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ORBITS ON REAL AFFINE SYMMETRIC SPACES.
PART I: THE INFINITESIMAL CASE

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Orbits on real affine symmetric spaces. Part I: the infinitesimal case*)

by

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ABSTRACT

An exposition is given of the infinitesimal orbit theory on real affine symmetric spaces. Main references are the results by Kostant and Rallis on complex symmetric spaces and the treatment by Varadarajan of the theory of orbits under the adjoint group. Partial results are due to Oshima and Matsuki. In a final chapter we consider the problem of the existence of invariant measures on the orbits in symmetric spaces.

KEY WORDS & PHRASES: *real symmetric spaces, Cartan subspaces, orbits, q-regular orbits, invariant measures*

*) This report will be submitted for publication elsewhere.

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INTRODUCTION

We generalize part of the results of KOSTANT and RALLIS [7] on complex symmetric spaces to real spaces. Our selection is primarily based on applications in analysis on symmetric spaces. We are therefore concerned with the theory of Cartan subspaces and the orbit structure. The main reference for our results is the above work, together with VARADARAJAN'S notes [11] on orbits in real reductive Lie algebras under the action of the adjoint group.

The results on Cartan subspaces are also obtained by OSHIMA and MATSUKI [9] by a different method, probably without being aware of the existence of [7] . Our starting point was section 1.13 in DIXMIER'S book [1] .
Let us now list the main results of this paper. Let $\mathfrak{g}$ be a real reductive Lie algebra and σ an involutive automorphism of $\mathfrak{g}$. Put $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, the decomposition of $\mathfrak{g}$ in $+1$ and -1 eigenspaces of σ. Let G be the connected adjoint group of $\mathfrak{g}$ and H the connected Lie subgroup of G with Lie algebra ad$\mathfrak{h}$. Then $\mathfrak{q}$ is stable under the action of H. A subspace $\mathfrak{a} \subset \mathfrak{q}$ is called a Cartan subspace of $\mathfrak{g}$ with respect to σ if

(i) $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{q}$

(ii) for each $x \in \mathfrak{a}$, adx is a semisimple endomorphism of $\mathfrak{g}$.

We shall prove the following:

- There are finitely many H-conjugacy classes of Cartan subspaces in $\mathfrak{q}$. All Cartan subspaces have the same dimension.
- Any semisimple element in $\mathfrak{q}$ is contained in a Cartan subspace.
- Let θ be a Cartan involution commuting with σ. Then any Cartan subspace is H-conjugate to a θ-invariant Cartan subspace ($\mathfrak{g}$ semisimple).
- H-orbits are closed in $\mathfrak{q}$ if and only if they consist of semisimple elements .
- Let N be the set of nilpotent elements in $\mathfrak{q}$. Then N is H-stable and splits into finitely many H-orbits.

More precise statements and additional results are given below. We shall not make use a priori of similar results for the special case, where $\mathfrak{g}$ is replaced by $\mathfrak{g} \times \mathfrak{g}$ and σ is given by $\sigma(x,y) = (y,x)$. This case is well-known since it amounts to the study of the G-space $\mathfrak{g}$. On the contrary, we will consider this case purely as a special case of our situation. We shall however use the results known for Riemannian symmetric spaces, i.e. for the case where σ is a Cartan involution and g a semisimple Lie algebra (see [4]).

In a final chapter we consider the problem of the existence of invariant measures on H-orbits. It turns out that "regular" H-orbits admit an invariant measure, but for general H-orbits the answer is negative. This is in contrast with the known affirmative answer for the special case of G-orbits in g .

Part II, which shall appear later, will be concerned with the H-orbit theory on G/H. We refer to OSHIMA and MATSUKI'S paper [9] for partial results in this case. I wish to acknowledge my indebtness to M.T. Kosters, who worked out the greater part of the content of Chapter I, following a suggestion of mine.

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1. CARTAN SUBSPACES

We first collect a few technical lemmas which are basic for this section. The proofs can be found in DIXMIER'S book [1] .

A Lie subalgebra h of a Lie algebra g is called reductive in g if the representation ad_g of h on g is semisimple. If h is reductive in g, then h is a reductive Lie algebra.

LEMMA 1. *Let g be a real Lie algebra. The following statements are equivalent:*

(i) *g is reductive;*

(ii) *g is a direct sum of an abelian and a semisimple ideal;*

(iii) *there exists a finite-dimensional representation ρ of g such that the bilinear form* $(x,y) \to \mathrm{tr}\,\rho(x)\rho(y)$ *is non-degenerate.*

(see [1] , Prop. 1.7.3)

LEMMA 2. *Let g be a real semisimple Lie algebra, a an abelian real Lie algebra, ρ a finite-dimensional representation of $g \oplus a$ on a real vector space. The following conditions are equivalent:*

(i) ρ *is semisimple;*

(ii) *for each* $x \in a$, $\rho(x)$ *is a semisimple endomorphism.*

(see [1] , Cor. 1.6.4)

PROPOSITION 3. *Let* g *be a real semisimple Lie algebra with Killing form* B, $m \subset g$ *a subalgebra satisfying the following two conditions:*

(i) $B|_{m \times m}$ *is non-degenerate ;*

(ii) *if* $x \in m$ *and* $x = s + n$ *its Jordan decomposition, then* $s, n \in m$.

Then m *is reductive in* g .

(see [1] , Prop. 1.7.6).

From now on let g be a real reductive Lie algebra and σ an involutive automorphism of g. Let h be the + 1 and q the −1 eigenspace of σ on g. Then h is a subalgebra of g and

$$g = h \oplus q \qquad \text{(direct sum)}$$

From Proposition 3 one easily sees that h is reductive in g.

DEFINITION 4. A set $a \subset q$ is called a Cartan subspace of g (with respect to σ) if

(i) a is an abelian subalgebra of g;

(ii) for all $x \in a$, $ad_g x$ is a semisimple endomorphism of g;

(iii) the centralizer of a in q equals a.

Given $x \in g$ we put

$$\overset{\circ}{g}(x) = \{ y \in g \ : \ \exists\, n \in \mathbb{N} \text{ such that } (adx)^n y = 0\},$$

an call it the nilspace of x.

An element $x \in q$ is called generic if $\dim (g^\circ(x) \cap q) \le \dim (g^\circ(y) \cap q)$ for all $y \in q$. Clearly $x \in q$ is generic if and only if the multiplicity of the eigenvalue zero of $(adx)^2|_q$ is minimal. One can show that $x \in q$ is generic if and only if the multiplicity of the eigenvalue zero of $ad_g x$ is minimal (cf.[7] , Prop.7).

Clearly the set of generic elements of q is a non-empty Zariski open subset of q , hence dense in q in the Euclidean topology. Let q^1 denote the set of generic elements of q .

THEOREM 5. (cf. [1] , Théorème 1.13.6).
Let x *be a generic element in* q. *Then* $a = g^{\circ}(x) \cap q$ *is a Cartan subspace of* g.

PROOF. Put $g^{*}(x) = \bigcap_{n=1}^{\infty} (\mathrm{ad}x)^{n} g$. Then $g = g^{\circ}(x) \oplus g^{*}(x)$, the Fitting decomposition of g with respect to adx. Since $\sigma x = -x$, we have $\sigma g^{\circ}(x) = g^{\circ}(x)$ and $\sigma g^{*}(x) = g^{*}(x)$. Therefore $g^{\circ}(x) = a \oplus (g^{\circ}(x) \cap h)$. For $y \in a$, both $g^{\circ}(x)$ and $g^{*}(x)$ are stable under ady. Therefore $(\mathrm{ad}y)^2 a \subset a$ and $(\mathrm{ad}y)^2(g^{*}(x) \cap q) \subset g^{*}(x) \cap q$ for all $y \in a$. Let S be the set of all $y \in a$ such that $(\mathrm{ad}y)^2$ is bijective on $g^{*}(x) \cap q$. Obviously $x \in S$. S is a Zariski open subset of a. Let R be the set of all $y \in a$ such that $(\mathrm{ad}y)^2|_a$ is not nilpotent. R is Zariski open in a. Suppose R is not empty. Then $R \cap S$ is not empty. This would imply the existence of $y \in a$ with $\dim g^{\circ}(y) \cap q < \dim a$, which contradicts the fact that x is a generic element of q. Therefore R is empty. Let $y \in a$. Then $(\mathrm{ad}y)^n a = (0)$ for some n. Moreover, since $(\mathrm{ad}y)(g^{\circ}(x) \cap h) \subset g^{\circ}(x) \cap q = a$ we obtain, $(\mathrm{ad}y)^{n+1} g^{\circ}(x) = (0)$. Let L be the Killing form of $g^{\circ}(x)$. Then $L(y,y) = 0$ for all $y \in a$, hence by polarization, $L = 0$ on $a \times a$. Since L is σ-invariant, $L(a, g^{\circ}(x) \cap h) = 0$, hence $L(a, g^{\circ}(z)) = 0$. From Proposition 3 we see that $g^{\circ}(x)$ is reductive in g, hence reductive. Hence a is contained in the center of $g^{\circ}(x)$ (by Lemma 1) and ady is semisimple endomorphism of g for all $y \in a$ (by Lemma 2). Finally let $z \in q$ be in the centralizer of a. Then in particular $[z,x] = 0$, so $z \in g^{\circ}(x) \cap q = a$. Thus a is a Cartan subspace of g. □

Let G denote the connected adjoint group of g and let H be the connected Lie subgroup of G with Lie algebra ad$h$.

THEOREM 6. *Any Cartan subspace of* g *is of the form* $g^{\circ}(x) \cap q$ *where* x *is a generic element of* q.

PROOF. (see also [1], Prop. 1.13.13).
Let a be a Cartan subspace and denote by m the centralizer of a in h. There exist non-zero linear forms $\lambda_1, \ldots, \lambda_n$ on the complexification $a_{\mathbb{C}}$ of a such that

$$\mathfrak{g}_{\mathbb{C}} = (\mathfrak{m}+\mathfrak{a})_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{\lambda_1} \oplus \dots \oplus \mathfrak{g}_{\mathbb{C}}^{\lambda_n},$$

where, as usual, $\mathfrak{g}_{\mathbb{C}}^{\lambda} = \{y \in \mathfrak{g}_{\mathbb{C}} : [t,y] = \lambda(t)y \text{ for all } t \in \mathfrak{a}_{\mathbb{C}}\}(\lambda \in \mathfrak{a}_{\mathbb{C}}^{*})$. Choose $y \in \mathfrak{a}$ such that $\lambda_i(y) \neq 0$ for all i. Then clearly $\mathfrak{q} = \mathfrak{a} \oplus (\mathrm{ad}y)^2 \mathfrak{q}$. Since $(\mathrm{ad}y)\, \mathfrak{q} \subset \mathfrak{h}$, we also have $\mathfrak{q} = \mathfrak{a} + [\mathfrak{h},y]$. Let f be the mapping from $H \times \mathfrak{a}$ into $\mathfrak{q}$ given by $f(h,t) = h.t$ and let T be the tangent map of f at the point $(1,y)$. Then $T: \mathfrak{h} \oplus \mathfrak{a} \longrightarrow \mathfrak{q}$ is given by $T(z,t) = t + [z,y]$ $(z \in \mathfrak{h}, t \in \mathfrak{a})$. So T is surjective and hence $H.\mathfrak{a}$ contains an open neighborhood of y in $\mathfrak{q}$. Since the set of generic elements in $\mathfrak{q}$ is dense in $\mathfrak{q}$ and H-stable, we obviously have that $\mathfrak{a}$ contains a generic element, say x. Clearly $\mathfrak{a} \subset \mathfrak{q}^{\circ}(x) \cap \mathfrak{q}$. But $\mathfrak{g}^{\circ}(x)$ is a Cartan subspace by Theorem 5, hence $\mathfrak{a} = \mathfrak{g}^{\circ}(x) \cap \mathfrak{q}$. □

By an observation like the above, one easily shows that the map $f: H \times \mathfrak{a} \to \mathfrak{q}$ given by $f(h,t) = h.t$ is submersive on $H \times \mathfrak{a}'$, where $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{q}'$.

<u>THEOREM 7</u>. *There are finitely many Cartan subspaces $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ such that any Cartan subspace is* H-*conjugate to some $\mathfrak{a}_i$.*

<u>PROOF</u>. By a theorem of Whitney, $\mathfrak{q}'$ has finitely many connected components, say $\mathfrak{q}_1, \dots, \mathfrak{q}_n$. Fix $x_i \in \mathfrak{q}_i$ and put $\mathfrak{a}_i = \mathfrak{g}^{\circ}(x_i) \cap \mathfrak{q}$ $(i=1,\dots,n)$. For any $x \in \mathfrak{q}'$, put $\mathfrak{a}_x = \mathfrak{g}^{\circ}(x) \cap \mathfrak{q}$, $\mathfrak{a}_x' = \mathfrak{a}_x \cap \mathfrak{q}'$ and $\mathfrak{a}_x^{+}$ the connected component of $\mathfrak{a}_x'$ containing x. Clearly $\mathcal{O}_x = H.\mathfrak{a}_x^{+}$ is an open and connected subset of $\mathfrak{q}'$, hence $\mathcal{O}_x \subset \mathfrak{q}_i$ for some i. Furthermore, if $z \in \mathfrak{a}_x^{+}$ then $\mathfrak{a}_x^{+} = \mathfrak{a}_z^{+}$, hence $\mathcal{O}_x = \mathcal{O}_z$. Let $x,y \in \mathfrak{q}_i$. Assume $\mathcal{O}_x \cap \mathcal{O}_y$ is non-empty. Then we can find $h_1, h_2 \in H$ and $x' \in \mathfrak{a}_x^{+}$, $y' \in \mathfrak{a}_y^{+}$ such that $h_1 x' = h_2 y'$. Then we have $\mathcal{O}_x = \mathcal{O}_{x'} = \mathcal{O}_{y'} = \mathcal{O}_y$. Since $\mathfrak{q}_i$ is connected we get $\mathfrak{q}_i = \mathcal{O}_{x_i}$ for all i. Now let $\mathfrak{a}$ be a Cartan subspace. Then $\mathfrak{a} = \mathfrak{a}_x$ for some $x \in \mathfrak{a}'$. Let i be such that $x \in \mathfrak{q}_i$. Then $x = hz$ for some $z \in \mathfrak{a}_{x_i}^{+}$ and some $h \in H$. Hence $\mathfrak{a} = h\,\mathfrak{a}_i$. □

<u>REMARK</u>. If $\mathfrak{g}$ is a complex reductive Lie algebra and σ a complex involution, then $\mathfrak{q}'$ is connected and all Cartan subspaces of $\mathfrak{g}$ with respect to σ are conjugate under H.

We recall that an element $x \in \mathfrak{q}$ is said to be semisimple if $\mathrm{ad}x$ is a semisimple endomorphism of $\mathfrak{g}$.

PROPOSITION 8. *Any semisimple element of q is contained in a Cartan subspace of g with respect to σ.*

PROOF. Let $x \in q$ be a semisimple element. Put $P_x = \{y \in g^{\circ}(x) \cap q : (\mathrm{ad}y)^2$ is non-singular on $g^*(x) \cap q\}$. Then $x \in P_x$ and P_x is Zariski open in $g^{\circ}(x) \cap q$. The mapping $(h,y) \mapsto h.y$ from $H \times P_x$ to q is everywhere submersive. Hence $H.P_x$ is open in q and thus contains a generic element. Therefore P_x itself contains a generic element $z \in q$. Since $[x,z] = 0$, we have $x \in g^{\circ}(z) \cap q$. □

A subalgebra of g consisting of semisimple elements is called a torus. Any torus is an abelian subalgebra. A torus which is contained in q is called a q-torus.

PROPOSITION 9. *Any q-torus is contained in a Cartan subspace.*

PROOF. Let b be a torus, $b \subset q$. We can certainly find $x \in b$ such that $Z(b) = Z(x)$ (Z denoting centralizer). By Proposition 8, $Z(x)$ contains a Cartan subspace a. Since $[b,a] = (0)$, we have $b \subset a$. □

COROLLARY 10. *Any maximal q-torus is a Cartan subspace.*

2. ORBIT STRUCTURE ON q

In this chapter we follow [10], Part I, 1.

1. PRELIMINARIES

Throughout what follows g is a real reductive Lie algebra and g_c its complexification. G (resp. G_c) is the connected adjoint group of g (resp. g_c). Let σ be an involutive automorphism of g and $g = h \oplus q$ the corresponding decomposition in +1 and −1 eigenspaces. Extending σ to g_c in the natural manner, $g_c = h_c \oplus q_c$ is the corresponding decomposition in +1 and −1 eigenspaces. Denote H (resp. H_c) the connected Lie subgroup of G (resp. G_c) with Lie algebra $\mathrm{ad}h$ (resp. $\mathrm{ad}h_c$).

If g is semisimple, we can find a compact real form u of g_c which is

σ-invariant ([8] ,p.153). Write $k_{\mathbb{R}} = h \cap u$ and $p_{\mathbb{R}} = q \cap i\, u$. Then $q_{\mathbb{R}} = k_{\mathbb{R}} \oplus p_{\mathbb{R}}$ is the Cartan decomposition of a real form $g_{\mathbb{R}}$ of g_c, such that $h_c = (k_{\mathbb{R}})_c$, $q_c = (p_{\mathbb{R}})_c$. If $a_{\mathbb{R}}$ is a maximal abelian subspace of $p_{\mathbb{R}}$, then $a_c = (a_{\mathbb{R}})_c$ is a Cartan subspace of g_c with respect to σ.

Let g be as before, a real reductive Lie algebra. Let z be the center of g. Clearly z is σ-stable. Put $z_c = z_c^+ \oplus z_c^-$, the decomposition in +1 and -1 eigenspaces with respect to σ. Write $g = z \oplus g_1$ with $g_1 = [g,g]$. Clearly g_1 is σ-stable too. Choose a Cartan subspace a_c in $g_{1,c}$ as above. Then, $a_c \oplus z_c^-$ is a special Cartan subspace of g_c with respect to σ. Every Cartan subspace of g_c is H_c-conjugate to this special one (Theorem 7,Remark), and hence "special".

Put I the algebra of all H_c-invariant polynomials on q_c. For any indeterminate T and $x \in q_c$, let

$$\det(T-(\mathrm{ad}x)^2)_{q_c} = \sum_{i=0}^{m} q_i(x)\, T^i .$$

where $m = \dim q_c$. Put ℓ = dimension of a Cartan subspace of q_c, and call it the q - rank of g . Then we have $q_i \in I$, q_i homogeneous of degree $2(m-i)$, $q_m = 1$, $q_s = 0$ for $0 \le s \le \ell$ and $q_i \neq 0$. Let $q_\ell = \xi$. Then, as before, $x \in q_c$ is generic if $\xi(x) \neq 0$. As usual, let q_c' denote the set of generic elements of q_c and put $a_c' = a_c \cap q_c'$ for any subset a_c of q_c . If $a_c \subset q_c$ is a Cartan subspace and Δ the set of roots of (g_c,a_c) , then

$$\xi(t) = \prod_{\alpha\in\Delta} \alpha(t)^{m_\alpha} \quad (t \in a_c)$$

where $m_\alpha = \dim g_\alpha$.

Let $x \in g_c$. x is called *semisimple* (s.s) (resp. *nilpotent*) if adx is a semisimple endomorphism of g_c(resp. $x \in [g_c,g_c]$ and adx a nilpotent endomorphism of g_c). Any $x \in g_c$ can be written uniquely as $x_s + x_n$ where $x_s, x_n \in g_c$, x_s is s.s., x_n is nilpotent and $[x_s,x_n] = 0$ (Jordan decomposition of x). If $x \in g$, then $x_s,x_n \in g$. Moreover, if $x \in q_c$ then x_s and x_n in q_c . x_s(resp.x_n) is called the s.s. (resp. nilpotent) component of x.

<u>LEMMA 11.</u> *Let* $x \in q$ *be nilpotent. Then there exists* $t \in [g,g] \cap h$ *such that* $[t,x] = 2x$.

PROOF. By the Jacobson-Morozow theorem, there are t_0 and y_0 in $[g,g]$ such that $[t_0,x] = 2x$, $[t_0,y_0] = -2y_0$ and $[x,y_0] = t_0$. Put $t_0 = t + t_1$ where $t \in h$, $t_1 \in q$. Then $t \in [g,g] \cap h$. Since $2x = [t_0,x] = [t,x] + [t_1,x]$, we get $[t_1,x] = 0$ and $[t,x] = 2x$. □

REMARK. One can prove a stronger result,saying that there exist $t,y \in [g,g]$, $t \in h$, $y \in q$ such that $[t,x] = 2x$, $[t,y] = -2y$, $[x,y] = t$. For details we refer to [7].

LEMMA 12. *Let* $x \in q$. *Then* $x_s \in CL(H.x)$ (CL=closure) *and* $p(x) = p(x_s)$ *for all* $p \in I$. x *is nilpotent if and only if* $c\,x \in H.x$ *for some* $c \neq 1$; *in this case* $cx \in H.x$ *for all* $c > 0$ *and* $0 \in CL(H.x)$. *If* Ω *is a* H-*invariant open subset of* q *containing all s.s. points of* q , *then* $\Omega = q$.

PROOF. Let z denote the centralizer of x_s in g . Since $x_s \in q$, z is σ-invariant. Furthermore x and x_n belong to $z \cap q$ and z is reductive in g. By Lemma 11, there is $t \in z \cap h$ such that $[t,x_n] = 2x_n$. Hence

$$e^{\lambda\,\mathrm{ad}t}\, x = x_s + e^{2\lambda}\, x_n \longrightarrow x_n \quad \text{when } \lambda \to -\infty .$$

So $x_s \in CL(H.x)$ and $p(x) = p(x_s)$ for all $p \in I$. If x is nilpotent, then $e^{2\lambda}x \in H.x$ by the above calculation, for all $\lambda \in \mathbb{R}$. Conversely if $cx \in H.x$ for some $c \neq 1$, then $x \in [g,g]$ and $\mathrm{ad}(cx)$ and $\mathrm{ad}x$ have the same eigenvalues, which must be zero. The remaining statements are clear. □

COROLLARY 13. *Let* $x \in q$ *be such that* $H.x$ *is a closed subset of* q. *Then* x *is s.s.* .

This is clear, from Lemma 12.

2. ORBITS IN q_c

The greater part of this section is known (see [7]. We include it for the sake of completeness and as preparation for the structure theory of the orbits in q.

Let N (resp. N_c) be the set of nilpotent elements of q (resp. q_c).

PROPOSITION 14. *N_c is the set of common zeros of all* $p \in I$ *with* $p(o) = 0$. *N_c is H_c-stable and splits into finitely many orbits.*

As to the first assertion, observe that if $(\mathrm{ad}x)^2$ is nilpotent on q_c, then $\mathrm{ad}x$ is nilpotent on g_c, for $x \in q_c$. Moreover $p(x) = p(o)$ for all nilpotent $x \in q_c$, since $o \in Cl(H.x)$ (Lemma 12). The proof of the second assertion is due to KOSTANT and RALLIS. We shall need it in the real case also and prove it in Theorem 23. The result for g_c follows by applying this theorem to the real Lie algebra underlying g_c. We omit the proof of the second assertion therefore at this time.

Let us fix a special Cartan subspace a_c of q_c as in section 1. Let W denote the Weyl group of the root system associated with the pair (g_c, a_c). Then $W \simeq$ normalizer of a_c in H_c / centralizer of a_c in H_c.[1)] If $I(a_c)$ denotes the algebra of W-invariant polynomials on a_c, then $I(a_c)$ is isomorphic to a polynomial algebra in ℓ variables. Furthermore, the restriction map $p \longrightarrow p|_{a_c}$ from I to $I(a_c)$ is an algebra isomorphism. All this is an easy consequence of the similar results for the "Riemannian" case (see [3])).

Let $p_1, \ldots, p_\ell$ be algebraically independent homogeneous polynomials such that $I = \mathbb{C}[p_1, \ldots, p_\ell]$. Let

$$\phi(x) = (p_1(x), \ldots, p_\ell(x)), \quad M_x = \phi^{-1}(\phi(x)) \quad (x \in q_c) .$$

ϕ is constant on H_c- orbits and each M_x is H_c - stable.

LEMMA 15. *Let a_c be a Cartan subspace of q_c and put $\phi_{a_c} = \phi|_{a_c}$. Then $\phi_{a_c} : a_c \longrightarrow \mathbb{C}^\ell$ is surjective and proper.*

PROOF. The surjectivity can be shown by the method of [5], 23, exercise 9. If Δ is the set of roots of (g_c, a_c), then for $t \in a_c$ the numbers $\alpha(t) (\alpha \in \Delta)$ are the roots of the equation (in z)

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1) This can be shown similar to: N.R. Wallach, Harmonic analysis on homogeneous spaces, Marcel Dekker, Inc. New York(1973); Proposition 8.9.6. Observe that the right-hand-side is a finite group.

$$z^{2m} + \sum_{i=0}^{m-1} q_i(t)\, z^{2i} = 0 \; .$$

Hence $|\alpha(t)| \leq \sum_{i=0}^{m} |q_i(t)| \quad (\alpha \in \Delta, t \in a_c)$.

The proof of the properness is now easily completed as in [11], Part 1, Lemma 6. □

For any orbit γ in q_c , put $\phi_\gamma = \phi(x) \quad (x \in \gamma)$.

PROPOSITION 16. *The s.s. orbits in* q_c *are precisely the closed ones, and the map* $\gamma \longmapsto \phi_\gamma$ *is a bijection of the set of all s.s. orbits onto* $\mathbb{C}^\ell$. *If* $a_c \subset q_c$ *is a Cartan subspace and* W *is the Weyl group of the pair* (g_c, a_c), *the correspondence* $\gamma \longmapsto \gamma \cap a_c$ *is a bijection of the set of all s.s. orbits onto the set of all* W*-orbits in* a_c. *Let* $x \in q_c$. *Then* M_x *is a finite union of orbits, exactly one of which is closed, and this one consists of all s.s. elements of* M_x. *Moreover,* $x' \in M_x$ *if and only if* $x'_s \in H_c.x_s$. *Furthermore, we can write* $M_x = O_1 \cup \ldots \cup O_s$ *where the* O_i *are disjoint orbits,* $O_i \cup \ldots \cup O_s$ *is closed and contains* O_i *as an open subset for* $i = 1, \ldots, s$. *Finally, if* x *is generic then* $M_x = H_c.x$.

PROOF. This is similar to the proof of [11], Part I, Proposition 7. □

PROPOSITION 17. *Let* $x \in q_c$. *Then* $H_c.x$ *is open in its closure in* q_c *and is a regularly imbedded submanifold of* q_c. *If* Z_c *is the centralizer of* x *in* H_c, *then* $h\,Z_c \mapsto h\,x$ *is an analytic diffeomorphism of* H_c/Z_c *onto* $H_c.x$.

This can be shown similar to [11] , Part I, Proposition 8.

LEMMA 18. *Let* $a_c \subset q_c$ *be a Cartan subspace. For* $B \subset a_c$, *let* $< B >$ *be the set of all* $x \in q_c$ *such that* $x_s \in H_c.B$. *Then* $< B >$ *is open (resp. closed) in* q_c *if* B *is open (resp. closed) in* a_c.

PROOF. Similar to the proof of [11] , Part I, Lemma 9.

LEMMA 19. *Let* a_c *be a Cartan subspace and* $a_0 \subset a_c$ *any subset. Then the centralizer of* a_0 *in* G_c *is connected.*

For the proof, we refer to [11], Part I, Lemma 10.

3. H_c-INVARIANT OPEN SETS IN q_c. THE SETS U_ω, V_ω.

We start with a result by KOSTANT & RALLIS ([7], Proposition 1). Let $H_{c,\sigma}$ denote the subgroup of elements $a \in G$ which commute with σ. Then h_c and q_c are obviously stable under $H_{c,\sigma}$ and the identity component of $H_{c,\sigma}$ is just H_c. Let a_c be a Cartan subspace of g_c with respect to σ, put $A = \exp \operatorname{ad} a_c$ and let F be the finite group of all elements of order 2 in A. Clearly $F \subset H_{c,\sigma}$ and therefore normalizes $H_{c,\sigma}$.

<u>PROPOSITION 20</u>. *One has* $H_{c,\sigma} = F.H_c$.

See also Theorem 31.

Fix a s.s. element $x \in q_c$. Clearly $g_c^0(x)$ is the centralizer of x in g_c. Let Z_c be the centralizer of x in H_c. Clearly $Z_c^0 \subset Z_c \subset Z(x)_{c,\sigma}$ where $Z(x)_\sigma$ is the centralizer of x in G_c and $Z(x)_{c,\sigma} = Z(x)_c \cap H_{c,\sigma}$. Obviously Z_c^0 is the identity component of $Z(x)_{c,\sigma}$. As in the proof of Proposition 8, we put
$P_x = \{y \in g_c^0(x) \cap q_c : (\operatorname{ad} y)^2$ is non-singular on $g_c^*(x) \cap q_c\}$. Then $x \in P_x$, P_x is Z_c stable and is an open dense subset of $g_c^0(x) \cap q_c$. The mapping $\pi : H_c \times P_x \longrightarrow q_c$ defined by $\pi(h,y) = h.y$ is everywhere submersive. Hence $H_c.U$ is open in q_c for every open subset U of P_x. Let $a_c \subset g_c^0(x) \cap q_c$ be a Cartan subspace of $g_c^0(x)$. Then a_c is also a Cartan subspace of g_c. Furthermore, $x \in a_c$. Let W (resp. W_x) denote the Weyl group of (g_c, a_c) (resp. $g_c^0(x), a_c$). The W_x is the centralizer of x in W. We can find an open set ω_0 in a_c, containing x, such that $\omega_0 \subset P_x$, $\omega_0^s = \omega_0$ for all $s \in W_x$, $\omega_0^s \cap \omega_0 = \emptyset$ if $s \in W \setminus W_x$.
For any subset $\omega \subset a_c$ we put
$U_\omega = \{y : y \in g_c^0(x) \cap q_c$, s.s. component of y lies in $Z_c\omega\}$. Note that $Z_c\,\omega = Z_c^0\,\omega$. Moreover, if ω is open (resp. closed) then U_ω is open (resp. closed) in $g_c^0(x) \cap q_c$. Let $V_\omega = H_c.U_\omega$. Observe that for $\omega \in a_c \cap P_x$ we have that $y \in V_\omega$ implies $M_y \subset V_\omega$. Indeed, if $y \in U_\omega$ then $y_s \in U_\omega$; hence $y \in V_\omega$ implies $y_s \in V_\omega$. Since $M_y = M_{y_s}$ we may assume that y is semisimple abd also that $y \in U_\omega$. Let $y' \in M_y$. Then $h\,y'_s = y$ for some $h \in H_c$. Put $n = h\,y'_n$. Since $y \in P_x$ we have $g_c^0(y) \cap q_c \subset g_c^0(x) \cap q$. Therefore

$n \in g_c^0(x) \cap q_c$, hence $hy' \in g_c^0(x) \cap q_c$ and $(hy')_s = y$. Consequently $hy' \in U_\omega$ and hence $y' \in V_\omega$.

PROPOSITION 21. *For any open set* $\omega \subset \omega_0$, U_ω *is open in* P_x *and* V_ω *is open in* q_c . U_ω *is* Z_c *- invariant and contains, along with any element, its s.s. component. Moreover,*

$$h \in H_c \ , \quad h\,U_\omega \cap U_\omega \neq \emptyset \;\Rightarrow\; h \in Z_c \ .$$

PROOF. We only have to prove the last statement. Let $h \in H_c$ be such that $h\,U_\omega \cap U_\omega \neq \emptyset$. Since the generic elements in P_x are dense in P_x, there is a generic element $y \in U_\omega$ such that $hy \in U_\omega$. There exist $h_1, h_2 \in Z_c$ such that $t = h_1^{-1} y \in \omega$ and $h_2 h h_1 t \in \omega$ and both generic. Put $h_0 = h_2 h h_1$. Then $h_0 \cdot a_c = a_c$, so $h_0|_{a_c} \in W$. By the definition of ω_0, $h_0|_{a_c} = k_0|_{a_c}$ for some $k_0 \in H_c$ with $k_0 x = x$. Consequently , $h_0 x = x$, so $h_0 \in Z_c$, hence $h \in Z_c$.
□

4. H-ORBITS IN q

THEOREM 22. *Let* $x \in q$. *Then* H.x *is closed if and only if* x *is semisimple. In this case,* $(H_c.x) \cap q$ *has finitely many connected components; each component is a closed* H-*orbit and* H.x *is the component containing* x.

PROOF. H.x can be closed only when x is s.s. by Cor. 13. Conversely, let $x \in q$ be semisimple. Define ω_0 as before (with respect to x) and let $U = U_{\omega_0} \cap g^0(x)$, $V = H.U$. Then V is an open subset of q. We assert that $V \cap H_c^0.x = H.x$. If $y = hx = kx' \in V$ for some $x' \in U$, $h \in H_c$ and $k \in H$, then we get from Proposition 21, $h^{-1}k \in Z_c$, hence $h.x = k.x$, and so $y = k.x \in H.x$. So H.x is open in $H_c.x \cap q$. This argument can be used for all $x' \in H_c.x \cap q$. Therefore, each H-orbit in $H_c.x \cap q$ is open in $H_c.x \cap q$, showing that they are precisely the connected components of $H_c.x \cap q$, and that they are all closed also, since $H_c.x \cap q$ is closed in q. They are finite in number since $H_c.x \cap a$ is finite for all Cartan subspaces $a \subset q$ and since there are only finitely many H-conjugacy classes of Cartan subspaces in q. □

We now come to nilpotent orbits in q and prove the theorem alluded to under Proposition 14, for the real case.

THEOREM 23. *Let N be the set of nilpotent elements in q. N splits into finitely many* H-*orbits. Moreover, we can write* $N = O_1 \cup \ldots \cup O_s$ *where the* O_i *are disjoint orbits and for* $1 \le i \le s$, $O_i \cup \ldots \cup O_s$ *is a closed set containing* O_i *as an open subset;* $O_s = (0)$.

The closed orbit in N is (0) by Cor. 13. It is enough to show that N splits into finitely many H-orbits. The other assertions are direct consequences of the Baire category theorem.

The proof of Theorem 23 is due to KOSTANT and RALLIS in the complex case. The arguments for the proof in the real case are quite similar. For completeness we include the headlines of the proof in the form of three lemmas.

A set of three linearly independent elements (t,x,y) in g is said to be an S-triple if the relations: $[t,x] = 2x$, $[t,y] = -2y$ and $[x,y] = t$ are satisfied. An S-triple (t,x,y) will be called a *normal* S-triple if $t \in h$ and $x,y \in q$. H operates on the set of normal S-triples by $h(t,x,y) = (ht,hx,hy)$.

LEMMA 24. *Any* $0 \neq x \in N$ *can be embedded in a normal* S-*triple* (t,x,y). *Moreover this sets up a one-to-one correspondence between the set of all* H-*orbits in* $N-(0)$ *and the set of all* H- *conjugacy classes of normal* S-*triples in* g.

PROOF. Similar to [7], Proposition 4. In fact, everything stated is valid if we replace $\mathbb{C}$ by any field of characteristic zero. □

It is well-known that any two elements of an S-triple uniquely determine the third (cf. [6] , Cor. 3.5)

LEMMA 25. *Let* (t,x,y) *be a normal* S-*triple. There exist finitely many* $x_1,\ldots,x_n$ *and* $y_1,\ldots,y_n$ in $N-(0)$ *such that*

(i) $(t,x_1,y_1), \ldots, (t,x_n,y_n)$ *are normal* S-*triples*

(ii) *any normal* S-*triple of the form* (t,x',y') *is* H-*conjugate to one of the normal* S-*triples* (t,x_i,y_i) , $1 \le i \le n$.

PROOF. Let g°, h° and q° be the centralizers of t in g, h and q respectively. Obviously $g^{\circ} = h^{\circ} \oplus q^{\circ}$, since $\sigma t = t$. Let $H^{\circ} \subset H$ be the Lie subgroup corresponding to ad h°. Now let g^2, h^2 and q^2 be respectively the spaces of all $z \in g$, h and q such that $[t,z] = 2z$. Then $g^2 = h^2 \oplus q^2$.
The relation $[g^{\circ},g^2] \subset g^2$ implies $[h^{\circ},q^2] \subset q^2$, so that q^2 is H°-invariant. Let V be the Zariski open set of all $z \in q^2$ such that $[h^{\circ},z] = q^2$.
V is not empty since $x \in V$. V is clearly H°-stable. On the other hand if $z \in V$, then the tangent space to the orbit $H^{\circ}.z \subset V$ at z is just $[h^{\circ},z] = q^2$, so $H^{\circ}.z$ is open in V. By a theorem of Whitney, V has finitely many connected components, which are all H°-invariant, hence they are all of the form $H^{\circ}.z$ for some $z \in V$. So there exist $x_1,\ldots,x_n \in V \subset N$ such that for all normal S-triples (t,x',y') we have: $hx' = x_i$ for some i $(1 \le i \le n)$ and some $h \in H^{\circ}$. Hence $H(t,x',y') = (t,x_i,y_i)$ where $y_i = hy'$. Note that y_i is determined by t and x_i. □

LEMMA 26. *Let* X *be the set of all* $t \in h$ *which appear in normal* S*-triples of the form* (t,x,y). *Then* X *is* H*-stable and splits into finitely many* H*-orbits.*

PROOF. This is similar to the proof of [7], Theorem 2, observing that any $t \in X$ is contained in one of the finitely many H-conjugacy classes of Cartan subalgebras of the reductive Lie algebra h. □

The proof of Theorem 23 follows now easily from the above lemmas.

THEOREM 27. *Let* $x_0 \in q$ *and let* Z *be the centralizer of* x_0 *in* H. *Then* $H.x_0$ *is open in its closure in* q, *is a regularly embedded analytic submanifold of* q *and* $hZ \longmapsto h.x_0$ *is an analytic diffeomorphism of* H/Z *onto* $H.x_0$.

PROOF. Similar to the proof of [11], Part I, Theorem 17. □

3. COMPLEMENTS

In this chapter, g is a real semisimple Lie algebra with Killingform B and σ an involutive authomorphism of g. Let θ be a Cartan involution of g which commutes with σ (such θ exist, see for instance [8], p.153). Let h,q and k,p be the +1 and −1 eigenspaces of σ and θ respectively. Then we have

$$g = h \oplus q = k \oplus p .$$

Let H (resp.K) be the connected Lie subgroup of G with Lie algebra ad h (resp.ad k).

<u>THEOREM 28</u>. *Let* $a \subset q$ *be a Cartan subspace of* g *with respect to* σ. *There exist* $h_0 \in H$ *such that* $h_0 . a$ *is a* θ*-stable Cartan subspace of* g *with respect to* σ.

<u>PROOF</u>. Fix a generic element $x \in q$ such that $a = g^0(x) \cap q$. The orbit H.x is closed in q by Theorem 22. Put

$$f(h) = - B(hx, \theta(hx)) \quad (h \in H).$$

Then f is a positive C^∞-function on H, which takes its minimum in a point $h_0 \in H$. In particular,

$$\frac{d}{dt} f(h_0 . e^{t\,ad_y}) = 0 \quad \text{for all } y \in h.$$

So

$$B(h_0[y,x], \theta(h_0 .x)) + B(h_0 x, \theta(h_0 . [y,x]) = 0 ,$$

hence, since $\theta^2 = 1$, $B(\theta(h_0.x), [y, h_0.x]) = 0$ for all $y \in h$, and also $B([\theta(h_0.x), h_0.x], y) = 0$ for all $y \in h$.
Since B is non-degenerate on $h \times h$, we get $[\theta(h_0.x), h_0.x] = 0$. Note that $\theta(q) = q$. Therefore $\theta(h_0.x) \in h_0.a$ and hence $\theta(h_0.a) = h_0.a$. □

<u>THEOREM 29</u>. *Let both* a *and* b *be* θ*-stable Cartan subspaces of* g *with respect to* σ *which are* H*-conjugate* . *There is* $h_0 \in H \cap K$ *such that* $h_0\, a = b$.

<u>PROOF</u>. Choose $h_1 \in H$ such that $h_1\, a = b$.
Then clearly $h_1(a \cap k) = b \cap k$ and $h_1(a \cap j) = b \cap p$. Since $H = H \cap K$. $\exp ad(h \cap p) = \exp ad(h \cap p) . H \cap K$, being just the Cartan decomposition of H, we can write

$$h_1 = \exp \operatorname{ad} t \,.\, h_0 \quad \text{where } t \in \mathfrak{h} \cap \mathfrak{p},\ h_0 \in H \cap K.$$

Let $z \in \mathfrak{a} \cap \mathfrak{k}$ be arbitrary and put $u = h_1 . z \in \mathfrak{b} \cap \mathfrak{k}$. From

$$h_1 .\ \exp \operatorname{ad} z \,.\, h_1^{-1} = \exp \operatorname{ad} u$$

we get

$$\exp \operatorname{ad} t \,.\, h_0 \,.\, \exp \operatorname{ad} z\ h_0^{-1}\ \exp(-\operatorname{ad} t) = \exp \operatorname{ad} u \quad \text{and}$$

also, by applying θ (which can be lifted to G),

$$\exp(-\operatorname{ad} t) \,.\, h_0 \,.\, \exp \operatorname{ad} z\ h_0^{-1}\ \exp \operatorname{ad} t = \exp \operatorname{ad} u$$

and thus,

$$\exp \operatorname{ad}(-u) \,.\, \exp 2 \operatorname{ad} t \,.\, \exp \operatorname{ad} u = \exp 2 \operatorname{ad} t.$$

By diagonalizing $\operatorname{ad} t$, we see that $\exp \operatorname{ad} u$ and $\exp \operatorname{ad} t$ commute. Consequently, $\exp \operatorname{ad} h_0 . z = \exp \operatorname{ad} u$ for all $z \in \mathfrak{a} \cap \mathfrak{k}$. Therefore $h_0 . z = u$ and thus $h_0(\mathfrak{a} \cap \mathfrak{k}) = \mathfrak{b} \cap \mathfrak{k}$. Similarly $h_0(\mathfrak{a} \cap \mathfrak{p}) = \mathfrak{b} \cap \mathfrak{p}$ and hence $h_0\, \mathfrak{a} = \mathfrak{b}$. □

PROPOSITION 30. *Any θ-stable $\mathfrak{q}$-torus is contained in a θ-stable Cartan subspace of $\mathfrak{g}$ with respect to σ.*

PROOF. Let $\mathfrak{b}$ be a θ-stable $\mathfrak{q}$-torus. Denote by $Z(\mathfrak{b})$ the centralizer of $\mathfrak{b}$ in $\mathfrak{g}$. $Z(\mathfrak{b})$ is both σ and θ-stable. Also $\mathfrak{q}$-rank $Z(\mathfrak{b})$ = $\mathfrak{q}$-rank of $\mathfrak{g}$. Let Z denote the center of $Z(\mathfrak{b})$. Clearly Z is both σ and θ-stable and $\mathfrak{b} \subset Z \cap \mathfrak{q}$. Let $\mathfrak{a}_1$ be a θ-invariant Cartan subspace of $[Z(\mathfrak{b}),Z(\mathfrak{b})] \cap \mathfrak{q}$. Such $\mathfrak{a}_1$ exist by Theorem 28 and the fact that the restriction of θ to $[Z(\mathfrak{b}),Z(\mathfrak{b})]$ is a Cartan involution of $[Z(\mathfrak{b}),Z(\mathfrak{b})]$. Then $\mathfrak{a} = Z \cap \mathfrak{q} \oplus \mathfrak{a}_1$ is a θ-invariant Cartan subspace of $\mathfrak{g}$ with respect to σ, containing $\mathfrak{b}$. □

Let H_σ be the subgroup of G consisting of all $g \in G$ which commute with σ. Clearly H is the connected component of e in H_σ. Let $\mathfrak{a} \subset \mathfrak{q} \cap \mathfrak{k}$ be a torus of maximal possible dimension.
Put $A = \exp \operatorname{ad} \mathfrak{a}$ and let $F = \{a \in A : a^2 = e\}$.
Note that $\operatorname{card} F = 2^{r_0}$ if $r_0 = \dim A$.

THEOREM 31. $H_\sigma = F\,H = H\,F$.

PROOF. Clearly $F \subset H_\sigma$. Indeed, if $x \in \mathfrak{a}$, then $\sigma x = -x$ and hence $a^\sigma = a^{-1}$ for all $a \in A$. For $a \in F$ we have $a = a^{-1}$, so $a^\sigma = a$. Conversely let $h \in H_\sigma$. Put $P = \exp \operatorname{ad} \mathfrak{p}$. Then $G = K\,P$ and $K \cap P = (e)$. Write

$$h = k.\exp \operatorname{ad} x \qquad (k \in K, x \in p).$$

Applying σ to both sides gives

$$h = k^\sigma \exp \operatorname{ad} \sigma(x) ,$$

hence $k = k^\sigma$ and $\sigma(x) = x$. Therefore $k \in K \cap H_\sigma$ and $x \in \mathfrak{h} \cap \mathfrak{p}$, so $h \in (K \cap H_\sigma).H$. The pair $(K, K \cap H_\sigma)$ is a compact symmetric pair. Let L be the connected component of the identity of $K \cap H_\sigma$. Then $L \subset H$. It is known that $K = L\,A\,L$ (see [4] , Theorem 6.7). So, any $y \in K \cap H_\sigma$ can be written as $y = \ell_1\, a\, \ell_2$ with $a \in A$, $\ell_1, \ell_2 \in L$. Applying σ to both sides we get $y = \ell_1^\sigma\, a^\sigma \ell_2^\sigma = \ell_1\, a^\sigma\, \ell_2$, hence $a = a^\sigma = a^{-1}$. Thus $y \in L\,F\,L$, so $h \in L\,F\,L\,H = L\,F\,H = L\,H\,F = H\,F$. □

Theorem 31 generalizes Proposition 20.

4. INVARIANT MEASURES ON H-ORBITS

In this chapter $\mathfrak{g}$ is a real reductive Lie algebra with involution σ . We keep to the notation of the previous chapters.

It is well-known that any G-orbit in $\mathfrak{g}$ admits an invariant measure, which even can be viewed as a tempered Radon measure on $\mathfrak{g}$. Here we present some (partial) results on the existence and properties of invariant measures on H-orbits in $\mathfrak{q}$. It turns out that in general not every H-orbit admits an invariant measure. Let us therefore consider the following example.

Let $G = SL(n,\mathbb{R})$, $H_0 = S(GL(1,\mathbb{R}) \times GL(n-1,\mathbb{R}))$, $(n \geq 3)$.

- Let J be the matrix given by $J = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$ and σ the involution on G given by $\sigma x = JxJ$. Then $H_0 = \{x \in G : \sigma x = x\}$.
- Lifting σ to the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, we get the usual decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with $\mathfrak{q}$ the space of matrices

$$x(p,q) = \begin{pmatrix} 0 & p_1 \cdots p_{n-1} \\ q_1 & \\ \vdots & \Theta \\ q_{n-1} & \end{pmatrix}, \text{ where}$$

$p = (p_1,\ldots,p_{n-1}) \in \mathbb{R}^{n-1}$, $q = (q_1,\ldots,q_{n-1}) \in \mathbb{R}^{n-1}$.

- H_0 acts on q . If

$$g \in H_0 , \quad g = \left(\begin{array}{c|c} \det h^{-1} & 0 \\ \hline 0 & h \end{array}\right) \quad (h \in GL(n-1,\mathbb{R})) ,$$

then

$$g.\ x(p,q) = x\,(\det h^{-1}.p\, h^{-1}, \det h . h\, q)$$

Here we regard p as a (n-1) x 1 matrix and q as a 1 x(n-1) matrix.

- For $x = x(p,q)$, put $Q(x) = \sum_{i=1}^{n-1} p_i q_i$.

Denote H the identity component of H_0 . Then

$$H = \left\{ \left(\begin{array}{c|c} \det h^{-1} & 0 \\ \hline 0 & h \end{array}\right) : h \in GL_+ (n-1,\mathbb{R}) \right\} .$$

- The H-orbits in q are:
  (i) $Q(x) = \alpha$ $(\alpha \neq 0)$ (generic orbits)
  (ii) the four nilpotent orbits

$$0_1 = \{ x(p,0) : p \neq 0 \} , \quad 0_2 = \{ x(0,q) : q \neq 0 \} ,$$

$$0_3 = \{ x(p,q) : Q(x) = 0,\ p \neq 0,\ q \neq 0 \} \text{ and}$$

$$0_o = \{ x(0,0) \} .$$

- The orbits 0_1 and 0_2 do *not* admit an invariant measure, but 0_3 does (0_3 is a so-called q-regular H-orbit).

Since $0_1 = H\, x(e_1,0)$, we have to compute the Haar modulus Δ_1 of Stab $x(e_1,0) =$

$$\left\{ g = \begin{pmatrix} \det h^{-1} & 0 \ . \ . \ . \ 0 \\ 0 & \det h^{-1} \quad * \text{----} * \\ \vdots & 0 \\ 0 & \vdots \qquad * \\ & 0 \end{pmatrix} : \det g = 1,\ h \in GL_{+}(n-1,\mathbb{R}) \right\} .$$

Identifying this group with the group

$$\left\{ \begin{pmatrix} \alpha & * \ \cdots \ * \\ 0 & \\ \vdots & u \\ 0 & \end{pmatrix} : \alpha^2 \det u = 1,\ \alpha > 0 \right\}$$

one easily gets $\Delta_1(g) = \alpha^{-n} = (\det h)^n$.
A similar observation gives $\Delta_2(g) = (\det h)^{-n}$ for $g \in \operatorname{Stab} x(0,e_1)$. Finally, $\operatorname{Stab} x(e_2,e_1)$ is unimodular. We leave the proof to the reader (do it!) .

What can be said about the general situation.
Call $x \in q$, q-regular if $\dim H.x = n - \ell$, where $n = \dim q$. Denote R the set of q - regular elements. Obviously $\underset{\sim}{q}' \subset \underset{\sim}{R}$ and $\underset{\sim}{R}$ is a Zariski open subset of $\underset{\sim}{q}$. Let I be, as before, the algebra of H_c- invariant polynomials on q_c and let $p_1,\dots,p_\ell$ be algebraically independent homogeneous elements of I such that $I = \mathbb{C}[p_1,\dots,p_\ell]$. We may ssume that $p_1,\dots,p_\ell$ are real-valued on q. By a result of KOSTANT and RALLIS ([7] , Theorem 13), the differentials $dp_1,\dots,dp_\ell$ are linearly independent in each point of R. Let $Q : q \longrightarrow \mathbb{R}^\ell$ be the mapping defined by $Q(x) = (p_1(x),\dots,p_\ell(x))$. Then $Q : R \longrightarrow \mathbb{R}^\ell$ is a submersion, hence in particular, $Q(R)$ is an open subset of $\mathbb{R}^\ell$. Fix a translation invariant measure dx (resp. dy) on $\mathbb{R}^\ell$ (resp. $\mathbb{R}$). If $\Omega \subset \mathbb{R}^k$ is an open set, we put $C_0(\Omega)$ the space of continuous functions f on $\mathbb{R}^k$ with compact support and $\operatorname{Supp} f \subset \Omega$.

THEOREM 32. *There exists a well-defined map* $f \longmapsto M_f$ *of* $C_0(R)$ *onto* $C_0(Q(R))$ *such that for all* $\phi \in C_0(Q(R))$ *one has*

$$\int_q \phi(Q(x))\, f(x)\, dx = \int_{Q(R)} \phi(y)\, M_f(y)\,dy .$$

Moreover

$$\text{Supp } (M_f) \subset Q \text{ (Supp f)}.$$

This theorem is a special case of a general theorem by HARISH-CHANDRA (see [3] , p.274).
For $y \in \mathbb{R}^{\ell}$ put $\Gamma_y = \{ x \in R : Q(x) = y \}$. Γ_y is a closed subset of R. If $y \in Q(q')$, then $\Gamma_y = \{ x \in q : Q(x) = y \}$ and hence a closed subset of q. Note that $\Gamma_0 = R \cap N$. Both Γ_0 and Γ_y $(y \in Q(q'))$ are H-stable and splits into finitely many (open) H-orbits of the same dimension (cf.Theorem 22,23). Let $y \in Q(R)$ be fixed. Then $f \longmapsto M_f(y) (f \in C_0(R))$ defines a positive measure on R with support contained in Γ_y. This measure is clearly H-invariant and non-zero, since the map $f \longmapsto M_f$ is surjective. Therefore this measure defines an H-invariant non-zero positive measure on Γ_y and also on each H-orbit, contained in Γ_y. Resuming :

THEOREM 33. *Any* q*-regular* H*-orbit in* q *carries an* H*-invariant positive measure.*

On generic orbits in q, these measures can of course be considered as Radon measures on q. For q-regular nilpotent orbits this is still an open problem (except in special cases). Let $\ell = 1$ and dim $q > 2$. If H.x is a q-regular orbit in q, then the invariant measure on H.x defines a *tempered* Radon measure on q. This cas be shown by the method used in ([10], Proposition 2-5.) .

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