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ON REGULAR NEAR POLYGONS

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On regular near polygons *)

by

A.E. Brouwer & H.A. Wilbrink

ABSTRACT

We develop a structure theory for regular near 2d-gons. Main results are the existence of sub 2j-gons for $2 \leq j \leq d$ and the nonexistence of sporadic 2d-gons for $d \geq 4$ with $s > 1$ and $t_2 > 1$ and $t_3 \neq t_2(t_2 + 1)$.

KEY WORDS & PHRASES: *near n-gon*

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*) This report will be submitted for publication elsewhere.

INTRODUCTION

A *near polygon* is a partial linear space (X,L) such that for any point $p \in X$ and line $\ell \in L$ there is a unique point on ℓ nearest p.

A *regular* near polygon with parameters $(s,t_2,t_3,\ldots,t_d)$ is a near polygon of diameter d such that all lines have s+1 points, each point is on t+1 lines and any point at distance i from a given point x_0 is adjacent to t_i+1 points at distance i-1 from x_0. (Here distances and adjacency are interpreted in the point graph: two points are adjacent iff they are collinear.) Note that $t_0 = -1$, $t_1 = 0$, $t_d = t$.

A subset $Y \subset X$ is called *geodetically closed* if for any two points $y_1, y_2 \in Y$ all shortest paths between y_1 and y_2 are contained in Y. A *quad* is a geodetically closed subset of X of diameter two which is nondegenerate (i.e., not all of its points are adjacent to one fixed point); it follows that a quad is a generalized quadrangle.

SHULT & YANUSHKA [5] showed that if lines have more than two points then any two points $x,y \in X$ with at least two common neighbours determine a unique quad $Q(x,y)$ containing them.

On the other hand, a near polygon with all lines of length two is just a connected bipartite graph. Thus, this paper has two parts: the first part is about *thick* near polygons ($\forall \ell \in L : |\ell| \geq 3$) and the second part (to be published separately) about *thin* near polygons ($\forall \ell \in L : |\ell| = 2$).

In the first case one would like to generalize Yanushka's lemma and obtain the existence of sub 2j-gons for $2 \leq j \leq d$. SHAD & SHULT [4] showed that if each point at distance two from a quad has distance two to exactly one point of this quad then the near polygon contains hexes (geodetically closed sub near hexagons). Here we show that if a near polygon is regular then it contains sub 2j-gons for $2 \leq j \leq d$ (provided $t_2 > 0$). Moreover we prove that there are only very few possibilities for the parameter set of a near polygon.

<u>NOTATION.</u> $\sim$ denotes adjacency;

$\Gamma_i(x)$ is the set of all points at distance i from the point x, and similarly for $\Gamma_i(Y)$.

A. THICK NEAR POLYGONS

Let (X,L) be a fixed near polygon and assume that not all lines are thin.

a) Relation between two lines.

LEMMA 1. *Let* ℓ, m *be two lines. Then either* (i) *or* (ii) *holds.*

(i) *There is an integer* i *such that each point of* ℓ *has distance* i *to* m *and each point of* m *has distance* i *to* ℓ. *It follows that* $|\ell| = |m|$. *In this case* ℓ *and* m *are called parallel.*

(ii) *There are points* $x_0 \in \ell$ *and* $y_0 \in m$ *such that for all* $x \in \ell$ *and* $y \in m$ *we have* $d(x,y) = d(x,x_0) + d(x_0,y_0) + d(y_0,y)$.

PROOF. Trivial. □

Note that being parallel need not be an equivalence relation.

LEMMA 2. *If some shortest path between* x *and* y *contains a line of length* a *then all paths do. In particular, if we remove all lines of size* a *then distances remain the same or become infinite: we get a disjoint union of (geodetically closed) near polygons.*

PROOF. (i) No two edges in a shortest path are parallel.
(ii) Let α be a geodesic between x and y containing the edge uv on a line uv of size a. Let β be any path between x and y not containing lines of size a. Then $\alpha \circ \beta^{-1}$ is a circuit not containing a line parallel to the line uv. But this is impossible: Let us walk around the circuit starting at u. By induction we see that for any vertex of the circuit u is the nearest point on uv. When we reach v we find a contradiction. □

THEOREM 1. *Suppose that any two points at distance two have at least two common neighbours. (In fact it is enough to suppose that if* u *is a common neighbour of* x *and* y *and not both* ux *and* uy *have size two then* x *and* y *have another common neighbour.) If lines of several sizes occur then* (X,L) *is a direct product of near polygons with fixed line sizes:* $(X,L) = \prod_i (X_i, L_i)$,
i.e.,

$$X = \prod_i X_i \text{ and } L = \bigcup_i \{\{z \mid z_i \in \ell \text{ and } \forall j \neq i: z_j = y_j\} \mid y \in X, \ell \in L_i\}.$$

PROOF. Let a be one of the line sizes. All components that arise when all lines of size a are removed are isomorphic since the quads connecting them are rectangular grids. Also, there cannot be paths only using lines of size a from a given point to two distinct points of some component (by Lemma 2). Now all is clear. □

REMARK. Clearly, a direct product of near polygons is again a near polygon. It is regular only if each of the factors is a Hamming cube $(s+1)^{e_i}$ (the direct product of e_i lines of size s+1), and now the product is a Hamming cube $(s+1)^e$ with $e = \Sigma_i e_i$. Hamming cubes are characterized by $t_i = i-1$ $(0 \le i \le d)$.

Now assume that any two points at distance two have at least two common neighbours. By the theorem above we may assume that all lines have the same size s+1 where $s > 1$. (That the line size is constant is not so important, but we often need the presence of three points on a line.)

LEMMA 3. *Let* S(x,y) *be the set of all lines on* x *in a geodesic from* x *to* y. *Then if* $y \sim z$ *and* $d(x,y) = d(x,z)$ *we have* $S(x,y) = S(x,z)$.

PROOF. Let x' be the point on yz closest to x (so that $x' \neq y$ and $x' \neq z$). Let $\ell \in S(x,y)$. Then either ℓ is on a geodesic from x to x' and hence in S(x,z), or ℓ is parallel to yz and again in S(x,z). □

b) Relation between a point and a quad.

As Shult & Yanushka proved, there are two possible relations between a point x and a quad Q: *either* there is a unique point πx in Q closest to x, and $d(x,z) = d(x,\pi x) + d(\pi x,z)$ for all $z \in Q$, *or* the collection of points in Q closest to x forms an *ovoid* in Q, that is, a set of points meeting each line of Q exactly once. In the first case x is called *classical* and in the second case x is called *of ovoid type* with respect to Q.

Let $N_i := N_i(Q) := \{x \in X \mid d(x,Q) = i\}$,

$N_{i,C} := \{x \in N_i \mid x \text{ is classical w.r.t. } Q\}$,

$N_{i,O} := \{x \in N_i \mid x \text{ is of ovoid type w.r.t. } Q\}$.

Note that $N_0 = Q$, $N_d = \emptyset$, $N_{d-1,C} = \emptyset$, $N_{1,O} = \emptyset$.

A regular near polygon is called *classical* if all its point-quad relations are classical; otherwise it is called *sporadic*. CAMERON [2] shows that classical near polygons are dual polar spaces.

Let us first look at the structure of a near polygon in terms of these sets $N_{i,C}$ and $N_{i,O}$ for a fixed quad Q. Most of the following lemmas are due to Shad & Shult. (No regularity is assumed.).

LEMMA 4. *There are no edges between* $N_{i,O}$ *and* $N_{i,C}$ $(2 \le i \le d-2)$.

PROOF. If there exist points of ovoid type then Q is regular with parameters (s,t_2). A point $x \in N_{i,O}$ determines an ovoid O of size $1+st_2$. If x is adjacent to $y \in N_{i,C}$ then each point of O has distance at most one to πy. If $\pi y \in O$ then $O = \{\pi y\}$ and $t_2 = 0$, a contradiction. If $\pi y \notin O$ then πy is incident with $1+st_2 > 1+t_2$ lines, again a contradiction. □

LEMMA 5. *Let* x,y *be adjacent points in* $N_{i,C}$ *such that* $\pi x \ne \pi y$. *Then* $\pi x \sim \pi y$, *the line* $\ell = \langle x,y\rangle$ *is contained in* $N_{i,C}$, *and* $\pi\ell = \langle \pi x, \pi y\rangle$.

PROOF. $d(y,\pi x) = d(y,\pi y) + d(\pi y,\pi x) = i + d(\pi y,\pi x)$. But $d(y,\pi x) \le i+1$, so $\pi y \sim \pi x$. If $z \in \ell$ then $z \notin N_{i-1}$, otherwise $\pi x = \pi y$. Now since z has distance at most i+1 to two points of $\langle \pi x,\pi y\rangle$, it has distance i to some point on this line, so that $\pi\ell \subset \langle \pi x,\pi y\rangle$. Conversely, if $u \in \langle \pi x,\pi y\rangle$ then u has distance at most i+1 to two points of $\ell \subset N_{i,C}$, so it has distance i to some point on that line, i.e., $\pi\ell = \langle \pi x,\pi y\rangle$. □

LEMMA 6. *Let* x,y *be adjacent points in* $N_{i,O}$ *and* N_{i+1}, *respectively. Then* $y \in N_{i+1,O}$ *and* x *and* y *determine the same ovoid.*

PROOF. Obvious. □

LEMMA 7. *Let* x,y *be adjacent points in* $N_{i,O}$. *Then either* $\ell = \langle x,y\rangle$ *intersects* $N_{i-1,O}$ *and* x,y *determine the same ovoid, or* ℓ *does not meet* $N_{i-1,O}$ *and* x,y *determine distinct ovoids.*

PROOF. Obvious. □

LEMMA 8. *Let* ℓ *be a line meeting both* N_i *and* N_{i+1}. *Then* $|\ell \cap N_i| = 1$.

PROOF. Let $x,y \in \ell \cap N_i$. If both x and y are classical then by Lemma 5 we have $\ell \subset N_{i-1} \cup N_i$. Contradiction. If both x and y are of ovoid type, and $z \in \ell \cap N_{i+1}$ then by Lemma 6 the points x,z,y determine the same ovoid, while according to Lemma 7 the points x,y determine distinct ovoids, contradiction. □

LEMMA 9. (i) *Let ℓ be a line contained in $N_{i,O}$. Then the points of ℓ determine $|\ell|$ pairwise disjoint ovoids partitioning* Q.
(ii) *Let ℓ be a line meeting $N_{i-1,C}$ and $N_{i,O}$. Then the points of $\ell \cap N_{i,O}$ determine $|\ell|-1$ ovoids, pairwise intersecting in $p := \pi(\ell \cap N_{i-1,C})$ and partitioning the points at distance two from* p *in* Q.

PROOF. Obvious. □

REMARK. Note that the lines of type considered in this Lemma have s+1 points.

LEMMA 10. *Let $x \in N_{i-1,C}$ and $y \in N_{i+1,O}$. Then* x *and* y *have at most one common neighbour in $N_{i,O}$.*

PROOF. Suppose $u,v \in N_{i,O}$ are common neighbours of x and y. Lines are thick, so let $z \in N_{i+1,O}$ be a third point on the line <u,y>. Let w be the neighbour of z on the line <x,v> (in the quad Q(x,y)). Now by lemma 6 the points u,y,z,v,w all determine the same ovoid, while by Lemma 9(ii) the points v and w determine distinct ovoids. Contradiction. □

LEMMA 11. *Let ℓ,m be lines with $m \subset Q$. We have $\ell \parallel m$ exactly in the following cases:*

(i) $\ell \subset N_{i,C}$, $m = \pi\ell$, $d(\ell,m) = i$,
(ii) $\ell \subset N_{i,C}$, $m \cap \pi\ell = \emptyset$, $d(\ell,m) = i+1$,
(iii) $\ell \subset N_{i,O}$, m *arbitrary*, $d(\ell,m) = i$,
(iv) ℓ *meets* $N_{i,O}$ *and* $N_{i-1,C}$, $\ell \cap N_{i-1,C} = \{x\}$, $\pi x \notin m$, $d(\ell,m) = i$.

PROOF. Obvious. □

c. Relation between two quads.

Let Q be a fixed quad. We shall write N_i for $N_i(Q)$ etc.

LEMMA 12. *Let* Q' *be a quad meeting* N_{i-1}, N_i *and* N_{i+1}. *Then* $Q' \cap N_{i-1} = \{x\}$ *and* $Q' \cap N_i \subset \Gamma_1(x)$. *In particular* $Q' \cap N_i$ *does not contain a line.*

PROOF. $Q' \cap (N_{i-1} \cup N_i)$ is linearly closed and hence a subquadrangle of Q'. If it were nondegenerate it would coincide with Q' (because it contains all neighbours of a point in N_{i-1}). Therefore it must be degenerate and consist of a number of lines through one point. □

{In this case Q' cannot intersect both $N_{i+1,0}$ and $N_{i+1,C}$.}

LEMMA 13. *Let* ℓ *be a line contained in* $N_{i,0}$. *Let* Q' *be a quad containing* ℓ. *Then either* (i) $Q' \subset N_{i,0} \cup N_{i+1,0}$ *and* $Q' \cap N_{i,0} = \ell$,

or (ii) $Q' \subset N_{i-1,0} \cup N_{i,0}$

or (iii) $Q' \subset N_{i-1,C} \cup N_{i,0}$.

PROOF. (i) Assume $Q' \subset N_{i,0} \cup N_{i+1,0}$ and $\{x\} \cup \ell \subset Q' \cap N_{i,0}$ where $x \notin \ell$. Let m be the line joining x to some point of ℓ, so that $m \subset N_{i,0}$. Let n be some line meeting ℓ and $N_{i+1,0}$. Every point of $n \backslash \ell$ determines the same oval and hence is joined to the same point of m. But this is impossible unless ℓ,m,n are concurrent in a point y. Any line through x distinct from m now serves to find a contradiction.

(ii) Now assume $Q' \subset N_{i-1} \cup N_{i,0}$ and $x \in Q' \cap N_{i-1,0}$ and $y \in Q' \cap N_{i-1,C}$. Let $x \sim v \in \ell$, $y \sim w \in \ell$.

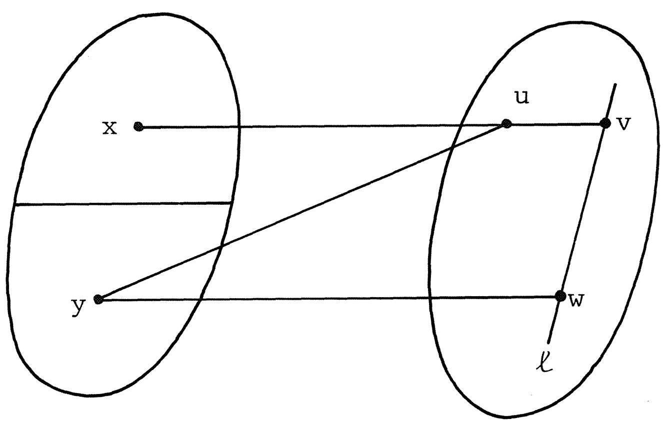
a) If $v \neq w$ then let $y \sim u \in xv$.

Now x,u,v determine the same ovoid O and $\pi y \in O$.

But w determines a disjoint ovoid also containing πy.

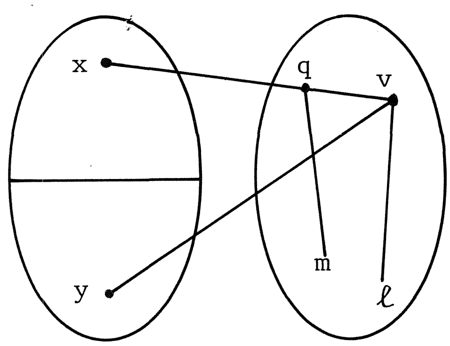
Contradiction.

L13(ii)a

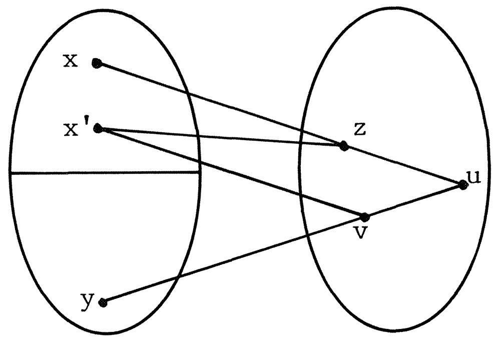


b) Consequently $v = w$, i.e., all points of $Q' \cap N_{i-1}$ are neighbours of v. Let q be a third point on xv and m a line through q in Q', $m \neq xv$.
Now m cannot meet N_{i-1}, so $m \subset N_{i,0}$ and all points of $N_{i-1} \cap Q'$ are neighbours of some point $r \in m$, where $0_r = 0_x$. But $0_q = 0_x$ so $r = q$ and y has two neighbours on the line xqv. Contradiction. □

L13(ii)b



L14



LEMMA 14. *Let* Q' *be a quad meeting* $N_{i,0}$ *and* $N_{i,C}$ *and* $N_{i+1,0}$ *but not* N_{i-1}. *Then* $|Q' \cap N_{i,0}| = 1$ *and* $Q' \cap N_i$ *is an ovoid in* Q'.

PROOF. Let $x, x' \in Q' \cap N_{i,0}$ and $y \in Q' \cap N_{i,C}$. Clearly $Q' \cap N_i$ is a coclique, so $d(x,x') = 2$ and x and x' have a common neighbour $z \in N_{i+1,0}$. It follows that $0_x = 0_{x'}$. Let $y \sim u \in xz$. If $u \neq z$ then let $x' \sim v \in uy$. Now we find that u and v determine the same ovoids, a contradiction. Thus all points in $Q' \cap N_i$ are neighbours of z. Choose a third point $q \in xz$ and a line ℓ in Q' through q, $\ell \neq xz$. By the previous lemma we arrive at a contradiction. □

LEMMA 15. *Let* Q' *be a quad contained in* $N_{i-1} \cup N_{i,0}$. *Then* $Q' \cap N_{i-1,0}$ *is empty, a single point, a line or an ovoid in* Q'.

PROOF. By the previous lemma we may assume $Q' \cap N_{i-1,C} = \emptyset$. If $Q' \cap N_{i-1,0}$ is not a coclique then it contains a line and we are done by lemma 13. If

$Q' \cap N_{i,0}$ does not contain a line then $Q' \cap N_{i-1,0}$ is an ovoid. Therefore, let ℓ be a line in $Q' \cap N_{i,0}$ and let $x,x' \in Q' \cap N_{i-1,0}$. As before it follows that all points in $Q' \cap N_{i-1,0}$ are neighbours of the same point $z \in \ell$. Choose a third point y on the line xz and a second line m in Q' on y, then $m \subset N_{i,0}$ and all points on $Q' \cap N_{i-1,0}$ are neighbours of the same point of m, and this point must be y. Contradiction. □

Since obviously a quad Q' cannot intersect N_j for more than three values of j, or both $N_{i-1,0}$ and $N_{i,C}$ for some i, the lemma's 12-15 give a reasonable idea of the possible relations between Q and Q'. It would be easy but boring to give a complete description of all possibilities.

d. Some more regularity.

LEMMA 16. *Each point is in the same number* t+1 *of lines.*

PROOF. (i) Observe that two intersecting lines determine a unique quad. By our assumption on line length this quad is not thin, i.e. is not $K_{m,n}$, so that each point of the quad Q is in a constant number t_Q+1 of lines.
(ii) Let $x \sim y$, and consider all quads containing the line xy. We find that

$$t(x) + 1 = \Sigma t_Q + 1 = t(y) + 1$$

so that x and y are on the same number of lines. By connectedness of a near polygon we are done. □

LEMMA 17. *Let* t(x,y) *be the number of lines on* x *in some geodesic from* x *to* y. *Then if* $d(x,y) = d(x,z)$ *and* $y \sim z$ *we have* $t(x,y) = t(x,z)$ *and* $t(y,x) = t(z,x)$.

PROOF. (i) $t(x,y) = |S(x,y)|$, so $t(x,y) = t(x,z)$ follows from Lemma 3.
(ii) Consider the quads Q containing the line yz. If Q contains a line of S(y,x) then if x is of classical type w.r.t. Q then *either* $d(x,y) = d(x,Q) + 1$ so that πx,y and z are collinear and Q contains exactly one line from S(y,x) and S(z,x), *or* $d(x,y) = d(x,Q) + 2$ and Q contains exactly $t_Q + 1$ lines from each of S(y,x) and S(z,x). If x is of ovoid type w.r.t. Q then y and z are not in the ovoid O_x determined by x, and again Q contains

exactly $t_Q + 1$ lines from both $S(y,x)$ and $S(z,x)$. Summing up we find $t(y,x) = t(z,x)$. □

THEOREM 2. *Let* $d(x,y) = i$. *Then given a geodesic* $x = x_0, x_1, \ldots, x_i = y$, *there is a geodesic* $y = y_0, y_1, \ldots, y_i = x$ *such that* $d(x_j, y_j) = i$ $(0 \le j \le i)$.

PROOF. Induction on i, $i \le 2$ being clear. Choose points z_j $(1 \le j \le i)$ with $z_1 = x$ and z_j a common neighbour of z_{j-1} and x_j different from x_{j-1} $(2 \le j \le i)$. Put $y_1 = z_i$. By induction hypothesis there is a geodesic $y_1, \ldots, y_i = x$ such that $d(z_j, y_j) = i - 1$ $(1 \le j \le i)$. We now prove by induction on j that $d(x_j, y_j) = i$ and that y_j is of classical type with distance $\min(i+j-k-2, i-j+k-1)$ to the quad $Q(x_k, x_{k+1}, z_k, z_{k+1}) =: Q_k$ $(0 \le k \le i-1)$, with nearest point z_k if $j > k$ and z_{k+1} otherwise.

The induction step goes like this: look at the relation between y_j and the quad Q_{j-1}. The path $y_j, y_{j+1}, \ldots, y_i = x = z_1, \ldots, z_{j-1}$ shows that the distance is at most $i-2$. On the other hand, y_{j-1} is classical at distance $i-2$ (with closest point z_j) w.r.t. Q_{j-1}. It follows that y_j is also classical, and if $d(y_j, Q_{j-1}) = i-3$ then y_j and y_{j-1} would have the same nearest point, but $d(y_j, z_j) = i-1$. Thus $d(y_j, Q_{j-1}) = i-2$ and $d(y_j, x_j) = d(y_j, z_{j-1}) + d(z_{j-1}, x_j) = i-2+2 = i$. Now that $d(x_j, y_j) = i$ we see that y_j has three distinct distances to points of Q_k for each k $(0 \le k \le i-1)$ so that y_j is classical (with the stated distance and nearest point) w.r.t. Q_k.

Remains to start the induction for $j = 1$. It suffices to prove $d(x_1, y_1) = i$. By downward induction on k $(i \ge k \ge 1)$ we show that $d(x_k, y_1) = i-k+1$. For $k \ge i-1$ this is clear. Look at the relation of y_1 w.r.t. Q_k. The distance is at most $i-k-1$, while y is classical w.r.t. Q_k at distance $i-k-1$ with nearest point x_{k+1}. Therefore y_1 is also classical, and since by induction $d(y_1, x_{k+1}) = i-k$ the points y and y' have different nearest points, so $d(y_1, Q_k) = i-k-1$ and y_1 has nearest point z_{k+1} in Q_k so that $d(y_1, x_k) = i-k-1+2 = i-k+1$. This completes the proof. □

COROLLARY. $t(x,y) = t(y,x)$.

PROOF. Choose geodesics as in the theorem. We prove by induction on j that $t(x,y) = t(x_j, y_j)$. For $j = 0$ this is obvious. Let u_j be a third point on $x_{j-1}x_j$. Then $t(x_{j-1}, y_{j-1}) = t(u_j, y_{j-1}) = t(u_j, y_j) = t(x_j, y_j)$ by Lemma 17. □

e) The linear spaces S(x,y)

In Lemma 3 we defined the set S(x,y) as the set of all lines through x in a geodesic from x to y. This set has the structure of a linear space if we take the sets $\{\ell \mid x \in \ell \subset Q\}$ (Q is a quad on x) as lines. We can indicate a lot of subspaces:

LEMMA 18. *Let* $x = x_0, x_1, \ldots, x_i = y$ *be a geodesic from* x *to* y. *Then*

$$\emptyset = S(x,x_0) \subset S(x,x_1) \subset \ldots \subset S(x,y)$$

is a strictly ascending chain of subspaces of S(x,y). *In particular, if we write* $L_x := \{\ell \in L \mid x \in \ell\}$ *then every* S(x,y) *is a subspace of* L_x.

PROOF. Only 'strictly' requires proof, but this follows from Theorem 2. □

LEMMA 19. *Let for some quad* Q, $x \in N_{i+1,0}(Q)$. *Then* $\{\ell \mid x \in \ell$ *and* ℓ *meets* $N_{i,0}\}$ *is a subspace of* L_x.

PROOF. Let ℓ, m be two lines on x meeting $N_{i,0}$. Let $Q' = Q(\ell,m)$. If Q' meets N_{i-1} then $Q' \cap N_{i-1} = \{y\}$ and by Lemma 10 we have $y \in N_{i-1,0}$ so that all lines on x in Q' meet $N_{i,0}$. If $Q' \cap N_{i-1} = \emptyset$ then we are done by Lemma 14. □

LEMMA 20. *Let* $\ell_1, \ldots, \ell_r$ *be* r *lines on* x. *Then there is a point* y *with* $d(x,y) \le r$ *such that* $\{\ell_1, \ldots, \ell_r\} \subset S(x,y)$.

PROOF. Induction on r. □

REMARK. In case our near polygon is regular with parameters $(s,t_2,t_3,\ldots,t)$, our linear spaces are block designs with $\lambda = 1$ (Steiner systems), and we find some restrictions such as $t_2 | t_i$ and $t_2(t_2+1) | t_i(t_i+1)$ for $1 \le i \le d$.

f) Counting with respect to a quad

Thus far we considered the not necessarily regular case. Now assume that our near polygon has parameters $(s,t_2,t_3,\ldots,t_d)$.

LEMMA 21. *Fix* $x \in N_{i,C}$.

(i) x *is incident with* $1+t_i$ *lines meeting* $N_{i-1,C}$.

(ii) x *is incident with* $(1+t_2)(t_{i+1}-t_i)$ *lines contained within* $N_{i,C}$.

(iii) x *is incident with* $t-t_{i+2}$ *lines meeting* $N_{i+1,C}$.

(iv) x *is incident with* $t_{i+2}-t_i-(1+t_2)(t_{i+1}-t_i)$ *lines meeting* $N_{i+1,0}$.

PROOF. (i) is obvious. According to Lemma 2 we find for any line ℓ in Q incident with πx, $t_{i+1}-t_i$ lines contained within $N_{i,C}$ and projecting onto ℓ. This proves (ii). Fix a point $y \in Q$ with $d(y,\pi x) = 2$. Then $d(x,y) = i+2$. Claim: The lines through x meeting $N_{i+1,C}$ are exactly those not meeting $\Gamma_{i+1}(y)$.

For: any neighbour of x in $N_{i+1,C}$ has distance i+1 to πx and to no other point of Q. Conversely, let ℓ be a line on x not meeting $N_{i+1,C}$. If ℓ meets $N_{i+1,0}$ then by Lemma 6(ii) one of the points of $\ell \cap N_{i+1,0}$ determines an ovoid containing y. If ℓ is contained in $N_{i,C}$ then $\pi\ell$ is a line through πx and contains a neighbour of y. Finally, if ℓ meets $N_{i-1,C}$ then y has distance i+1 to the point $\ell \cap N_{i-1}$. □

This proves (iii). Now (iv) follows since x is incident with 1+t lines and our four cases exhaust all possibilities. □

COROLLARY.

$$|N_{i,C}| = (1+s)(1+st_2).s^i \prod_{j=2}^{i+1} (t-t_j) / \prod_{j=2}^{i} (1+t_j).$$

LEMMA 22.

(i) *The number of lines incident with* $x \in N_{2,0}$ *and meeting* $N_{1,C}$ *is* $(1+t_2)(1+st_2)$.

(ii) *The number of lines incident with* $x \in N_{2,0}$ *and contained in* $N_{2,0}$ *is* $(1+t_3) - (1+t_2)(1+st_2)$.

(iii) $|N_{2,0}| = s^2(1+s)(t-t_2)(t_3-(1+t_2)t_2)/(1+t_2)$.

PROOF. Let x determine the ovoid $0 \subset Q$, so that $|0| = 1+st_2$. Now (i) is clear. For each point $y \in Q\backslash 0$ there are $1+t_3-(1+t_2)(1+st_2)$ lines through x in $N_{2,0}$ containing a point of $\Gamma_2(y)$. There are $s(1+st_2)$ choices for y, and each line is counted $s(1+st_2)$ times. □

COROLLARY. *If* $t_3 \neq t_2(1+t_2)$ *then* $1+t_3 \geq (1+t_2)(1+st_2)$.

LEMMA 23. $|N_{i,0}| = s^i(s+1)(t_{i+1} - t_2(1+t_i)) \Pi_{j=2}^{i} (t-t_j)/\Pi_{j=2}^{i}(1+t_j)$.

PROOF. Count triples (x,y,z) with $x,y \in Q$, $d(x,y) = 1$, $d(y,z) = i$, $d(x,z) = i+1$. We find

$$|Q|.s(t_2+1).p^1_{i,i+1} = |N_{i,C}|.s(t_2+1) + |N_{i-1,C}|.s(t_2+1).st_2$$
$$+ |N_{i,0}|.(st_2+1).s(t_2+1).$$

But

$$p^1_{i,i+1} = |\Gamma_{i+1}|.\frac{(t_{i+1}+1)}{s(t+1)}$$

and

$$|\Gamma_i| = s^i \prod_{j=0}^{i-1} \frac{t-t_j}{1+t_{j+1}} \qquad \text{(where } t_0 = -1,\ t_1 = 0\text{)},$$

and $|N_{i,C}|$ is known by the corollary to Lemma 21. Substitution now gives the result. □

REMARK. Similar counting proves that

$$|N_{i,C}| = p^2_{i,i+2}.(1+s)(1+st_2)$$

and

$$p^2_{i,i+2} = |\Gamma_{i+2}|.\frac{(1+t_{i+1})(1+t_{i+2})}{s^2t(t+1)}$$

which is equivalent to our previous result.

COROLLARY. $t_{i+1} \geq t_2(1+t_i) \qquad (1 \leq i \leq d-1)$.

LEMMA 24. *Fix* $x \in N_{i,0}$.
(i) x *is incident with* $t - t_{i+1}$ *lines meeting* $N_{i+1,0}$.
(ii) *Let* x *be incident with* a_C, a_0 *and* a_I *lines meeting* $N_{i-1,C}$, $N_{i-1,0}$ *and contained within* $N_{i,0}$, *respectively. Then*

a) $a_c + a_0(1+st_2) = (1+t_i)(1+st_2)$

b) $a_c + a_0 + a_I = 1 + t_{i+1}$.

PROOF. Let O be the ovoid determined by x. For (i) choose a point $y \in Q\backslash O$ and observe that the lines through x meeting $N_{i+1,0}$ are exactly the lines through x going away from y. For (iia) count pairs (p,ℓ) where $p \in O$ and ℓ is a line incident with x and meeting $\Gamma_{i-1}(p)$. (iib) is obvious. □

COROLLARY. *If* $N_{i,0} \neq \emptyset$ *then* $N_{j,0} \neq \emptyset$ *for* $i \leq j \leq d-1$.

REMARK. Averaging over $x \in N_{i,0}$ we find

$$\bar{a}_0 = \frac{(1+t_i)(t_i-t_2(1+t_{i-1}))}{t_{i+1}-t_2(1+t_i)},$$

$$\bar{a}_c = \frac{(1+st_2)(1+t_i)(t_{i+1}-t_i-t_2(t_i-t_{i-1}))}{t_{i+1}-t_2(1+t_i)}$$

Lemma 21 shows that we know everything about points of classical type. Unfortunately we see no way to determine a_c and a_0 for $i \geq 3$ and x of ovoid type, except in some special cases. For example, if Q does not admit a partition into ovoids then no set $N_{i,0}$ contains a line and for each $x \in N_{i,0}$ we have $a_I = 0$. Now it follows that

$$a_c = (t_{i+1} - t_i)(1 + st_2)/st_2 \quad \text{and} \quad a_0 = \frac{(1+t_i)(1+st_2)-(1+t_{i+1})}{st_2},$$

but we know $\bar{a}_0$, and thus find a quadratic equation for t_{i+1}:

$$st_2(1+t_i)(t_i-t_2(1+t_{i-1})) = (t_{i+1}-t_2(1+t_i))((1+t_i)(1+st_2)-(1+t_{i+1})).$$

For example, if $N_{2,0} \neq \emptyset$ and $d \geq 4$ then by Lemma 22(ii) we have

$$1 + t_3 = (1 + t_2)(1 + st_2)$$

and the above equation yields for $i = 3$ the existence of two integers with sum $(1+t_3)(1+st_2-t_2) - 1$ and product $(1+t_3)st_2(t_3-t_2(1+t_2))$.

If $s = t_2 = q$ then one easily verifies that the discriminant can be a square only for $q = 2$. But if $q = 2$ the quadratic reduces to $(t_4-50)(t_4-54) = 0$, hence $t_4 \in \{50,54\}$. If $t_4 = 50$ and $d = 4$ one finds that the multiplicity of the eigenvalue $-t-1$ is nonintegral. If $t_4 = 50$ and $d > 4$ then we again have a quadratic for t_5:

$$(t_5-102)(254-t_5) = 2.2.51.20 = 4080$$

which does not have an integral solution. Therefore $t_4 = 54$. But below we shall show that $(t_3-t_2)|(t_4-t_3)$. In this case we find $(14-2)|(54-14)$, a contradiction.

Thus we proved that if a regular near polygon has $d \geq 4$, $s = t_2 > 1$ and one if its quads does not admit a partition into ovoids than the near polygon is classical. In particular this holds for $s = t_2 \in \{2,3,4\}$.

[In fact the situation seems to be as follows: the classical generalized quadrangle corresponding to $O_5(q)$ (called $Q(4,q)$) has $s = t_2 = q$. For $q = 2,3,4,5,7$ all ovoids of the quadrangle are intersections of the quadric with a 3-space (hyperplane) - consequently no two ovoids are disjoint. For $q = 8$ there are two kinds of ovoids: those on a hyperplane and those corresponding to a Tits-ovoid, but any two ovoids intersect. (In general, if q is even and N is the nucleus of the quadric then for any ovoid in the quadrangle we find an ovoid in th 3-space $N^{\perp}/N$.) Kantor constructed large classes of ovoids for odd prime powers q as follows: Let $Q(x,y,z,u,v) = xv + yu + z^2$. Let $\sigma \in Aut(\mathbb{F}_q)$. Let $-k$ be a nonsquare in $\mathbb{F}_q$. Then $\{<1,y,z,ky^{\sigma},-z^2-ky^{\sigma+1}>\} \cup \{<0,0,0,0,1>\}$ is an ovoid in $O_5(q)$. (Such ovoids are intersections of the quadric with a hyperplane iff $\sigma = 1$.) For $q = 9$ we found several sets of five pairwise disjoint ovoids, but no partition into ovoids. $Q(4,q)$ is selfdual when q is even. For odd q its dual $Q(4,q)^*$ does not possess ovoids. No other generalized quadrangles with $s = t_2$ are known. For $s = t_2 \in \{2,3,4\}$ it is known that there are no others. For us this means that for no quad with $s = t_2$ it is known that there is a partition into ovoids, while for $s = t_2 \in \{2,3,4\}$ there certainly isn't.]

Below we shall prove that all quads in a sporadic regular near polygon of diameter ≥ 3 do admit partitions into ovoids, with the unique exception of GQ(2,2) in the near hexagon with parameters $(s,t_2,t) = (2,2,14)$ on 759 points.

g) Eigenvalues

A regular near polygon defines a distance regular graph $(X,\sim)$, and the usual eigenvalue techniques are applicable. We have

$$v = |X| = \sum_{i=0}^{d} \frac{s^i \prod_{j=0}^{i-1}(t-t_j)}{\prod_{j=1}^{i}(1+t_j)} = (1+s)\left(\sum_{i=0}^{d-1} s^i \frac{\prod_{j=1}^{i}(t-t_j)}{\prod_{j=1}^{i}(1+t_j)}\right).$$

Let A be the adjacency matrix, and A_i the matrix with entries

$$(A_i)_{xy} = \begin{cases} 1 \text{ if } d(x,y) = i \\ 0 \text{ otherwise} \end{cases}.$$

Now $A_0 = I$, $A_1 = A$, $\Sigma_{i=0}^{d} A_i = J$. All A_i are polynomials in A and hence simultaneously diagonisable. Number the eigenspaces in some arbitrary way, but such that those corresponding to the minimal idempotents $\frac{1}{v}J$ and $\frac{1}{v(s^{-1})} \Sigma_{i=0}^{d} (-\frac{1}{s})^i A_i$ are numbered 0 and 1 respectively. (Here $v(s^{-1})$ denotes v with s^{-1} substituted for s.) In these eigenspaces A_i has eigenvalues

$$|\Gamma_i(x)| = \frac{s^i \prod_{j=0}^{i-1}(t-t_j)}{\prod_{j=1}^{i}(1+t_j)} \quad \text{and} \quad \frac{(-1)^i \prod_{j=0}^{i-1}(t-t_j)}{\prod_{j=1}^{i}(1+t_j)}, \text{ respectively.}$$

In particular we find for $i = 1$ that A has eigenvalues $s(t+1)$ and $-(t+1)$ here; the first is the largest eigenvalue and has multiplicity one since the graph is connected. The second is the smallest eigenvalue [for: let N be the point-line incidence matrix. Then $NN^t = A + (t+1)I$ is positive semi-definite].

Write $P_{ij} = \lambda_i(A_j)$ = eigenvalue of A_j in i-th eigenspace. Then the Krein condition $q_{11}^r \geq 0$ is equivalent to $\Sigma_{i=0}^{d} s^{-2i} \lambda_r(A_i) \geq 0$. Thus:

PROPOSITION. $\Sigma_{i=0}^{d} s^{-2i} A_i$ *is positive semidefinite.* □

For $r = 1$ we find

PROPOSITION. $\Sigma_{i=0}^{d} \frac{(-1)^i}{s^{2i}} \frac{\prod_{j=0}^{i-1}(t-t_j)}{\prod_{j=1}^{i}(1+t_j)} \geq 0$. *Factoring out a factor* $(1-\frac{1}{s^2})$ *we find that either* $s = 1$ *or*

$$\sum_{i=0}^{d-1} \frac{(-1)^i}{s^{2i}} \prod_{j=1}^{i} \frac{t-t_j}{1+t_j} \geq 0.$$

In particular:

- *if* d = 2 *then* s = 1 *or* $t \leq s^2$.
- *if* d = 3 *then* s = 1 *or* $t^2 - ((s^2+1)(t_2+1)-1)t + s^4(t_2+1) \geq 0$.

(Very roughly this last condition says that $t \gtrsim s^2 t_2$ or $t \lesssim s^2$.)

- *if* d = 4 *then* s = 1 *or* $t^3 - (s^2(t_3+1) + t_3 + t_2)t^2 + {}$
$$+ (s^4(t_2+1)(t_3+1) + s^2(t_3+1)t_2 + t_2 t_3)t - s^6(t_2+1)(t_3+1) \leq 0.$$

  It follows that $t \leq s^2(t_3-t_2+1) + t_3$.
- *if* d *is even then* $1 + t \leq (s^2+1)(1+t_{d-1})$. □

In the special case of a generalized quadrangle (d = 2) we have

$$P = \begin{pmatrix} 1 & s(t+1) & s^2 t \\ 1 & -(t+1) & t \\ 1 & s-1 & -s \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} 1 \\ s^2(st+1)/(s+t) \\ st(s+1)(t+1)/(s+t) \end{pmatrix}$$

where μ_j is the rank of the j-th eigenspace.

In the special case of a near hexagon (d = 3) we have

$$\begin{pmatrix} 1 & s(t+1) & \frac{s^2 t(t+1)}{t_2+1} & \frac{s^3 t(t-t_2)}{t_2+1} \\ 1 & -(t+1) & \frac{t(t+1)}{t_2+1} & -\frac{t(t-t_2)}{t_2+1} \\ 1 & \alpha & (s-1)\alpha - (s^2-s+1) & -s\alpha + s(s-1) \\ 1 & \beta & (s-1)\beta - (s^2-s+1) & -s\beta + s(s-1) \end{pmatrix}$$

where the numbers α and β are the roots of

$$x^2 - (s-1)(t_2+2)x + (s^2-s+1)(t_2+1) - s(t+1) = 0$$

and, say, $\alpha > \beta$. [By SHAD & SHULT [4] α and β are integers. Consequently $(s-1)^2(t_2+2)^2 - 4(s^2-s+1)(t_2+1) + 4s(t+1)$ is a square.] The multiplicity of the eigenvalue $-(t+1)$, i.e, the rank of the first eigenspace, is

$$\frac{(v/(s+1)).s^3(t_2+1)}{s^2(t_2+1) + st(t_2+1) + t(t-t_2)}$$

where $v = (s+1)(1 + st + \frac{s^2t(t-t_2)}{t_2+1})$.

The Krein condition $q_{11}^3 \geq 0$ yields for $s > 1$ that $t + 1 \leq (s^2-s+1)(s+1+t_2)$, or, equivalently, $t \leq s^3 + t_2(s^2-s+1)$. (This is the MATHON bound.)

In the case of a classical near hexagon ($t_3 = t_2(t_2+1)$) we can be somewhat more explicit: we have

$$\alpha = s(t_2+1) - 1,$$

$$\beta = s - (t_2+1),$$

$$\text{rank } E_1 = \frac{s^3(1+st_2)(1+st_2^2)}{(s+t_2)(s+t_2^2)},$$

$$v = (1+s)(1+st_2)(1+st_2^2),$$

We have the following possibilities:

name	$O^+(6,q)$	$O(7,q)$	$O^-(8,q)$	$Sp(6,q)$	$U(6,q^2)$	$U(7,q^2)$
s	1	q	q^2	q	q	q^3
t_2	q	q	q	q	q^2	q^2

In the case of a classical near octagon ($t_3 = t_2(t_2+1)$ and $t_4 = t_2(t_3+1)$) we find (with $q := t_2$):

$$q_{11}^1 = C.(s^2-1)(s^2-q)(s^2-q^2)(s^2-q^3)$$

for some positive constant C, so that $q_{11}^1 = 0$ for all classical near octagons, except those with $s = q^2$.

In the case of a near octagon with classical hexes ($t_2 = q$, $t_3 = q^2+q$)

we know that $t = \frac{q^e-1}{q-1}$ with $e \geq 4$ (since we have a projective space locally, cf. section e). If $e = 4$ the near octagon is classical; if $e > 4$ (and $s > 1$, $t_2 > 0$) then $q^1_{11} < 0$, a contradiction. Consequently a near classical near octagon is classical.

h) The case $1 + t_3 = (1+t_2)(1+st_2)$

THEOREM 3. *If a regular near hexagon satisfies* $s > 1$, $t_2 > 0$, $1 + t_3 = (1+t_2)(1+st_2)$ *then it is the unique regular near hexagon with* $s = t_2 = 2$, $v = 759$.

PROOF. First suppose that $s = t_2$. Considering μ_1 the multiplicity of the eigenvalue $-(t+1)$ we see that $\mu_1 \in \mathbb{N}$ implies $s \in \{1,2\}$. By assumption $s > 1$ so that $s = t_2 = 2$. It is known that the regular near hexagon with parameters $(s,t_2,t) = (2,2,14)$ is unique (see BROUWER [7]).

Now return to the general case; by counting things we shall see that both $s \geq t_2$ and $s \leq t_2$, a contradiction.

Consider the possible relations of a quad Q' to a fixed quad Q. If $Q \cap Q' = \emptyset$ then $Q \cap \Gamma_1(Q')$ is a subquadrangle of Q meeting all lines of Q [note that $1+t = (1+t_2)(1+st_2)$ implies that $N_2(Q')$ does not contain any lines by Lemma 22(ii)] so is A. an ovoid, B. a point and its neighbours, C. a subquadrangle $GQ(s,t_2/s)$ or D. all of Q. the other possibilities are E. $|Q \cap Q'| = 1$, F. $Q \cap Q'$ is a line, G. $Q = Q'$.

By Mathon's bound $t \leq s^3 + t_2(s^2-s+1)$ while in any sporadic regular near hexagon $1 + t \geq (1+t_2)(1+st_2)$. Combining these we see that $\frac{t_2}{s} < \frac{1+\sqrt{5}}{2} < 2$. Since we assumed $s \neq t_2$ it follows that case C. does not occur.

Choose a point $x \in N_2(Q)$. It is incident with $\frac{t(t+1)}{t_2(t_2+1)} = (1+s+st_2)(1+st_2)$ quads, $1 + st_2$ of which intersect Q.

Write n_T for the number of quads of type T on x, $T \in \{A,B,E\}$. We have

$$n_A + n_B + n_E = (1+s+st_2)(1+st_2)$$

$$n_E = 1 + st_2,$$

$$(t_2+1)n_B = t(1+t_2)^2(1+st_2),$$

where the last equation is obtained by counting pairs (ℓ,Q') with

$\ell \subset N_1(Q) \cap Q'$. It follows that $n_A = (s-t_2)(1+t_2)(1+st_2)$, and hence $s \geq t_2$.

Write N_T for the total number of quads of type T, $T \in \{A,B,D,E,F,G\}$. We have

$$N_A = \frac{|N_2(Q)|.n_A}{s(1+st_2)} = s(s+1)t_2(1+t_2)(s-t_2)(st-t+t_2),$$

$$N_B = \frac{|N_2(Q)|.n_B}{s^2t_2} = (s+1)t_2(1+t_2)(1+st_2)(st-t+t_2).$$

Counting pairs of interesting lines in $N_1(Q)$ we find

$$t_2(t_2+1).N_B + (s+1)(st_2+1)t_2(t_2+1).N_D = (s+1)(st_2+1).s(t-t_2).(1+t_2)t_2^3,$$

so

$$N_D = st_2^2(1+t_2)(t_2-s).$$

It follows that $t_2 \geq s$ and we proved the theorem. □

We shall see that any regular near polygon contains sub near hexagons. It follows that in any sporadic near polygon we have $1 + t_3 > (1+t_2)(1+st_2)$, for otherwise we would have $(s,t_2,t_3) = (2,2,14)$, and we already saw that this is impossible when $d > 3$.

i) A divisibility condition

In this section we prove a rather strong divisibility condition and introduce the methods used in the next section to construct hexes.

THEOREM 4. *Let* $s > 1$ *and* $t_2 > 0$. *Then* $\frac{t_j-t_{j-1}}{t_i-t_{i-1}}$ *is integral for all* i,j *with* $1 \leq i \leq j \leq d$.

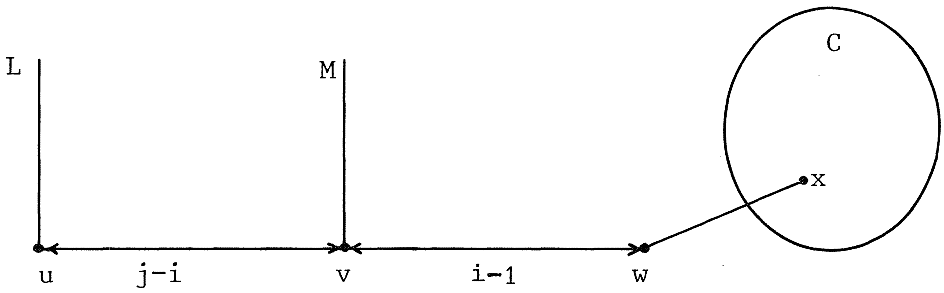
PROOF. Fix three points u,v,w with $d(u,v) = j-i$, $d(v,w) = i-1$ and $d(u,w) = j-1$. Fix a line L through u such that $d(w,L) = d(w,u)$.

CLAIM. (i) w is incident with t_j-t_{j-1} lines parallel to L.
(ii) Every line through w parallel to L intersects exactly one com-

ponent C of $\Gamma_j(u) \cap \Gamma_i(v)$.

(iii) If one line through w meeting a component C of $\Gamma_j(u) \cap \Gamma_i(v)$ is parallel to L, then all lines on w meeting C are parallel to L.

(iv) Given a component C of $\Gamma_j(u) \cap \Gamma_i(v)$, there are either 0 or $t_i - t_{i-1}$ lines on w meeting C.



Clearly (i) - (iv) imply the lemma.

Ad(i): Choose a point $z \in L\backslash\{u\}$. Now $d(z,w) = j$ and there are $t_j + 1$ lines on w meeting $\Gamma_{j-1}(z)$. $t_{j-1}+1$ of these lines also meet $\Gamma_{j-2}(u)$. The remaining $t_j - t_{j-1}$ are parallel to L.

Ad(ii): Let N be a line on w parallel to L. Then $N\backslash\{w\} \subset \Gamma_j(u) \cap \Gamma_i(v)$.

Ad(iii): Let $x \in C$, $d(x,L) = j-1$ and $x \sim x' \in C$. If $z \in L$ with $d(x,z) = j-1$ then $z \neq u$ and $d(x',z) \leq j$ so that $d(x',L) = j-1$. Thus $d(x',L) = j-1$ for all $x' \in C$.

Ad(iv): Note that any component C of $\Gamma_j(u) \cap \Gamma_i(v)$ also is a component of $\Gamma_i(v)$. Let $x \in C$, $x \sim w$. Let M be a line on v parallel to wx. Now $M \in S(v,x)$ and hence (by Lemma 3) $M \in S(v,x')$ for all $x' \in C$. Consequently any line on w meeting C is parallel to M and by (i) there are at most $t_i - t_{i-1}$ such lines.

Consider the t_i lines on x distinct from wx and meeting $\Gamma_{i-1}(v)$. Each of these determines together with wx a quad Q, and we find t_i/t_2 such quads. Note that Q cannot intersect $\Gamma_{i+1}(v)$, so either Q is of classical type w.r.t. v (and $d(v,Q) = i-2$) or Q is of ovoid type w.r.t. v (and $d(v,Q) = i-1$).

The first case occurs exactly $t_{i-1}+1$ times and the latter $\frac{t_i}{t_2} - (t_{i-1}+1)$ times.

<u>LEMMA 25</u>. *Let* Q *be a generalised quadrangle with thick lines. Let* X *be a*

point and O *an ovoid in* Q. *Then* $Q\setminus O$ *and* $Q \cap \Gamma_2(x)$ *are connected.*

PROOF. Easy exercise. (Or see [8].) □

Consequently, for each of our quads Q we have $Q \cap \Gamma_i(v) \subset C$. Counting lines through w meeting C we find at least

$$1 + (t_{i-1}+1)(t_2-1) + (\frac{t_i}{t_2} - (t_{i-1}+1))t_2 = t_i - t_{i-1}$$

such lines. But we already know that there are no more. □

As an application we see that there are no regular near octagons with parameters $(s,t_2,t_3,t) = (2,1,11,39)$ or $(2,2,14,54)$. It follows that there are no sporadic regular near octagons with $s = 2$, $t_2 > 0$.

j. The existence of sub near polygons

DEFINITION. If $A \subset \mathcal{L}_x$ then rank $A = \min\{i \mid \exists y \in \Gamma_i(x): A \subset S(x,y)\}$. Clearly $0 \leq \text{rank } A \leq d$ and rank $A = 0$ iff $A = \emptyset$.

THEOREM 5. *Suppose* $s > 1$, $t_2 > 0$. *Suppose rank* $(S(x,y) \cap S(x,z)) \geq d(x,z)$. *Then there is a point* y' *such that* y' *and* y *lie in the same component of* $\Gamma_i(x)$ *and* z *is on a geodesic from* x *to* y'. *In particular* $S(x,z) \subseteq S(x,y)$.

PROOF. Induction on $d(x,y)$ - for $d(x,y) = 0$ the assertion is trivial. For fixed $d(x,y)$ induction on $d(x,z)$ - for $d(x,z) = 0$ we put $y' = y$ and the theorem is true.

By induction on k ($0 \leq k \leq d(x,z)$) we find points z_k such that $d(x,z_k) = k$ and $S(x,z_k) \cap S(x,y) \cap S(x,z) \supsetneq S(x,z_{k-1})$ (for $k > 0$). (As follows: put $z_0 = x$. Having found z_k ($k < d(x,z)$) we see that rank $(S(x,y) \cap S(x,z_k)) \geq k$ and rank $(S(x,z) \cap S(x,z_k)) \geq k$ [for: $|S(x,y) \cap S(x,z_k)| > t_{k-1} + 1$ etc.], so that by the theorem $S(x,z_k) \subset S(x,y) \cap S(x,z)$. Now choose a line $L \in S(x,y) \cap S(x,z)\setminus S(x,z_k)$. Choose z_{k+1} such that $z_{k+1} \sim z_k$ and $z_k z_{k+1} \parallel L$. It follows that $z_{k+1} \in \Gamma_{k+1}(x)$ and $S(x,z_k) \cup \{L\} \subset S(x,z_{k+1}) \cap S(x,y) \cap S(x,z)$.)

If $h := d(x,z)$ then put $u = z_{h-1}$ and $v = z_h$.

Suppose we know the truth of the theorem in the special case where z has a neighbour z' with d(x,z') = d(x,z)-1 and $S(x,z') \subset S(x,y)$. Then we find $S(x,v) \subset S(x,y)$ and $S(x,v) \subset S(x,z)$ hence (since their cardinalities are equal) S(x,v) = S(x,z) so that $S(x,z) \subset S(x,y)$ and by our assumption we are done.

Therefore it suffices to prove the theorem in this special case. Let h := d(x,z) and i := d(x,y).

Let $y = y_0, y_1, \ldots, y_r = z$ be a path from y to z with the following properties:

(i) $d(x,y_\alpha) \le i$ for $0 \le \alpha \le r$.

(ii) If $d(x,y_\alpha) = \delta$ then rank $(S(x,y_\alpha) \cap S(x,y)) \ge \delta$.

(iii) Under the conditions (i) and (ii), the path keeps as far from x as possible, i.e., it is impossible to replace a point of the path by one or more points, each farther from x than the original point.

The only function of the assumption $S(x,y) \supseteq S(x,z')$ is to ensure that such paths exist: a geodesic from y to x followed by a geodesic from x to z over z' satisfies (i) and (ii), and therefore there also is a path satisfying (i) - (iii).

Suppose our path contains two successive points at the same distance j from x, where j < i. Then we can find three successive points w_0, w_1, w_2 such that $d(x,w_0) = j+1$ and $d(x,w_1) = d(x,w_2) = j$. Let w be a common neighbour of w_0 and w_2 distinct from w_1. The line w_1w_2 contains a point of $\Gamma_{j-1}(x)$, so x is classical w.r.t. $Q(w_0,w_1,w_2)$ and it follows that d(x,w) = j+1 and $S(x,w_0) = S(x,w)$. But this means that we can replace w_1 by w in the path, violating (iii). Contradiction.

Next suppose that $d(x,y_\alpha) \le d(x,y_{\alpha-1})$ for $1 \le \alpha \le e$ and $d(x,y_{e+1}) > d(x,y_e)$. Write $(w_0,w_1,w_2) = (y_{e-1},y_e,y_{e+1})$, $j = d(x,y_e)$ so that we have $d(x,w_1) = j$, $d(x,w_0) = d(x,w_2) = j+1$, and $S(x,w_1) \subset S(x,w_0) \subset S(x,y)$.

If j = i-1 then let L be a line in $S(x,y) \cap S(x,w_2) \backslash S(x,w_1)$. Then both w_1w_0 and w_1w_2 are parallel to L and by the proof of theorem 4 it follows that w_0 and w_2 are in the same component of $\Gamma_i(x)$, contradiction.

If j < i-1 then consider the quad $Q = Q(w_0,w_1,w_2)$. If rank$(S(x,w_0) \cap S(x,w_2)) \ge j+1$ then by induction w_0 and w_2 lie in the same component of $\Gamma_{j+1}(x)$, and we can replace w_1 by some points in $\Gamma_{j+1}(x)$, contradiction. Therefore $S(x,w_0) \cap S(x,w_2) = S(x,w_1)$. Since $S(x,w_0) \ne S(x,w_2)$ it follows that x is classical w.r.t. Q and $\pi x = w_1$. Let w be another common neighbour of w_0 and w_2. Then $d(x,w) = j+2$ and $\mathrm{rank}(S(x,w) \cap S(x,y)) \geq \mathrm{rank}(S(x,w_0) \cup (S(x,w_2) \cap S(x,y))) > j+1$ so that we can replace w_1 by w in the path, again a contradiction.

This proves that our path has the form: y....y' within $\Gamma_i(x)$ followed by y'....z, part of a geodesic from y' to x. This proves the theorem. □

Now we can compute the size of the components of $\Gamma_i(x)$ - they have just the right size to be $\Gamma_i(x)$ in a near 2i-gon.

<u>PROPOSITION</u>. *If* C *is a component of* $\Gamma_i(x)$ *then* $|C| = \dfrac{s^i \prod_{j=0}^{i-1} (t_i - t_j)}{\prod_{j=1}^{i} (t_j+1)}$.

The number of components of $\Gamma_i(x)$ *is* $\prod_{j=0}^{i-1} \dfrac{t-t_j}{t_i-t_j}$.

<u>PROOF</u>. Choose $y \in \Gamma_i(x)$. Construct paths $x = u_0, u_1, \ldots, u_i$ such that $u_j \sim u_{j-1}$, $d(x,u_j) = j$ and $S(x,u_j) \cap S(x,y) \supsetneq S(x,u_{j-1})$ $(1 \leq j \leq i)$. By the previous theorem $S(x,u_j) \subset S(x,y)$ $(0 \leq j \leq i)$ so that y and u_i lie in the same component of $\Gamma_i(x)$. The number of choices for u_j given u_{j-1} is $s(t_i - t_{j-1})$ [for: there are $t_i - t_{j-1}$ choices for a line $L \in S(x,y)\backslash S(x,u_{j-1})$. Given L there are $t_j - t_{j-1}$ lines on u_{j-1} parallel to L, but each such line is parallel to $t_j - t_{j-1}$ lines in $S(x,y)\backslash S(x,u_{j-1})$: in fact there are $t_j - t_{j-1}$ lines in L_x parallel to L, so at most $t_j - t_{j-1}$ in $S(x,y)\backslash S(x,u_{j-1})$ - in this way we find a lower bound on |C|; suppose M is a line on u_{j-1} parallel to L, and $L' \in L_x$, $L' \,//\, M$. Then $L' \notin S(x,u_{j-1})$ since parallel lines are not together in a geodesic. Choose $v \in M\backslash\{u_{j-1}\}$. By the previous theorem v is on a geodesic from x to y' for some y' with $S(x,y) = S(x,y')$. But $L' \in S(x,v)$, so $L' \in S(x,y)$. Thus all lines in L_x parallel to M are in $S(x,y)\backslash S(x,u_{j-1})$.] and each point u_j is found $1+t_j$ times. Now $|C| = |\{u \mid S(x,u) = S(x,y)\}|$ has the required value. Dividing $|\Gamma_i(x)|$ by |C| yields the number of components. □

<u>THEOREM 6</u>. *Let* $d(x,y) = i$. *Then there is a unique geodetically closed sub near* 2i*-gon* H(x,y) *containing* x *and* y.

<u>PROOF</u>. Define

$$H(x,y) = \{u \mid S(x,u) \subset S(x,y)\} = \{z \mid z \text{ on a geodesic from } x \text{ to } C\},$$

where C is the component of $\Gamma_i(x)$ containing y.

(i) Clearly H(x,y) contains all geodesics from x to each of its points.

(ii) H(x,y) is linearly closed. (For: if a line ℓ has two points u,v in H(x,y), and $w \in \ell$ then either $d(x,w) = d(x,\ell)$ and w is on a geodesic from u to x, or $d(x,w) > d(x,\ell)$ and if $d(x,u) \geq d(x,v)$ then $S(x,w) = S(x,u)$; in both cases $w \in H(x,y)$.)

(iii) Let $x \sim x' \in H(x,y)$ and $d(x',y) = i$. We prove that $H(x',y) = H(x,y)$.

A. Let ℓ be a line in $H(x',y)$ having a point $u \in H(x,y)$, and suppose $\ell \not\subset H(x,y)$. Now $d(x,\ell) = d(x,u)$. Let $d(x',\ell) = d(x',v)$ with $v \in \ell$ and suppose $v \notin H(x,y)$. Then $xx' \mathbin{//} uv$ and S(x,v) contains both S(x,u) and the line xx' and by the previous theorem it follows that $S(x,v) \subset S(x,y)$. Contradiction. This shows that u = v.

B. Let C' be the component of y in $\Gamma_i(x')$. Then $C' \subset H(x,y)$: C' is connected, and if ℓ is a line with two points in C' then by induction and A. we have $\ell \subset H(x,y)$.

C. Let $z \in H(x',y)$, i.e., z on a geodesic from x' to $y' \in C$. Then $z \in H(x,y)$: suppose z is the last point of the geodesic not in H(x,y). By B. $z \notin C'$. Let ℓ be the line connecting z with its successor in the geodesic. By A we find a contradiction.

(iv) Let $u \in H(x,y)$. Then $\exists v$: $H(u,v) = H(x,y)$.

(For: let $x = x_0, x_1, \ldots, x_i$ be a geodesic containing u with $x_i \in C$. By theorem 2 there is a geodesic $x_i = y_0, y_1, \ldots, y_i = x$ such that $d(x_j, y_j) = i \ \forall_j$. If $u = x_j$ then set $v = y_j$. Note that all x_j and y_j are in H(x,y) since they are on geodesics from a point of C to x. Now by (iii) we see that $H(x,y) = H(u,v)$ [just as in the proof of the corollary to theorem 2].)

Now everything is clear. □

COROLLARY. *Let* $A \subset L_x$ *with* rank A = i. *Then there is a unique sub near* 2i-*gon containing* A. □

k. On a_0.

In Lemma 19 we saw that given a quad Q and a point $x \in N_{i+1,0}(Q)$, the set $O(x,Q) := \{\ell \mid x \in \ell \text{ and } \ell \text{ meets } N_{i,0}\}$ is a subspace of L_x.

LEMMA 26. rank $O(x,Q) < d(x,Q)$.

PROOF. Otherwise we could find a subset $A \subset O(x,Q)$ with rank $A = d(x,Q)$. Let H be the $2(i+1)$-gon determined by A. Let O be the ovoid in Q determined by x. Then $O \subset H$, and since H is geodetically closed, $Q \subset H$. Now $d(x,y) = i+2$ for $y \in Q\backslash O$, but this is impossible in a $2(i+1)$-gon. □

In particular it follows for $d(x,Q) = 3$ that $a_0 = |O(x,Q)| \in \{0,1,1+t_2\}$. [We know that $t_4 \le t_3 + s^2(t_3-t_2+1)$. But on the other hand $1 + t_4 = a_0 + a_C + a_I \ge a_0 + a_C = (1+t_3)(1+st_2) - st_2a_0$, so that $s(t_3-t_2+1) \ge t_2(t_3+1-a_0)$. Thus $(s-t_2)(t_3+1) \ge t_2(s-a_0) > -t_2(t_2+1) > -t_3$ and therefore $s \ge t_2$ and if $s = t_2$ then only $a_0 = t_2+1$ occurs. In this last case we find from $\overline{a_0}$ that $t_4 = t_3(t_3+1)/(t_2+1)$. - However, we shall see that $t_2 = 1$ without using these estimates, and for $t_2 = 1$ they are not interesting.]

ℓ. Relation between a point a hex

A *hex* is a geodetically closed sub near hexagon. Let H be a geodetically closed sub near 2j-gon.

DEFINITION. A point x is called *of classical type* with respect to H if there exists a point $\pi x \in H$ such that $d(x,y) = d(x,\pi x) + d(\pi x,y)$ for all $y \in H$.

A point x is called *of ovoid type* w.r.t. H if x has the same distance to all lines of H.

LEMMA 27. *Let* $d(x,H) = 1$. *Then* x *is of classical type w.r.t.* H.

PROOF. Let $x \sim x' \in H$. Let $y \in H$. We must show that if $d(x',y) = i$ then $d(x,y) = i+1$. But if $d(x,y) \le i$ then the line xx' contains a point x" at distance i-1 from y, and since H is geodetically closed and x'x"...y is a geodesic from x' to y we have $x'x'' \subset H$ and thus $x \in H$, contradiction. □

As a consequence we have

LEMMA 28. *Let* $d(x,H) = d(u,H) = 1$ *and* $x \sim u$. *Then* $\pi x \sim \pi u$ *or* $\pi x = \pi u$. □

LEMMA 29. *Let* $d(x,H) = i$ *and suppose* $\Gamma_{i+j}(x) \cap H \ne \emptyset$. *Then* x *is of classical type w.r.t.* H.

PROOF. Let $d(x,x') = i$ for some $x' \in H$. Then $\Gamma_j(x') \cap H$ is connected and contained in $\Gamma_{i+j}(x)$. Let $y \in H$. By the first line of the proof of Theorem 6 (i.e., by Theorem 5) there is a geodesic from x' to some point of $\Gamma_j(x') \cap H$ in H containing y. It follows that $d(x,y) = d(x,x') + d(x',y)$. □

Now assume that H is a hex.

LEMMA 30. *Let* $d(x,H) = 2$. *Then any two points in* $\Gamma_2(x) \cap H$ *have distance two.*

PROOF. Set $A := \Gamma_2(x) \cap H$. Clearly no two points of A can be adjacent (otherwise x would have a neighbour on the connecting line and $d(x,H) \leq 1$). Set $B := \Gamma_3(x) \cap H$. If $q \in B$ then $H(x,q) \cap H$ is geodetically closed and hence a point, line or quad. Thus, if q has more than one neighbour in A then $\Gamma_1(q) \cap A$ is contained in the quad $H(x,q) \cap H$.

Now suppose $u,v \in A$ with $d(u,v) = 3$. Let u p q v be a path of length 3 connecting u and v. Then $p,q \in B$. Let r be the unique point in $\Gamma_2(x) \cap$ $\cap\, pq$, so that $r \in A$. Now $H(x,q) \cap H$ contains the points q,v,r and hence p and therefore also u, a contradiction. □

We see that A carries a pairwise balanced design: the blocks are intersections of A with quads, i.e., are ovoids. In the regular case this gives us a Steiner system $S(2,1+st_2,|A|)$.

Much more can be said, but at this point we have assembled enough material to prove that regular near octagons almost never exist. In order to prepare for the next section let us set up some equations. Assume that (X,L) is a regular near octagon with parameters (s,t_2,t_3,t_4) where $s > 1$, $t_2 > 0$. Let H be a fixed hex and x a fixed point with $d(x,H) = 2$. Set $A = \Gamma_2(x) \cap H$, $B = \Gamma_3(x) \cap H$ and $C = \Gamma_4(x) \cap H$. Set $B_0 = \{y \in B \mid \Gamma_1(y) \cap A = \emptyset\}$, $B_1 = \{y \in B \mid |\Gamma_1(y) \cap A| = 1\}$, $B_2 = \{y \in B \mid |\Gamma_1(y) \cap A| = t_2 + 1\}$. Let $a := |A|$, $b := |B|$, etc. Then we have the following equations:

$$a + b_0 + b_1 + b_2 + c = (s+1)(1 + st_3 + \frac{s^2 t_3(t_3-t_2)}{1+t_2} \tag{1}$$

$$b_1 + (t_2+1)b_2 = sa(t_3+1) \tag{2}$$

$$b_0(t_3+1)+b_1 t_3+b_2(t_3-t_2) = s^{-1}c(t_3+1) \tag{3}$$

$$a(a-1)/(st_2+1)st_2 = b_2/(st_2+1)s \tag{4}$$

and consequently

$$b_2 = a(a-1)/t_2 \tag{4'}$$

$$b_0 + b_1 + b_2 = sa + s^{-1}c \tag{5}$$

$$a + s^{-1}c = 1 + st_3 + \frac{s^2 t_3(t_3-t_2)}{1 + t_2} \tag{6}$$

Mimicking the reasoning that produced $1 + t_3 \geq (1+t_2)(1+st_2)$ we have

$$s(t_4+1) \geq b_0(t_3+1)/a_{max} \tag{7}$$

where $a_{max} = \max\{|\Gamma_2(y) \cap H| \,|\, d(y,H) = 2\}$. And indeed counting pairs (y,z) with $y \sim x$, $z \in B_0$ and $d(y,z) = 2$ we find $b_0(t_3+1) \leq s(t_4+1).a_{max}$.

Estimating a little bit more carefully we find by counting pairs (y,z) with $y \sim x$, $z \in H$ and $d(y,z) = 2$:

$$(s(t_4+1) - (t_2+1)a)a_{max} + (t_2+1)\ as(t_3+1) \geq b(t_3+1) + a(t_2+1)(s-1). \tag{7'}$$

From (2) and (4') we see that $(t_2+1)(a-1) \leq st_2(t_3+1)$ so that

$$a-1 \leq s(t_3+1) - \frac{s(t_3+1)}{t_2+1} \leq s(t_3+1) - s(st_2+1) = st_3 - s^2 t_2. \tag{8}$$

From (2), (4'), (5) and (6) we see that

$$0 \leq b_0 = a^2 - a(st_3+2) + 1 + st_3 + \frac{s^2 t_3(t_3-t_2)}{1+t_2} \tag{8'}$$

One might wonder whether there are any points x in X with $d(x,H) = 2$. But such points exist if and only if X is not classical: since the projection of a line in $\Gamma_1(H)$ is a line in H we find that a point $y \in \Gamma_1(H)$ is on one line meeting H and on $(t_3+1)t_2$ lines within $\Gamma_1(H)$. If $\Gamma_2(H) = \emptyset$ then $t_4 = t_2(t_3+1)$. Now look at the design of quads and hexes on a fixed line. It has block size $K := \frac{t_3}{t_2}$ and replication number $R := \frac{t_4-t_2}{t_3-t_2}$. We have $R = 0$ (no blocks - ridiculous) or $R = 1$ (all points on a unique block - again ridiculous) or $R \geq K$. If $t_4 = t_2(t_3+1)$ then $R \geq K$ becomes $t_3 \leq t_2(t_2+1)$, so that $t_3 = t_2(t_2+1)$. Thus (X,L) is classical.

m. <u>The nonexistence of regular near octagons.</u>

THEOREM 7. *Let* (X,L) *be a regular near octagon with parameters* (s,t_2,t_3,t_4). *Then one of the following holds:*

(i) $s = 1$

or (ii) $t_2 = 0$

or (iii) $t_2 = 1$

or (iv) $t_3 = t_2(t_2+1)$ and $t_4 = t_2(t_3+1)$: (X,L) is classical.

PROOF. Let K and R be the block size and replication number of the design (Steiner system) of quads and hexes on a fixed line as discussed in the previous section. We need a slightly stronger result than Fisher's inequality $R \geq K$.

LEMMA 31. *In a Steiner system* $S(2,K,V)$ *we have* $R = 0$ *or* $R = 1$ *or* $R = K$ *or* $R = K+1$ *or* $R > K+\sqrt{K}$. *More precisely, if* $R = K+m$ *then* $K|m(m-1)$. □

Similarly we need an inequality for generalized quadrangles slightly sharper than $s \leq t_2^2$.

LEMMA 32. *In a generalized quadrangle* $GQ(s,t_2)$ *with* $t_2 > 1$ *we have* $s = t_2^2$ *or* $s = t_2^2-t_2$ *or* $s = t_2^2-t_2-1$ *or* $s \leq t_2(t_2-2)$.

PROOF. One of the eigenvalues has multiplicity $s^2(st_2+1)/(s+t_2)$ so that $(s+t_2)|t_2^2(t_2^2-1)$. If $s = t_2(t_2-2)+\sigma$ then $t_2^2 = (s+t_2)+(t_2-\sigma)$ and $(s+t_2)|(t_2-\sigma)(t_2-\sigma-1)$. For $0 < \sigma < 2t_2$, $\sigma \neq t_2, t_2-1$ it follows that $s+t_2 = t_2^2-t_2+\sigma \leq t_2^2-(2\sigma+1)t_2+\sigma(\sigma+1)$, a contradiction. □

The idea of the proof is that the Krein condition $q_{11}^1 \geq 0$ for octagons gives an upper bound for t_4 while $R \geq K$ gives a contradictory lower bound for t_4. We use the Krein condition $q_{11}^1 \geq 0$ for hexagons to give a lower bound for $K = \frac{t_3}{t_2}$. As we saw in section g we have

$$(s-1)(t_3^2-((s^2+1)(t_2+1)-1)t_3+s^4(t_2+1)) \geq 0. \qquad (9)$$

Assume $s > 1$, $t_2 > 1$, $t_3 \neq t_2(t_2+1)$. If $s \neq t_2^2$ then (9) implies

$$t_3 \geq s^2t_2 - \frac{s^2}{t_2-1}. \qquad (10)$$

(Indeed, if $s \le t_2^2-t_2-1$ then the left hand side of (9) is negative for $t_3 = s^2t_2 - \frac{s^2}{t_2-1}$ and if $s = t_2^2-t_2$ it is negative for $t_3 = s^2t_2 - \frac{s^2}{t_2-1} - 1$ and $t_2 \ge 4$. It is also negative for $t_3 = s^2\frac{t_2+1}{t_2-1}$ if $t_2 \ge 3$, regardless of the value of s. This shows that if $s \ne t_2^2$ and $t_2 \ge 4$ then either $t_3 < s^2\frac{t_2+1}{t_2-1}$ or $t_3 > s^2t_2 - \frac{s^2}{t_2-1} - 1$. The former possibility contradicts $t_3+1 \ge (t_2+1)(st_2+1)$ and $s \le t_2(t_2-1)$. In the latter case we obtain (10) using $t_2|t_3$. Remain the cases $t_2 = 2$ and $(t_2,s) = (3,6)$. If $t_2 = 3$ and $s = 6$ then (9) implies $t_3 \le 58$ or $t_3 \ge 89$; but $t_3 \ge 4.19-1 = 75$ and $t_2|t_3$ so $t_3 \ge 90$ as claimed in (10). If $t_2 = 2$ then $s \in \{1,2,4\}$ hence $s = 2$. But all near hexagons with $s = 2$ are known; the only sporadic one has $t_3 = 14$ and thus satisfies (10).)

Assume $s \ne t_2^2$, $R > K+1$. Write $R = K+m$. We saw already that there is no regular near octagon with parameters $(2,2,14,t_4)$, so $t_2 > 2$. By (10) we find

$$K = \frac{t_3}{t_2} \ge s^2 - \frac{s^2}{t_2(t_2-1)} \ge s^2-s,$$

so that

$$m \ge s$$

and using $t_4 \le s^2(t_3-t_2+1) + t_3$ it follows that

$$s^2 + 1 + \frac{s^2}{t_3-t_2} \ge \frac{t_4-t_2}{t_3-t_2} = R = K+m \ge s^2.$$

Now (regardless of the value of s) $s^2 < t_3 - t_2$, for otherwise $s^2 \ge t_3 - t_2 \ge st_2^2 + st_2$ so that $s \ge t_2^2 + t_2$, a contradiction. Therefore $R \le s^2 + 1$.

Now if $K > s^2-s$ then it follows that $m > s$ so that $R \ge s^2 + 2$, a contradiction. Consequently $K = s^2-s$, and $s = t_2(t_2-1)$. From $t_2(t_2+1)|t_3(t_3+1)$ we find (since $s \equiv 2 \pmod{t_2+1}$ and $K \equiv 2 \pmod{t_2+1}$ so that $t_3 \equiv -2 \pmod{t_2+1}$) that $2 \equiv 0 \pmod{t_2+1}$, a contradiction.

Thus we proved that any counterexample to the theorem satisfies $s = t_2^2$ or $R = K$ or $R = K+1$.

Next suppose $s = t_2^2$, $R = K+m$, $m(m-1) > K$. In this case $m(m-1) \ge 2K$.

From (9) we derive (for $t_2 \geq 3$)

$$t_3 > t_2^5 - t_2^3 - t_2^2 - 8t_2 \text{ or } t_3 < s^2 \frac{t_2+1}{t_2-1}. \tag{11}$$

Thus if $t_3 \geq s^2 \frac{t_2+1}{t_2-1}$ then

$$K = \frac{t_3}{t_2} \geq t_2^4 - t_2^2 - t_2 - 7. \tag{12}$$

Since $R \leq s^2 + 1 = t_2^4 + 1$ we have

$$m \leq t_2^2 + t_2 + 8,$$
$$2(t_2^4 - t_2^2 - t_2 - 7) \leq 2K \leq m(m-1) \leq t_2^4 + 2t_2^3 + 16t_2^2 + 15t_2 + 56,$$
$$t_2 \leq 5.$$

If $t_2 = 5$, $s = 25$ then $K \geq 588$, $R \leq 626$, $m \leq 38$, $K = \frac{1}{2}m(m-1)$, $R = \frac{1}{2}m(m+1)$ and $m(m-1) \geq 1176$, $m(m+1) \leq 1252$, impossible.

If $t_2 = 4$, $s = 16$ then $K \geq 229$, $R \leq 257$, $m \leq 28$. Now either $K = \frac{1}{3}m(m-1)$ and $R = \frac{1}{3}m(m+2)$ so that $m(m-1) \geq 687$ and $m(m+2) \leq 771$, impossible, or $K = \frac{1}{2}m(m-1)$ and $R = \frac{1}{2}m(m+1)$ so that $m(m-1) \geq 458$ and $m(m+1) \leq 514$, i.e., $m = 22$, $K = 231$, $R = 253$, $t_3 = 924$. But these parameters violate (9).

If $t_2 = 3$, $s = 9$ then $K \geq 62$, $R \leq 82$, $m \leq 20$. Since $62 \nmid m(m-1)$ we have $K \geq 63$ and $m \leq 19$, $m(m-1) \leq 342$. If $K = \frac{1}{5}m(m-1)$ then $m \leq 16$, $K \leq 48$, contradiction. If $K = \frac{1}{4}m(m-1)$ then $m = 17$, $K = 68$, $R = 85$, contradiction. If $K = \frac{1}{3}m(m-1)$ then $m = 15$, impossible. Finally, if $K = \frac{1}{2}m(m-1)$ then $m = 12$, $K = 66$, $R = 78$, $t_3 = 198$. These parameters satisfy (9) but die on the condition $t_2(t_2+1) \mid t_3(t_3+1)$.

If $t_2 = 2$, $s = 4$ then (9) does not yield any restriction, but by the Mathon bound we have $t_3 \leq s^3 + t_2(s^2-s+1) = 90$ and $t_3 \geq (t_2+1)(st_2+1) = 27$. Each of the intermediate values for t_3 dies on the condition ($t_2 \mid t_3$ and $t_2(t_2+1) \mid t_3(t_3+1)$ and the eigenvalues of H have integral multiplicities).

Thus we proved that if $s = t_2^2$ and $R > K+1$ and $t_3 \geq s^2 \frac{t_2+1}{t_2-1}$ then $K = m(m-1)$ and $R = m^2$. Also, that regardless of the value of s, $t_2 > 2$. Now $R \leq s^2+1$, so $m \leq s$, but by (12) we find $m > t_2^2-1$, i.e., $m = s$, so that

$K = s^2-s$, $t_3 = t_2^3(t_2^2-1)$. In other words, if we write $q := t_2$ then $(s,t_2,t_3) = (s,t_2,t_3) = (q^2,q,q^5-q^3)$. The multiplicity of the eigenvalue $-t_3-1$ of H is

$$\frac{(1+st_3 + \frac{s^2t_3(t_3-t_2)}{1+t_2}).s^3(t_2+1)}{s^2(t_2+1)+st_3(t_2+1)+t_3(t_3-t_2)} = \frac{(1+q^5(q^2-1)+q^7(q-1)(q^5-q^3-q))q^2}{1+(q^3-q)+(q-1)(q^4-q^2-1)} \equiv$$

$$\frac{(1+q^5(q^2-1) - q^8(q^3-q+1))q^2}{q^5-q^4+q^2-2q+2} \equiv \frac{(q^5+1)q^2}{q^5-q^4+q^2-2q+2} \equiv \frac{q^3-2}{q^5-q^4+q^2-2q+2} \not\equiv 0 \pmod 1,$$

a contradiction.

Thus we proved that if $s = t_2^2$ then $t_3 < s^2 \frac{t_2+1}{t_2-1}$ or $R \le K+1$. Suppose $s = t_2^2$ and $t_3 < s^2 \frac{t_2+1}{t_2-1}$ (and $t_2 \ge 3$). Again write $q := t_2$ so that $s = q^2$. From (9) derive

$$t_3 \le q^4 + q^3 + q^2 + 2q + 18,$$
$$K \le q^3 + q^2 + q + 2 + \frac{18}{2}.$$

On the other hand,

$$K = \frac{t_3}{t_2} \ge \frac{\lceil (q+1)(q^3+1)\rceil}{q} = q^3 + q^2 + 2.$$

Using the notation of the previous section we find from (8) that

$$a-1 \le s(t_3+1) - s\frac{t_3+1}{t_2+1} < q^6 + q^4 + q^3 + 18q^2.$$

Let R_0 and K_0 be the replication number and blocksize of the Steiner system $S(2,st_2+1,a)$ on A. Then $K_0 = st_2 + 1 = q^3 + 1$ and $R_0 = \frac{a-1}{q^3} < q^3 + q + 1 + \frac{18}{q}$. If $R_0 > K_0 + \sqrt{K_0}$ then $q \le 4$. If $q = 4$ then $K_0 = 65$, $R_0 \le 73$, $R_0 < K_0 + \sqrt{K_0}$. If $q = 3$ then $K_0 = 28$, $R_0 \le 36$, $m := R_0-K_0 \le 8$. Since $K_0 | m(m-1)$ we have $m = 8$, $R_0 = 36$, $a-1 = 27.36$, $t_3+1 \ge 4.36$, $143 \le t_3 \le 141$, contradiction. Thus $R_0 \le K_0 + 1$ and $a \le (q^3+1)^2$.

Let x be chosen such that $a = a_{max}$. From (8') and (7) we find $s(t_4+1) > (t_3+1)(a-q^3-2+q^5)$ so that $a < q^6+q^3+q^2+2-q^5$ (since $t_4+1 <$

$(s^2+1)(t_3+1))$.
If $R_0 = K_0+1$ then $a = q^6+2q^3+1$, impossible.
If $R_0 = K_0$ then $a = q^6+q^3+1$, impossible again.
If $R_0 = 1$ then $a = K_0 = q^3+1$, $q^6+q^2 = s(s^2+1) > \frac{s(t_4+1)}{t_3+1} > -1+q^5(q^3+1)$,
contradiction.
If $R_0 = 0$ then $a = 1$, $q^6+q^2 > -q^3-1+q^5(q^3+1)^2$, contradiction.

At this point we have shown that any counterexample to the theorem satisfies $R = K$ or $R = K+1$.

Suppose $R = K+1$. This means that the planar space of lines, quads and hexes on a given point is locally affine.

PROPOSITION. *A regular locally affine planar space has line size two; the points and planes form a Steiner system* $S(3,q+1,q^2+1)$, *i.e., a Möbius plane.* (cf. [1] Thm. 24 and [6]).

PROOF. If the space is locally $AG(2,q)$ then we have q^2 lines/point, q^2+q planes/point, $q+1$ planes/line, q lines/pt in a given plane. Let there be k points on each line. Then there are $1+(k-1)q$ points in each plane, $1+(k-1)q^2$ points in the whole space, and the total number of planes is

$$\frac{(1+(k-1)q^2).(q^2+q)}{1+(k-1)q} \equiv \frac{(1-q)(q^2+q)}{1+(k-1)q} \pmod 1.$$

Using that $(q,1+(k-1)q) = 1$ we find that $q^2-1 \equiv 0 \pmod{1+(k-1)q}$,

$$k+q-1 \equiv 0 \pmod{1+(k-1)q} \text{ so that } 1+(k-1)q \leq q+k-1,$$
$$\text{i.e., } (k-2)(q-1) \leq 0. \square$$

In our case $k = t_2+1$, $q = \frac{t_3}{t_2}$ so that $R = K+1$ can occur only when $t_2 = 1$.

Suppose $R = K$. This means that the planar space of lines, quads and hexes on a given point is locally projective. By DOYEN & HUBAUT [3] we have

$$q+1-k \in \{0,1,k^2-k+1,k^3+1\}$$

if the space is locally $PG(2,q)$. In our case $q+1 = \frac{t_3}{t_2}$ and $k = t_2+1$.

If $q+1-k = 0$ then $t_3 = t_2(t_2+1)$ and our hexes are classical, contradiction. If $q+1-k = 1$ then $t_3 = t_2(t_2+2)$ but $t_3+1 \geq (t_2+1)(st_2+1)$, a contradiction. If $q+1-k = k^2-k+1$ then $(t_2+1)^2+1 = \frac{t_3}{t_2} > st_2+s+1$ (recall that we already excluded $t_3+1 = (t_2+1)(st_2+1)$), so that $s \leq t_2$.

Above we saw $R \leq s^2+1$ so that $t_2+1 \leq s$, a contradiction. Consequently $q+1-k = k^3+1$, i.e.,

$$\frac{t_3}{t_2} = (t_2+1)^3 + (t_2+1) + 1 = t_2^3 + 3t_2^2 + 4t_2 + 3.$$

The fact that our planar space is locally projective means that two hexes intersect in $\emptyset$, a point or a quad but not in a line. Returning to the situation of the previous section this means that $b_1 = 0$ so that $a = 1 + \frac{st_2(t_3+1)}{t_2+1} = 1+st_2(t_2^3+2t_2^2+2t_2+1)$. In particular a is constant. If x does not have neighbours in $\Gamma_3(H)$ then we have equality in (7') (with $a_{max} = a$) so that

$$s(t_4+1) - a(t_2+1) + (t_2+1)(t_3+1)s = \frac{b(t_3+1)}{a} + (t_2+1)(s-1),$$

$$s(t_4+t_3-t_2+1)a = b(t_3+1) = \left((s-1)a+1+st_3+\frac{s^2t_3(t_3-t_2)}{1+t_2}\right)(t_3+1)$$

$$(t_3+1)\left(1+st_3+\frac{s^2t_3(t_3-t_2)}{1+t_2}\right) = a(s(t_4-t_2)+(t_3+1)) = a\left(s(t_3-t_2)\frac{t_3}{t_2} + (t_3+1)\right),$$

$$(t_3+1)(1+st_3) = s(t_3-t_2)\frac{t_3}{t_2} + (t_3+1) + \frac{st_2(t_3+1)^2}{t_2+1},$$

$$t_2 = t_3,$$

a contradiction. Therefore there exists a point y with $d(y,H) = 3$ in the near octagon X.

This point y must be of ovoid type w.r.t. H. If O is the ovoid $\Gamma_3(y) \cap H$ then $|O| = 1+st_3+\frac{s^2t_3(t_3-t_2)}{1+t_2}$.

Counting pairs (u,v) with $u \sim y$ and $v \in O$ and $d(u,v) = 2$ we find

$$s(t_4+1)a \geq |O|.(t_3+1),$$

$$s\left(\frac{t_3}{t_2}(t_3-t_2)+t_2+1\right)\left(1+\frac{st_2(t_3+1)}{t_2+1}\right) \geq (t_3+1)\left(1+st_3+\frac{s^2t_3(t_3-t_2)}{t_2+1}\right),$$

$$s\,\frac{t_4+1}{t_3+1} + s^2t_2 \geq 1 + st_3,$$

$$s\,\frac{t_3}{t_2} = s\frac{t_4-t_2}{t_3-t_2} > s\frac{t_4+1}{t_3+1} > st_3 - s^2t_2,$$

$$s > \frac{t_3(t_2-1)}{t_2^2} \geq t_2^3 + 2t_2^2 + t_2 - 2,$$

impossible (since $s \leq t_2^2$).

This completes the proof of theorem, except that we have not yet seen that $t_3 = t_2(t_2+1)$ implies $t_4 = t_3(t_2+1)$. But this was proved in the section on eigenvalues.

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