(sx,ty): U_0^2 \times U_0^2 \times X \times Y \to \mathbb{R}$ is continuous, and U_0 is compact, there exists an open neighbourhood of $(\bar{x},\bar{y})$ of the form $A \times B$, A open in X and B open in Y, such that $|f(sx,ty) - f(x,y)| < \varepsilon$ for all $s,t \in U_0^2$ and all $(x,y) \in A \times B$. That is,

(5) $$|f(x,y) - f(\bar{x},\bar{y})| < \varepsilon \text{ for } (x,y) \in U_0^2A \times U_0^2B.$$

Let i and j be two of the values of n in $\mathbb{N}$, $j > i$, for which (4.1) is valid with $O = A \times B$. Then

$$\exists t \in U_0, \quad x \in W_i,\ y \in V_i : (tx,ty) \in A \times B,$$
$$\exists t' \in U_0, \quad x' \in W_j,\ y' \in W_j : (t'x',t'y') \in A \times B.$$

However, $W_j \subseteq W_{j-1}' \subseteq W_i$, because $j-1 \geq i$. It follows, that $x' \in W_i'$, so that $(x',y) \in W_i' \times V_i$. Moreover, we have $t'y = t't^{-1}(ty) \in U_0U_0^{-1}B = U_0^2B$. Since obviously $t'x' \in A \subseteq U_0^2A$, this implies that

$$t'(x',y) \in U_0(W_i' \times V_i) \cap (U_0^2A \times U_0^2B).$$

We infer from this, that the neighbourhood $O_1 := U_0^2A \times U_0^2B$ of $(\bar{x},\bar{y})$ has the property, that

(6) $$O_1 \cap U_0(W_i' \times V_i) \neq \emptyset.$$

Observe, that (6) holds for those values i of n in $\mathbb{N}$ for which (4) holds with $O = A \times B$. Suppose i is such a value. Then for some point $(x',y') \in$ $\in (A \times B) \cap U_0(W_i \times V_i) \subseteq O_1 \cap U_0(W_i \times V_i)$ we have by (5), $(1)^*$ and (3):

$$|f(x_0,y_i) - f(\bar{x},\bar{y})| \geq |f(x_0,y_i) - f(x_i,y_i)| - |f(x_i,y_i) - f(x',y')|$$
$$- |f(x',y') - f(\bar{x},\bar{y})| > 3\varepsilon.$$

On the other hand, we have by (6) and $(1)^*$ for some point $(x'',y'') \in$ $O_1 \cap U_0(W_i' \times V_i)$:

$$|f(\bar{x},\bar{y}) - f(x_0,y_i)| \leq |f(\bar{x},\bar{y}) - f(x'',y'')| + |f(x'',y'') - f(x_0,y_i)| < 2\varepsilon.$$

This contradiction proves our lemma. $\square$

4.3. In order to prove, that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ it is, by 1.5, sufficient (and necessary) to prove that every $g \in E(X \times Y, C_c^*(G))$ can be extended to a continuous equivariant mapping $\tilde{g} : \beta_G X \times \beta_G Y \to C_c^*(G)$. The idea is, first to extend the mapping $x \mapsto g(x,-)(-): X \to C_c^*(Y \times G)$ to a mapping $\bar{\bar{g}} : \beta_G X \to C_c^*(Y \times G)$, and then to extend in a similar way the mapping $y \mapsto \bar{\bar{g}}(-)(y,-) : Y \to C_c^*(\beta_G X \times G)$ to $\beta_G Y$. In order to do so, we have to define a continuous action of G on $C_c^*(Y \times G)$.

4.4. Define $\xi : G \times C_c^*(Y \times G) \to C_c^*(Y \times G)$ by the rule

$$\xi(t,f)(y,s) := f(t^{-1}y,st)$$

for $(t,f) \in G \times C_c^*(Y \times G)$ and $(y,s) \in Y \times G$. It is easily seen, that $\xi^e f = f$ and that $\xi^s \xi^t f = \xi^{st} f$ for all $s,t \in G$ and $f \in C_c^*(Y \times G)$. In addition, using the inequality

$$|\xi^t f(y,s) - \xi^{t_0} f_0(y,s)| = |f(t^{-1}y,st) - f_0(t_0^{-1}y,st_0)| \leq$$
$$\leq |f(t^{-1}y,st) - f_0(t^{-1}y,st)| +$$
$$+ |f_0(t^{-1}y,st) - f_0(t_0^{-1}y,st_0)|$$

and a straightforward compactness argument, one may show that ξ is continuous (in fact, the proof is very similar to the proof of the continuity of the action ρ of G on $C_c^*(G)$; cf. [10], 2.1.3). Consequently, $\langle C_c^*(Y \times G),\xi\rangle$ is a G-space.

4.5. <u>PROOF OF THEOREM 4.1</u>. In the following lemma's let $g : X \times Y \to C_c^*(G)$ be a continuous, equivariant mapping such that $g[X \times Y]$ is relatively compact in $C_c^*(G)$, or, what amounts to the same because G is locally compact, such that $g[X \times Y]$ is an equicontinuous set of functions on G. For $x \in X$ and $(y,t) \in Y \times G$ we set

$$\bar{g}(x)(y,t) := g(x,y)(t).$$

4.6. <u>LEMMA</u>. *For every* $x \in X$, $\bar{g}(x)$ *is a continuous, bounded real valued* *function on* Y × G, *and* $\bar{g} : X \to C_c^*(Y \times G)$ *is continuous and equivariant w.r.t. the action* ξ *of* G *on* $C_c^*(Y \times G)$.

PROOF. Of course, boundedness of $\bar{g}(x)$ on Y × G is trivial. In addition, once one has shown that $\bar{g}(x) \in C_c^*(Y \times G)$, a straightforward calculation shows, that $\bar{g} : X \to C_c^*(Y \times G)$ is equivariant. So it remains to prove the continuity statements. At first glance one might be tempted to apply [2], Theorem 5.3: our lemma would be an immediate consequence of the homeomorphism of $C_c(X \times Y, C_c(G,\mathbb{R}))$ with $C_c(X \times Y \times G,\mathbb{R})$ and of $C_c(X \times Y \times G,\mathbb{R})$ with $C_c(X,C_c(Y \times G,\mathbb{R}))$. However, the latter homeomorphism requires either that Y × G is locally compact or that X × Y × G is a k-space, and therefore we can not apply this theorem. We shall indicate a direct proof, using equicontinuity of g[X × Y].

Consider $x_0 \in X$, $y_0 \in Y$ and $t_0 \in G$. Then for all $x \in X$ and $(y,t) \in Y \times G$ we have

$$\begin{aligned}(7)\qquad |\bar{g}(x)(y,t) - \bar{g}(x_0)(y_0,t_0)| &= |g(x,y)(t) - g(x_0,y_0)(t_0)| \leq \\ &\leq |g(x,y)(t) - g(x,y)(t_0)| + |g(x,y)(t_0) - g(x_0,y_0)(t_0)|.\end{aligned}$$

Let $\varepsilon > 0$. By equicontinuity of g[X × Y], there exists a neighbourhood W of t_0 in G such that

$$(8)\qquad |g(x,y)(t) - g(x,y)(t_0)| < \tfrac{1}{2}\varepsilon$$

for all $(x,y) \in X \times Y$ and all $t \in W$. Moreover, continuity of g implies that there are neighbourhoods U of x_0 and V of y_0 such that

$$|g(x,y)(t_0) - g(x_0,y_0)(t_0)| < \tfrac{1}{2}\varepsilon$$

for all $(x,y) \in U \times V$. Hence

$$(9)\qquad |\bar{g}(x)(y,t) - \bar{g}(x_0)(y_0,t_0)| < \varepsilon$$

for all $x \in U$ and all $(y,t) \in V \times W$. In particular, putting $x = x_0$ in (9) yields continuity of $\bar{g}(x_0)$ on $Y \times G$ for arbitrary $x_0 \in G$. Now in order to prove that $\bar{g} : X \to C_c^*(Y \times G)$ is continuous, use (9) and a standard compactness argument to show, that for given compact sets K_1 in Y and K_2 in G one has

$$|\bar{g}(x)(y,t) - \bar{g}(x_0)(y,t)| < 2\varepsilon$$

for all $(y,t) \in K_1 \times K_2$ and for all x in a suitable neighbourhood of x_0. Hence $\bar{g}$ is continuous. □

4.7. LEMMA. *The set* $\bar{g}[X]$ *is pointwise bounded and equicontinuous on* $Y \times G$, *hence it has compact closure in* $C_c^*(Y \times G)$.

PROOF. Putting $x_0 = x$ in formula (7) above, we obtain

$$|\bar{g}(x)(y,t) - \bar{g}(x)(y_0,t_0)| \leq |g(x,y)(t) - g(x,y)(t_0)| + |g(x,y)(t_0) - g(x,y_0)(t_0)|.$$

Taking into account equicontinuity of $g[X \times Y]$ as expressed by formula (8), it is sufficient to prove that there exists a neighbourhood V of y_0 such that

$$(10) \qquad |g(x,y)(t_0) - g(x,y_0)(t_0)| < \tfrac{1}{2}\varepsilon$$

for all $x \in X$ and all $y \in V$. To this end, consider the continuous mapping

$$F : (x,y) \mapsto g(x,y)(t_0) : X \times Y \to \mathbb{R}.$$

Then for all $(x,y) \in X \times Y$ and $t \in G$ we have, in view of equivariance of g:

$$|F(tx,ty) - F(x,y)| = |g(tx,ty)(t_0) - g(x,y)(t_0)|$$
$$= |g(x,y)(t_0t) - g(x,y)(t_0)|.$$

Thus, equicontinuity of $g[X \times Y]$ implies, that for every $\delta > 0$ we have $|F(tx,ty) - F(x,y)| < \delta$ for all $(x,y) \in X \times Y$ and all t in a suitable neighbourhood of e in G. Stated otherwise, $F \in UC^*\langle X \times Y,\tau\rangle$, and we may apply Lemma 4.2 to F. Hence there exists a neighbourhood V of y_0 such that

$$|F(x,y) - F(x,y_0)| < \tfrac{1}{2}\varepsilon$$

for all $x \in X$, $y \in V$. But this is exactly, what we need in (10). Hence $\bar{g}[X]$ is equicontinuous. As $\bar{g}[X]$ is also pointwise bounded (this follows from the fact that $g[X \times Y]$ is pointwise bounded on G), Ascoli's theorem implies that $\bar{g}[X]$ is relatively compact in $C_c^*(Y \times G)$. □

4.8. PROOF OF THEOREM 4.1. (continued). Note, that $\bar{g}[X]$ is an invariant subset of $C_c^*(Y \times G)$, because $\bar{g}:X \to C_c^*(Y \times G)$ is equivariant. Hence the closure Z of $\bar{g}[X]$ is invariant as well. Thus, Z is a compact (by 4.7) G-space, and $\bar{g} : X \to Z$ is a continuous morphism of G-spaces. This implies, that there exists a morphism of G-spaces $\bar{\bar{g}} : \beta_G X \to Z \subseteq C_c^*(Y \times G)$ which extends $\bar{g}$. Putting

$$\hat{g}(x,y)(t) := \bar{\bar{g}}(x)(y,t)$$

for $(x,y) \in \beta_G X \times Y$ and $t \in G$, it is clear that we obtain for every $(x,y) \in \beta_G X \times Y$ an element $\hat{g}(x,y)$ of $C^*(G)$. Thus, we have a function $\hat{g} : \beta_G X \times Y \to C^*(G)$ which obviously extends the original function $g : X \times Y \to C^*(G)$.

4.9. LEMMA. *The mapping* $\hat{g} : \beta_G X \times Y \to C_c^*(G)$ *is continuous, equivariant, and* $\hat{g}[\beta_G X \times Y]$ *has a compact closure in* $C_c^*(G)$.

PROOF. Consider $(x_0,y_0) \in \beta_G X \times Y$, $\varepsilon > 0$ and a compact subset K of G. We have to prove, that there exist neighbourhoods U of x_0 and V of y_0 such that

$$|\hat{g}(x,y)(t) - \hat{g}(x_0,y_0)(t)| < \varepsilon$$

for all $(x,y) \in U \times V$ and $t \in K$. First, observe that by the triangle inequality we have for all $(x,y) \in \beta_G X \times Y$ and $t \in G$:

$$|\hat{g}(x,y)(t) - \hat{g}(x_0,y_0)(t)| \le |\bar{\bar{g}}(x)(y,t) - \bar{\bar{g}}(x)(y_0,t)| + \tag{11}$$
$$|\bar{\bar{g}}(x)(y_0,t) - \bar{\bar{g}}(x_0)(y_0,t)|.$$

Consider the first term of the right-hand side of (11). Observe, that $\bar{\bar{g}}[\beta_G X]$ is equal to the closure of $\bar{\bar{g}}[X]$ in $C_c^*(Y \times G)$, and as $\bar{\bar{g}}[X]$ is equicontinuous, $\bar{\bar{g}}[\beta_G X]$ is equicontinuous on $Y \times G$ (cf. 4.7) (note that equicontinuity of $\bar{\bar{g}}[\beta_G X]$ does not follow from its compactness as $Y \times G$ is not locally compact). Hence for every $t' \in K$ there exists a neighbourhood U' of t' in G and a neighbourhood V' of y_0 in Y such that

$$|\bar{\bar{g}}(x)(y,t) - \bar{\bar{g}}(x)(y_0,t')| < \frac{\varepsilon}{4}$$

for all $x \in \beta_G X$, $y \in V'$ and $t \in U'$. Using compactness of K this implies that there exists $V \in \mathcal{V}y_0$ such that

$$|\bar{\bar{g}}(x)(y,t) - \bar{\bar{g}}(x)(y_0,t)| < \frac{\varepsilon}{2}$$

for all $x \in \beta_G X$ and $y \in V$. As to the second term of the right-hand side of (11), due to continuity of $\bar{\bar{g}} : \beta_G X \to C_c(Y \times G)$ there exists a neighbourhood U of x_0 in $\beta_G X$ such that this term is at most $\frac{1}{2}\varepsilon$ for all $x \in U$ and $t \in K$ (notice, that $\{y_0\} \times K$ is a compact subset of $Y \times G$). This concludes the proof that $\hat{g} : \beta_G X \times Y \to C_c^*(G)$ is continuous.

Now continuity of $\hat{g}$ implies, that $\hat{g}[\beta_G X \times Y]$ is included in the closure of $\hat{g}[X \times Y] = g[X \times Y]$ in $C_c^*(G)$, which is compact. Hence $\hat{g}[\beta_G X \times Y]$ has compact closure in $C_c^*(G)$. Finally, for all $t \in G$ and $(x,y) \in X \times Y$ we have

$$\hat{g}(t(x,y)) = g(t(x,y)) = \rho^t g(x,y) = \rho^t \hat{g}(x,y).$$

Stated otherwise, the continuous mappings $(x,y) \mapsto \hat{g}(t(x,y))$ and $(x,y) \mapsto \rho^t \hat{g}(x,y)$ from $\beta_G X \times Y$ into $C_c^*(G)$ are equal to each other on the dense subset $X \times Y$ of $\beta_G X \times Y$. Hence they are equal on all of $\beta_G X \times Y$. Thus, $\hat{g}$ is equivariant. □

4.10. PROOF OF THEOREM 4.1. (*continued*). We have shown in 4.5 through 4.9 that an arbitrary element g of $E(X \times Y, C_c^*(G))$ has a (unique, as $X \times Y$ is dense in $\beta_G X \times Y$) extension to an element $\hat{g}$ of $E(\beta_G X \times Y, C_c^*(G))$, provided $X \times Y$ is G-pseudocompact. However, in that case Y is G-pseudocompact by Proposition 2.6, hence $\beta_G X \times Y$ is G-pseudocompact by 2.7. Consequently, we may apply a similar procedure to $\hat{g}$, obtaining an equivariant continuous mapping $\hat{\hat{g}} : \beta_G X \times \beta_G Y \to C_c^*(G)$ which extends $\hat{g}$, hence also extends g. □

5. SOME OPEN PROBLEMS

There are two major open problems, the solution of which is required for a completely satisfying answer to the question of when $\beta_G(X \times Y)$ equals $\beta_G X \times \beta_G Y$.

5.1. The first problem concerns the additional condition which is needed in order to prove that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ implies G-pseudocompactness of $X \times Y$. In the classical case this condition (X and Y both infinite) is required because for X (or Y) *finite* one has always $\beta(X \times Y) = \beta X \times \beta Y$. In the case of a non-trivial, non-discrete group G the situation is different. Although some additional condition is required (cf. 3.6 above), the situation would be more satisfying when the condition of G-infiniteness which we employed would be sufficiently weak in order to prove the following result: *if one of the spaces* X *or* Y *is not* G-*infinite, then* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$. The following example shows that this statement is not generally true.

5.2. EXAMPLE. Let $G := \mathbb{R}$. We give an example of two $\mathbb{R}$-spaces $\langle X, \pi \rangle$ and $\langle Y, \sigma \rangle$ such that X is *not* $\mathbb{R}$-infinite, X is compact, and nevertheless $\beta_{\mathbb{R}}(X \times Y) \neq \beta_{\mathbb{R}} X \times \beta_{\mathbb{R}} Y$. Let $X =: S^1$, $Y := \mathbb{R}$ and consider the following actions of $\mathbb{R}$ on X and Y respectively

$$\pi(t,x) := x + t \pmod 1 \qquad \text{for } t \in \mathbb{R}, \qquad x \in [0,1),$$
$$\sigma(t,r) := r + t \qquad \text{for } t \in \mathbb{R}, \qquad r \in Y = \mathbb{R},$$

where S^1 is represented as $\mathbb{R}/\mathbb{Z}$ or, which amounts to the same, as the interval [0,1] with the endpoints identified. If $\beta_{\mathbb{R}}(X \times Y)$ were equal to $\beta_{\mathbb{R}}X \times \beta_{\mathbb{R}}Y$, then for *every* $f \in UC^*\langle X \times Y, \tau\rangle$ and every $\varepsilon > 0$ there would exist (cf. Lemma 3.2) $\delta > 0$ such that

$$(1) \qquad |f(t+x(\text{mod } 1), s+r) - f(x,r)| < \varepsilon$$

for all $x \in [0,1)$, $r \in \mathbb{R}$ and $s,t \in \mathbb{R}$ with $|s| < \delta$ and $|t| < \delta$. Consider $f : X \times Y \to \mathbb{R}$, defined by

$$f(x,r) := r \sin 2\pi(r-x), \quad x \in [0,1), \quad r \in \mathbb{R}.$$

Then for all $t \in \mathbb{R}$ and $(x,r) \in [0,1) \times \mathbb{R}$ we have

$$|f(t+x(\text{mod } 1), t+r) - f(x,r)| = |t \sin 2\pi(r-x)| \leq t.$$

From this, it is clear that $f \in UC^*\langle X \times Y, \tau\rangle$.

On the other hand, putting $x := 0$, $r := n \in \mathbb{N}$, $t := 1/n$ and $s := -1/n$ in (1) we obtain for all $n \in \mathbb{N}$:

$$|f(\tfrac{1}{n}, -\tfrac{1}{n} + n) - f(0,n)| = (-\tfrac{1}{n} + n) \sin 2\pi(n - \tfrac{2}{n}) =$$
$$= (n - \tfrac{1}{n}) \sin \frac{4\pi}{n} \xrightarrow{(n\to\infty)} 4\pi.$$

From this it follows, that (1) cannot hold for all suitably small s and t and all $r \in \mathbb{R}$ and $x \in [0,1)$.

5.3. PROBLEM. Is there a "non-triviality condition" (C) for G-spaces, expressible in topological properties of the space and the actions, such that the following is true for all G-spaces X and Y:

(i) If $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ and X and Y have (C), then $X \times Y$ is G-pseudocompact.

(ii) If one of the G-spaces X or Y does not have (C) then $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$.

5.4. Another way to fill the gap, indicated in 5.1 is, to replace the condition of G-pseudocompactness by a stronger property, and try to prove, that $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ implies this stronger property for $X \times Y$, under the additional hypothesis that X and Y are both infinite. A natural condidate for this "stronger property" would be ordinary pseudocompactness. In that case, Section 4 above could be replaced by the following sequence of statements:

5.5. LEMMA. *Assume that* G *is a topological group which is, as a topological space, merely a k-space, and let* $\langle X,\pi\rangle$ *be a* G*-space* (X *a Tychonov space*). *If* X *is pseudocompact, then* $\beta_G X = \beta X$, *the ordinary Stone-Čech compactification of* X.

PROOF. For every $t \in G$ the mapping $\pi^t : X \to X$ extends to a continuous mapping $\bar{\pi}^t : \beta X \to \beta X$. In this way we obtain a mapping $\bar{\pi} : G \times \beta X \to \beta X$ which is easily seen to have the properties of an action, except possibly continuity. We show that $\bar{\pi}$ is continuous if X is pseudocompact.

Let K be a compact subset of G and $\pi_K := \pi|_{K \times X}$. Then $\pi_K : K \times X \to X$ is continuous, hence it has a continuous extension $\tilde{\pi}_K : \beta(K \times X) \to \beta X$. However, $K \times X$ is pseudocompact, hence by Glicksberg's theorem, $\beta(K \times X) = \beta K \times \beta X = K \times \beta X$. Thus, π_K has a continuous extension $\tilde{\pi}_K : K \times \beta X \to \beta X$. Since for every $t \in K$ the continuous mappings $\tilde{\pi}_K^t$ and $\bar{\pi}^t$ are equal on X, they are equal on βX, that is, $\tilde{\pi}_K = \bar{\pi}|_{K \times \beta X}$. Consequently, $\bar{\pi}|_{K \times \beta X}$ is continuous for every compact subset K of G. It follows, that the restriction of $\bar{\pi}$ to an arbitrary compact subset $G \times \beta X$ is continuous. As $G \times \beta X$ is a k-space, this implies that $\bar{\pi}$ is continuous.

This shows that $\langle \beta X, \bar{\pi}\rangle$ is a G-space. Now it is easily seen, that this is the maximal G-compactification of X. This proves our lemma. □

5.6. REMARK. The result of Lemma 5.5 is stated without proof for locally compact groups G in [8].

5.7. COROLLARY. *Let* G *be as in 5.5 and let* $\langle X,\pi\rangle$ *and* $\langle Y,\sigma\rangle$ *be Tychonov* G*-spaces such that* $X \times Y$ *is pseudocompact. Then* $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$.

PROOF. For $Z = X$, $Z = Y$ or $Z = X \times Y$ we have $\beta_G Z = \beta Z$, by Lemma 5.5. Now apply Glicksberg's theorem. □

The observations above lead to the following

5.8. PROBLEM. Let G be a locally compact group, G not discrete. Is it true that every G-pseudocompact G-space X is pseudocompact? I believe the answer is no, even if G is locally connected and compact, but I was not able to find a counterexample.

5.9. The answer to the previous problem would be "yes" if the following version of Lemma 5.5 were true: if G is locally compact Hausdorff and $<X,\pi>$ is G-pseudocompact, then $\beta_G X = \beta X$ (use 4.1 above and necessity of Glicksberg's result for a G-space of the form $X \times Z$, X being G-pseudocompact and Z infinite, compact, having trivial action). Observe, that $\beta_G X = \beta X$ if and only if $UC^*<X,\pi> = C^*(X)$, i.e. every bounded continuous function on X is π-uniformly continuous. Thus, our next problem reduces to a question, studied among others in [13], if one considers the G-space $<G,\mu>$ ($\mu^t s = ts$).

5.10. PROBLEM. Find necessary and sufficient conditions for a G-space $<X,\pi>$ in order that $\beta_G X = \beta X$. In particular, is G-pseudocompactness sufficient?

5.11. REMARK. Necessity in the preceding problem is related to the implication (ii) $\Rightarrow$ (i) in 2.5. Indeed, suppose there exists a G-space $<X,\pi>$ such that X is G-pseudocompact, X is not pseudocompact, but $\beta_G X = \beta X$. Then there exists $f \in C^*(X)$ which has not a maximur or a minimum on X. Since $C^*(X) = UC^*<X,\pi>$, such an example would show that (ii) $\not\Rightarrow$ (i) in Proposition 2.5.

REFERENCES

[1] ANTONJAN, S.A., *Classification of compact G-extensions*, Dokl. Akad. Nauk ASSR 69 (1979), 260-264 (Russian).

[2] DUGUNDJI, J., *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[3] FROLÍK, Z., *The topological product of two pseudocompact spaces*, Czechoslovak Math. J. 10 (1960), 339-349.

[4] GILLMAN, L. & M. JERISON, *Rings of continuous functions*, GTM43, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[5] GLICKSBERG, I., *Stone-Čech compactification of products*, Trans. Amer. Math. Soc. 90 (1959), 369-382.

[6] SMIRNOV, Ju. M., *Some topological aspects of the theory of topological transformation groups*, in: General Topology and its Rel. to Modern Analysis and Algebra (Proc. 4th Prague Top. Sym., 1976), Part A, LNM 609, Springer-Verlag, Berlin, Heidelberg, New York, 1977, pp. 196-204.

[7] SMIRNOV, Ju. M., *Shape theory and continuous transformation groups*, Uspekhi Mat. Nauk 34 (1979), 119-123 (Russian), Engl. transl. in Russian Math. Surveys 34 (1979), 138-143.

[8] SMIRNOV, Ju. M., *Compactifications, dimension and absolutes of topological transformation groups*, in: Proc. of the Conference on Topology and Measure III (Vitte, DDR, october 1980), to appear.

[9] TODD, C., *On the compactification of products*, Canad. Math. Bull. 14 (1971), 591-592.

[10] VRIES, J. DE, *Topological transformation groups 1*, Math. Centre Tracts no. 65, Mathematisch Centrum, Amsterdam, 1975.

[11] VRIES, J. DE, *Equivariant embeddings of G-spaces*, in: General Topology and its rel. to modern analysis and algebra IV (Proc. 4th. Prague Top. Sym., 1976), Part B, pp. 485-493.

[12] VRIES, J. DE, *Linearization, compactification and the existence of nontrivial compact extensors for topological transformation groups*, in: Proc. of the conference on Topology and Measure III (Vitte, DDR, october 1980), to appear. Preprint available as Report ZW 149/80, Mathematisch Centrum, Amsterdam.

[13] COMFORT, W.W. & A.W. HAGER, *Uniform continuity in topological groups*, in: Symposia Mathematica, Vol. XVI, Academic Press, London, New York, 1975, pp. 269-290.

ONTVANGEN 2 2 OKT. 1981