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ON THE VALUES OF A FUNCTION RELATED TO EULER'S GAMMA FUNCTION

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On the values of a function related to Euler's gamma function

by

J. van de Lune & M. Voorhoeve

ABSTRACT

In this note it is shown that the meromorphic function $\beta(s) := \sum_{n=0}^{\infty}(-1)^n/(s+n)$ assumes every complex value infinitely many times.

KEY WORDS AND PHRASES: *Special functions*

0. INTRODUCTION

Section 40 of NIELSEN's Handbuch der Theorie der Gammafunktion [4;pp. 101-102] is devoted to the (possible) zeros of the meromorphic function

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{s+n}, \quad s = \sigma + it \in \mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}.$$

Since $\beta(s) > 0$ for $s > 0$,

$$\beta(s-1) = \frac{1}{s-1} - \beta(s),$$

and

$$\beta(s-2) = \frac{1}{(s-1)(s-2)} + \beta(s),$$

it is clear that for every $n \in \mathbb{N}_0 := \{0,1,2,3,\ldots\}$ we have

$$\beta(s) < 0 \quad \text{if} \quad -2n-1 < s < -2n$$

and

$$\beta(s) > 0 \quad \text{if} \quad -2n-2 < s < -2n-1,$$

so that $\beta(s)$ has no real zeros.

As to the (possible) complex zeros of $\beta(s)$ Nielsen shows that, in case of existence, they must lie in the half-plane $\mathrm{Re}(s) < -\frac{1}{2}$ and then states: "Es ist mir nicht gelungen allgemein zu beweisen, dass $\beta(s)$ ... wirklich komplexe Nullstellen hat; doch halte ich dies für wahrscheinlich".

In addition Nielsen recalls a claim by SCHLÖMILCH [5] that $\beta(s)$ does not assume the value -1, whereas CLAUSSEN [1] gave the numbers $-.5794 + i * .6950$ and $-2.51 + i * .63$ as approximate solutions of the of the equation $\beta(s) = -1$.

In this note we shall clarify these matters by showing that $\beta(s)$ assumes every complex value infinitely many times and we conclude this note by presenting a number of roots of the equations

$\beta(s) = 0$, $\beta(s) = 1$, $\beta(s) = -1$, $\beta(s) = i$ and $\beta(s) = -i$.

1. PRELIMINARIES

We recall that (cf.[6;p.221])

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^n \frac{2s}{s^2-n^2} = \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^n \{\frac{1}{s-n} + \frac{1}{s+n}\},$$

so that $\beta(s)$ satisfies the functional equation

$$\beta(s) + \beta(1-s) = \frac{\pi}{\sin \pi s} .$$

From this it follows that

$$\beta(s) = \frac{\pi}{\sin \pi s} - \sum_{n=1}^{\infty} (\frac{1}{2n-s-1} - \frac{1}{2n-s}) . \tag{1.1}$$

Hence, for $\sigma := \mathrm{Re}(s) < 1$, we may write

$$\beta(s) = \frac{\pi}{\sin \pi s} - \sum_{n=1}^{\infty} \{\int_0^{\infty} e^{-(2n-s-1)x}\, dx - \int_0^{\infty} e^{-(2n-s)x}\, dx\} ,$$

so that

$$\beta(s) = \frac{\pi}{\sin \pi s} - \int_0^{\infty} \frac{e^{sx}}{e^x+1}\, dx , \quad \sigma < 1 . \tag{1.2}$$

In section 2 we will use this formula in order to show that the equation $\beta(s) = c_0$, $c_0 \neq 0$, has infinitely many solutions.

<u>REMARK</u>. For $u > 0$ we obtain from (1.1) that

$$\beta(-u) = - \frac{\pi}{\sin \pi u} - \sum_{n=1}^{\infty} \frac{1}{(2n+u-1)(2n+u)} , \tag{1.3}$$

so that

$$|\beta(-u)| > \pi - \log 2 \ (> 2.448) , \quad u > 0,$$

a result somewhat more precise than saying that $\beta(s)$ has no zeros on the negative real axis.

Since $\frac{\pi}{\sin \pi u}$ is periodic and $\beta(1+u)$ is decreasing on $\mathbb{R}^+$ it is easily seen that $|\beta(-u)|$ is minimal on $\mathbb{R}^+$ in the interval $(1,2)$. The function $\beta(1+u)$ is easily computed by means of Euler's transformation of alternating series (cf. FICHTENHOLZ [2;Vol.II,p.401]) and we found that $|\beta(-u)|$, $1 < u < 2$, is minimal for $u = 1.498\,400\,476\,330\ldots$ with minimal value $2.988\,658\,431\,004\ldots$.

From (1.2) we obtain by integration by parts

$$\beta(s) = \frac{\pi}{\sin \pi s} + \frac{1}{2s} - \frac{1}{s}\int_0^\infty e^{sx}\frac{e^x}{(e^x+1)^2}\,dx\ .$$

In section 3 we will use this formula in order to show that the equation $\beta(s) = 0$ has infinitely many solutions.

2. THE EQUATION $\beta(s) = c_0$ with $c_0 \neq 0$.

We consider the equation $\beta(s) = c_0$, $s = \sigma + it$, where c_0 is a complex constant different from 0. By (1.2) this equation is (for $\sigma < 1$) equivalent to

$$f(s) := \frac{\pi}{\sin \pi s} - c_0 - \int_0^\infty \frac{e^{sx}}{e^x+1}\,dx = 0\ .$$

Since the function $\sin(.)$ assumes every (finite) complex value (infinitely many times)(cf.[6; p.323]), the periodic function

$$\phi(s) := \frac{\pi}{\sin \pi s} - c_0$$

has infinitely many zeros of the form $s_0 - 2n$, where $s_0 \in \mathbb{C}$ is fixed and $n \in \mathbb{N}_0$. Since s_0 is an isolated zero, there exist $d > 0$ and $r > 0$ such that $|\phi(s_0 + re^{i\theta})| \geq d$ for all $\theta \in \mathbb{R}$ so that due to the periodicity of $\phi(s)$, $|\phi(s_0 - 2n + re^{i\theta})| \geq d$ for all $n \in \mathbb{N}_0$ and all $\theta \in \mathbb{R}$.

Since, for $\sigma < 1$,

$$\left| \int_0^\infty \frac{e^{sx}}{e^x+1}\,dx \right| \leqq \int_0^\infty \frac{e^{\sigma x}}{e^x+1}\,dx < \frac{1}{1-\sigma}$$

it follows from Rouché's theorem that f(s) has infinitely many zeros in the half plane $\sigma < 0$, proving that the equation $\beta(s) = c_0$, with $c_0 \neq 0$, has infinitely many solutions.

3. THE EQUATION $\beta(s) = 0$

From section 1 it is clear that we may restrict ourselves to the half-plane $\sigma < 0$ and, since $\beta(\bar{s}) = \overline{\beta(s)}$, we may also assume that $t > 0$.

We recall the following theorem of HURWITZ (cf.[6;pp.156-157]): For $n \in \mathbb{N}$ let $F_n(s)$ be analytic in an open set $A \subset \mathbb{C}$ and let $F_n(s) \to F(s)$, uniformly in every compact subset of A as $n \to \infty$, F(s) not being identically zero. Then a (finite) point $s_0 \in A$ is a zero of F(s) if and only if it is an accumulation point of the set of zeros of the functions $F_n(s)$, points which are zeros for an infinity of values of n being considered as accumulation points.

By means of this theorem we now prove the following

LEMMA. *Let* G *be a compact set in* $\mathbb{C}$ *with interior* $G^0 \neq \emptyset$. *Let the functions* $\phi(s)$, $h(s)$ *and* $\psi(s)$ *be analytic on* G^0 *and continuous on* G, *and assume that* $\phi(s)$ *is not identically zero.*
If for some positive constant p, $|\psi(s)| \leqq p$ *for all* $s \in \partial G$ (:=*the boundary of* G) *and if* $\phi(s) + \lambda h(s)$ *has at least one zero in* G^0 *but not on* ∂G *for all* λ *satisfying* $|\lambda| \leqq p$, *then* $\phi(s) + h(s)\psi(s)$ *has at least one zero in* G.

PROOF. Suppose the lemma is false.

We consider

$$f_\theta(s) := \phi(s) + \theta\, h(s)\,\psi(s)$$

for $\Theta \geqq 0$ and $s \in G$.

Clearly $f_0(s) = \phi(s)$ has a zero in G^0 and hence in G, whereas by assumption $f_1(s) = \phi(s) + h(s)\ \psi(s)$ has no zeros in G. Define θ_0 as the infimum of all positive θ such that $f_\theta(s)$ has no zeros in G. Using the theorem of Hurwitz mentioned above we conclude that $\theta_0 > 0$.
We now claim that $f_{\theta_0}(s)$ has a zero on ∂G. If not, then $f_{\theta_0}(s) \not\equiv 0$ and we have, for some positive constant d, $|f_{\theta_0}(s)| \geqq d$ on a compact strip $S \subset G$ around ∂G. Now choose an increasing positive sequence $\{\theta_n\}_{n=1}^{\infty}$ tending to θ_0 and note that $f_{\theta_n}(s)$ tends uniformly to $f_{\theta_0}(s)$ on G as $n \to \infty$. Since $|f_{\theta_0}(s)| \geq d > 0$ on S there must be an $n_0 \in \mathbb{N}$ such that $f_{\theta_n}(s) \neq 0$ on S for all $n > n_0$. Since $f_{\theta_n}(s)$ has at least one zero in G, the zeros of $f_{\theta_n}(s)$ must lie in $G \setminus S$ for $n > n_0$. It follows (again by Hurwitz's theorem) that $f_{\theta_0}(s)$ has a zero in G^0. From this it is clear (by Rouché's theorem) that for all θ which are slightly larger than θ_0, the functions $f_\theta(s)$ must also have a zero in G^0. Since this contradicts the definition of θ_0, our claim has been proved.
Hence, there exists $s_1 \in \partial G$ such that

$$\phi(s_1) + \theta_0 h(s_1)\ \psi(s_1) = 0.$$

Defining $\lambda_0 := \theta_0\ \psi(s_1)$ we have $|\lambda_0| \leq |\theta_0| \cdot |\psi(s_1)| \leq 1$. $p = p$ and the function $\phi(s) + \lambda_0\ h(s)$ has the zero $s_1 \in \partial G$. This contradiction proves the lemma. □

We will apply this lemma to the functions

$$\phi(s) := \frac{\pi}{\sin \pi s} + \frac{1}{2s},$$

$$h(s) := \frac{1}{s},$$

and

$$\psi(s) := -\int_0^\infty \frac{e^{(s+1)x}}{(e^x+1)^2}\, dx,$$

with $a = \varepsilon_0$ and $G = R_i$, where ε_0 is some sufficiently small positive constant, whereas, for any $i \in \mathbb{N}$, R_i is some closed rectangle to be specified in what follows.

Let ε_0 be a fixed small positive number and consider the equation (in s)

$$\frac{\pi}{\sin \pi s} + \frac{1}{2s} - \lambda \frac{1}{s} = 0, \tag{3.1}$$

where $|\lambda| \le \varepsilon_0$ and $s = \sigma + it$. Define $c_\lambda = \lambda - \frac{1}{2}$ and take ε_0 so small that $|\arg(-c_\lambda)| \le 3\varepsilon_0$. Since the above equation has infinitely many solutions (cf.[6; p. 323]) there must be infinitely many with arbitrarily large absolute value. We only pay attention to those with negative real part and positive imaginary part.

As to the location of these solutions we note that

$$\frac{\pi s}{c_\lambda} = \sin \pi s = \frac{1}{2i}\left(e^{\pi s i} - e^{-\pi s i}\right),$$

so that $\dfrac{2\pi\, si}{c_\lambda} = e^{-\pi t} e^{\pi\sigma i} - e^{\pi t} e^{-\pi\sigma i}$,

from which we infer that

$$\frac{2\pi |s|}{|c_\lambda|} \le e^{-\pi t} + e^{\pi t} \le 2 e^{\pi t},$$

so that

$$t \ge \frac{1}{\pi} \log \frac{\pi |s|}{|c_\lambda|} \ge \frac{1}{\pi} \log \frac{\pi |\sigma|}{\frac{1}{2}+\varepsilon_0} . \tag{3.2}$$

Note that it follows that t is not bounded. Similarly we find that

$$t \le \frac{1}{\pi} \log \left(\frac{2\pi |s|}{|c_\lambda|} + 1\right),$$

from which it is easily seen that for large $|s|$ we have $t < |\sigma|$ and hence

$$t \le \frac{1}{\pi} \log \left(\frac{4\pi|\sigma|}{\frac{1}{2}-\varepsilon_0} + 1\right) . \tag{3.3}$$

It is clear now that the solutions of our equation (3.1) lie in a rather narrow strip described by (3.2) and (3.3) and that for s in this strip

$$\arg(s) \to \pi \text{ as } |s| \to \infty .$$

As to the horizontal distribution of these solutions we note that for large positive t

$$\arg(\sin \pi s) = \arg \frac{e^{-\pi t}e^{\pi\sigma i} - e^{\pi t}e^{-\pi\sigma i}}{2i} \sim$$

$$\sim -\frac{\pi}{2} + \pi - \pi\sigma + 2k\,\pi \quad \text{for some } k \in \mathbb{Z}$$

and that

$$\arg \frac{\pi s}{c_\lambda} = \arg s - \arg c_\lambda \sim 0$$

when ε_0 is small.
It follows that (uniformly) $\sigma \sim -2k+\frac{1}{2}$ for some $k \in \mathbb{N}$ if t is positive and large and if ε_0 is small. Hence, all solutions $s = \sigma + it$, with sufficiently large $t > 0$, and $\sigma < 0$ also lie in vertical strips of the form $-2k+\frac{1}{2}-\frac{1}{4} < \sigma < -2k+\frac{1}{2}+\frac{1}{4}$, with $k \in \mathbb{N}$, if $|\lambda| \le \varepsilon_0$ and ε_0 is small enough. Compare MAGNUS et al. [3; pp.17-18].

From these considerations it follows that we may construct infinitely many disjoint closed rectangles R_i in $\sigma < 0$ all of which contain solutions of our equation (3.1) in their interiors and not on their boundaries.
Since for $\sigma < -1$

$$|\psi(s)| = \left| \int_0^\infty \frac{e^{(s+1)x}}{(e^x+1)^2}\,dx \right| \le \frac{1}{|\sigma|+1} ,$$

it is clear that, if $|\sigma|$ is large enough, our lemma may be applied as announced above, proving that the equation $\beta(s) = 0$ has infinitely many solutions.

4. SOME NUMERICAL DATA

As indicated in Section 1, there exist excellent methods of computing $\beta(s)$ to a very high degree of accuracy. Utilizing two different methods we found (by means of Newton-approximation) the following approximate solutions of the equation $\beta(s) = c$, for $c = 0,1,-1$, and i .

Some solutions of $\beta(\sigma+it) = 0$

σ	t
- 1. 346 516	1. 055 160
- 3. 403 159	1. 258 497
- 5. 427 952	1. 382 406
- 7. 442 089	1. 471 712
- 9. 451 307	1. 541 528

Some solutions of $\beta(\sigma+it) = 1$

σ	t
- 1. 485 081	. 506 698
- 3. 495 174	. 537 892
- 5. 497 670	. 549 989
- 7. 498 637	. 556 361
- 9. 499 107	. 560 284

Some solutions of $\beta(\sigma+it) = -1$

σ	t
- . 579 415	. 694 980
- 2. 512 233	. 632 787
- 4. 504 305	. 610 889
- 6. 502 149	. 601 088
- 8. 501 281	. 595 611

Some solutions of $\beta(\sigma+it) = i$

σ	t
- 1. 073 106	. 558 394
- 3. 039 841	. 583 444
- 5. 026 586	. 588 698
- 7. 019 818	. 590 528
- 9. 015 763	. 591 362

Finally, from the solutions of $\beta(s) = i$ we obtain those of $\beta(s) = -i$ by observing that $\beta(\bar{s}) = \overline{\beta(s)}$ and $\bar{i} = -i$.

REFERENCES

[1] CLAUSSEN, Th., Grunerts Archiv, 13, (1849) 334-336.

[2] FICHTENHOLZ, G.M., *Differential- und Integralrechnung*, Berlin, 1979.

[3] MAGNUS, W., F. OBERHETTINGER & R.P. SONI, *Formulas and theorems for the special functions of mathematical physics*, Grundlehren Math. Wiss., Band 52, Springer, 1966.

[4] NIELSEN, N., *Handbuch der Theorie der Gammafunktion*, Teubner, 1906.

[5] SCHLÖMILCH, O., Grunerts Archiv, 12, (1849) 293-297.

[6] SANSONE, G. & J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, Noordhoff, 1960.

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