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AFDELING ZUIVERE WISKUNDE ZW 174/82 MEI
(DEPARTMENT OF PURE MATHEMATICS)

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ON SOME SEQUENCES DEFINED BY RECURRENCE RELATIONS OF INCREASING LENGTH

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Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 33A70, 30B40, 30C15, 30E20

On some sequences defined by recurrence relations of increasing length

by

J. van de Lune

ABSTRACT

This note contains a discussion of some convexity properties of the sequence $\{s_n\}_{n=0}^{\infty}$ defined by $s_0 := 1$ and the recurrence relation $s_n = s_0 n^{-\alpha} + s_1(n-1)^{-\alpha} + \ldots + s_{n-2}2^{-\alpha} + s_{n-1}$, $n > 0$, α being any positive constant.

KEY WORDS & PHRASES: *Recurrence relations, convexity, zeros, special functions*

0. INTRODUCTION

Recently, the error analysis of a numerical procedure for the solution of a certain type of integral equations led us to the problem of determining the asymptotic behaviour of some sequences defined by linear recurrence relations of increasing length.

More precisely, let $\{g_n\}_{n=0}^{\infty}$ be a given non-negative sequence such that $\Sigma_{n=0}^{\infty} g_n > 1$ (divergence being permitted) whereas the corresponding power series $\Sigma_{n=0}^{\infty} g_n z^n$, $z \in \mathbb{C}$, has radius of convergence 1 (say). Let $s_0 := 1$ and define recursively for $n > 0$

$$(1) \qquad s_n := s_0 g_{n-1} + s_1 g_{n-2} + \ldots + s_{n-2} g_1 + s_{n-1} g_0.$$

In particular, we had $g_n = (n+1)^{-\alpha}$ for $n \geq 0$, α being any (fixed) positive number, and we wanted to know the asymptotic behaviour of $\{s_n\}_{n=0}^{\infty}$ as $n \to \infty$. In order to attack this problem we define

$$G(z) := \sum_{n=0}^{\infty} g_n z^n, \qquad z \in \mathbb{C}, \ |z| < 1,$$

and observe that $x\, G(x)$ increases from 0 to $\Sigma_{n=0}^{\infty} g_n > 1$ as x increases from 0 to 1, so that there exists a unique $x_0 \in (0,1)$ such that $x_0\, G(x_0) = 1$. Now define $a_n := s_n x_0^n$ and $p_n := g_n x_0^{n+1}$ for $n \geq 0$, so that $a_0 = 1$, $p_n \geq 0$ for $n \geq 0$ and $\Sigma_{n=0}^{\infty} p_n = 1$. From this it is clear that we may define $P(z) := \Sigma_{n=0}^{\infty} p_n z^n$, $z \in \mathbb{C}$, $|z| < 1/x_0$, and that the recurrence relation (1) is equivalent to

$$(2) \qquad a_n := a_0 p_{n-1} + a_1 p_{n-2} + \ldots + a_{n-2} p_1 + a_{n-1} p_0 \quad \text{for } n > 0.$$

By mathematical induction it is easily shown that $0 < a_n \leq 1$ for all $n \geq 0$ so that we may define

$$A(z) := \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \mathbb{C}, \ |z| < 1.$$

Since $p_n \geq 0$ and $\Sigma_{n=0}^{\infty} p_n = 1$ it follows (by the maximum modulus theorem) that $|zP(z)| < 1$ for $|z| < 1$, so that $1 - zP(z)$ has no zeros in the disc $|z| < 1$.

By means of (2) it is easily shown that

$$A(z)P(z) = \frac{A(z)-1}{z}, \quad 0 < |z| < 1,$$

so that

$$A(z) = \frac{1}{1-zP(z)}, \quad |z| < 1,$$

or, equivalently,

$$S(z) := \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-zG(z)}, \quad |z| < x_0.$$

Note that the power series for $G(z)$ around $z = 0$ has radius of convergence 1 and that $p_n \geq 0$ for $n \geq 0$, so that, by a well-known theorem of Pringsheim (cf. TITCHMARSH [4; pp. 214-215]), the point $z = 1$ is a singular point of $G(z)$ and hence of $S(z)$. A similar remark holds true for $A(z)$ with respect to the point $z = 1/x_0$. In order to study the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ in greater detail we will make use of Cauchy's formula

$$s_n = \frac{1}{2\pi i}\oint \frac{S(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint \frac{1}{1-zG(z)}\frac{dz}{z^{n+1}},$$

where $\oint$ denotes counter clockwise integration along a circle around $z = 0$ with positive radius $\rho < x_0$.

In the following sections of this note we will be exclusively concerned with the specific case $g_n = (n+1)^{-\alpha}$, $\alpha > 0$, mentioned above. Besides determining the "main term" of the asymptotic behaviour of s_n, our main goal will be to show that a_n *tends logarithmically convex to a limit* L (to be specified later on) and we already note here that the analytic continuation of $P(z)$ will play an intriguing role in our discussion.

1. THE CASE $\alpha = 1$

As a *model* for our considerations we first consider the case in which $g_n = \frac{1}{n+1}$ for $n \geq 0$. It is clear that

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} = -\frac{1}{z}\log(1-z), \qquad |z| < 1,$$

from which it is easily seen that

$$x_0 = 1 - e^{-1} (\cong .632\ 120\ 559).$$

The generating function of $\{s_n\}_{n=0}^{\infty}$ is in this case

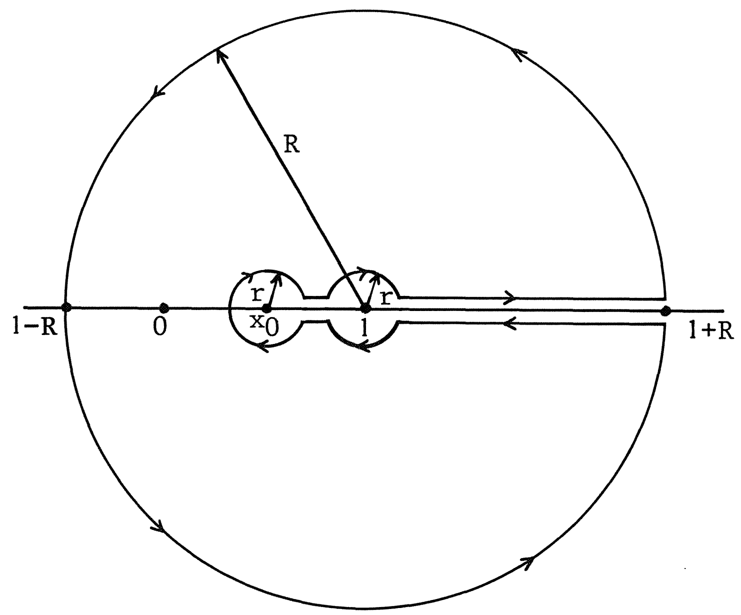
$$S(z) := \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-zG(z)} = \frac{1}{1+\log(1-z)},$$

so that Cauchy's formula for s_n reads

(3) $$s_n = \frac{1}{2\pi i}\oint \frac{1}{1+\log(1-z)}\,\frac{dz}{z^{n+1}},$$

where $\oint$ denotes counter clockwise integration along a circle around $z = 0$ with positive radius $\rho < x_0$.

Now choose a small positive r and a large positive R and deform the contour of integration in (3) as depicted below:



This deformation of the contour in (3) is possible due to the fact that $1 + \log(1-z)$ has $z = x_0$ as its only (simple) zero and $z = 1$ as its only singularity (on the entire Riemann surface corresponding to $1 + \log(1-z)$). Since $|\log(1-z)|$ tends uniformly to infinity for $z \to 1$ as well as for $|z| \to \infty$ (*so that* $\frac{1}{1+\log(1-z)}$ *is bounded for* $z \to 1$ *as well as for* $|z| \to \infty$), we have by a standard argument

$$s_n = \frac{\Lambda}{x_0^{n+1}} + \int_0^\infty \left(\frac{1}{1+\log u-\pi i} - \frac{1}{1+\log u+\pi i}\right)\frac{du}{(1+u)^{n+1}}$$

so that

$$(4) \qquad a_n = \frac{\Lambda}{x_0} + x_0^n \int_0^\infty \frac{1}{(1+\log u)^2+\pi^2}\,\frac{du}{(1+u)^{n+1}},$$

where Λ is such that $-\Lambda$ is the residue of $S(z) = \frac{1}{1+\log(1-z)}$ at $z = x_0$. It is easily verified that

$$(5) \qquad \Lambda = 1 - x_0 (\cong .367\ 879\ 441).$$

By the general theory of log-convex functions (cf. ARTIN [1]) it is immediately clear from (4) and (5) that $\{a_n\}_{n=0}^\infty$ tends log-convex to its limit

$$L := \frac{\Lambda}{x_0} = \frac{1-x_0}{x_0} = \frac{1}{e-1} \ (\cong .581\ 976\ 707).$$

2. THE CASE $\alpha \in (0,1)$

For any (fixed) $\alpha \in (0,1)$ we now take $g_n = (n+1)^{-\alpha}$ for $n \geq 0$. Since

$$G(z) = 1 + \frac{z}{2^\alpha} + \frac{z^2}{3^\alpha} + \ldots, \qquad |z| < 1,$$

the number x_0 is determined by

$$1 = x_0 + \frac{x_0^2}{2^\alpha} + \frac{x_0^3}{3^\alpha} + \ldots .$$

(Note that $x_0 = x_0(\alpha)$ increases from $\frac{1}{2}$ to 1 as α increases from 0 to ∞.)

In order to proceed here similarly as in Section 1 we first derive an expression for $G(z)$ which throws some light on the analytic continuation of $G(z)$. For $\alpha > 0$ we have

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt,$$

so that (by the substitution $t = nu$, $n > 0$)

$$\frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-nu} u^{\alpha-1} du,$$

and summation over n leads to

$$G(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{u^{\alpha-1}}{e^u - z} du. \tag{6}$$

From this representation it is clear that $G(z)$ can be continued analytically to the slit plane $\mathbb{C}^* := \mathbb{C}\backslash[1,\infty)$. Our first application of (6) is to show that x_0 is the only zero of $1 - zG(z)$ in $\mathbb{C}^*$. In order to see this we write $z = x + yi$ and consider imaginary parts in the equation $zG(z) = 1$.

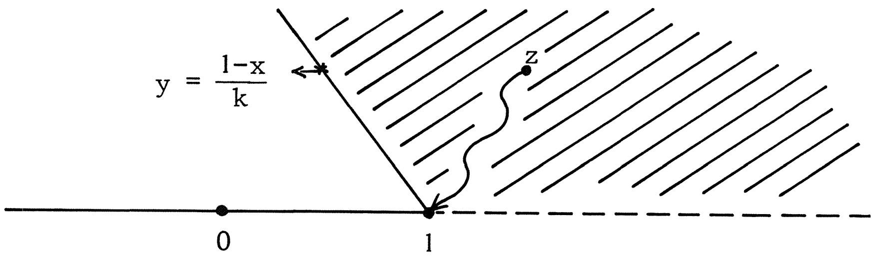
Since

$$\operatorname{Im}(zG(z)) = \frac{y}{\Gamma(\alpha)} \int_0^\infty \frac{e^u u^{\alpha-1}}{(e^u-x)^2+y^2} du,$$

it follows that $zG(z) = 1$ is impossible for $y \neq 0$. However, on the real axis, for $x < 1$, the function $xG(x)$ is clearly increasing so that x_0 is the only zero of $1 - zG(z)$ in the domain $\mathbb{C}^*$. Later on we will also show that the extension of $1 - zG(z)$ to the edges of the slit $(1,\infty)$ does not vanish there.

In Section 1 we used that $|\log(1-z)| \to \infty$ (uniformly) for $z \to 1$ as well as for $|z| \to \infty$. With this in mind we now show that $|zG(z)| \to \infty$ (uniformly) as $z \to 1$ ($z \in \mathbb{C}^*$).

It is clear that, where convenient, we may just as well show that $|G(z)| \to \infty$ (uniformly) as $z \to 1$. Again, write $z = x + yi$ and let z tend to 1 with z belonging to the sector: $y > 0$, $y \geq \frac{1-x}{k}$ where k is any positive constant. See figure below.



Considering the imaginary part of zG(z) we have

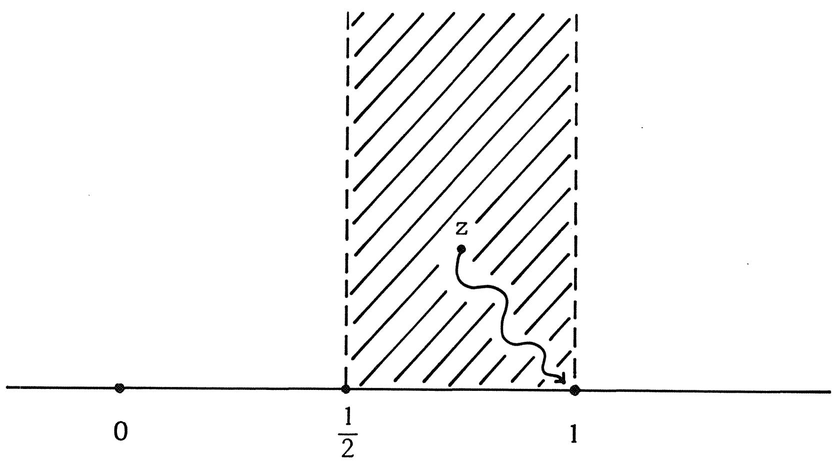
$$\mathrm{Im}(zG(z)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{ye^u}{(e^u-x)^2+y^2} u^{\alpha-1} du = (\text{substitute } u = \log(x+yt))$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\frac{1-x}{y}}^\infty \frac{\log^{\alpha-1}(x+yt)}{t^2+1} dt,$$

and, observing that the integrand is positive on the whole interval of integration and that $\frac{1-x}{y} \leq k$, it follows that

$$\Gamma(\alpha).\mathrm{Im}(zG(z)) > \int_k^{2k} \frac{\log^{\alpha-1}(x+yt)}{t^2+1} dt > \frac{k}{4k^2+1} \log^{\alpha-1}(x+2ky),$$

which (due to $x + 2ky > 1$ and $\alpha < 1$) tends to infinity as $x \to 1$ and $y \to 0$. Now let $z = x + yi$ be restricted to the strip $\frac{1}{2} < x < 1$, $y > 0$ and consider Re(G(z)).



By the substitution $u = \log(xt)$ in

$$\Gamma(\alpha).\mathrm{Re}(G(z)) = \int_0^\infty \frac{e^u - x}{(e^u-x)^2+y^2}\, u^{\alpha-1} du,$$

we obtain (note that we let $x \to 1$)

$$\Gamma(\alpha).\mathrm{Re}(G(z)) = \int_{\frac{1}{x}}^{\infty} \frac{xt-x}{(xt-x)^2+y^2} (\log xt)^{\alpha-1} \frac{dt}{t} > \int_{\frac{1}{x}}^{2} =$$

$$= \frac{1}{x} \int_{\frac{1}{x}}^{2} \frac{t-1}{(t-1)^2+(\frac{y}{x})^2} (\log xt)^{\alpha-1} \frac{dt}{t}$$

$$> \frac{1}{2} \log^{\alpha-1}(2x) \int_{\frac{1}{x}}^{2} \frac{t-1}{(t-1)^2+(\frac{y}{x})^2} dt$$

$$> c \cdot \int_{\frac{1-x}{y}}^{\frac{x}{y}} \frac{u}{u^2+1} du$$

$$= \frac{c}{2}(\log(x^2+y^2) - \log(y^2+(1-x)^2)), \quad (c = \frac{1}{2}\log^{\alpha-1} 2),$$

from which it is clear that $\mathrm{Re}(G(z)) \to \infty$ as $x \to 1$ and $y \to 0$.

The case $y = 0$, $x \uparrow 1$ can be dealt with directly from the original power series for $G(z)$. Since $G(\bar{z}) = \overline{G(z)}$, this proves that $|zG(z)| \to \infty$ (uniformly) for $z \to 1$. For another proof we refer to MAGNUS, OBERHETTINGER & SONI [3; pp. 32-35].

Our next goal is to show that $|zG(z)| \to \infty$ (uniformly) as $|z| \to \infty$. Although we realize that a substantial part of this rather technical problem has been considered before by FORD [2; Chapter III, Theorem IV, pp. 26-27], we choose to achieve our goal independently. First we restrict $z = x + yi$ to the half plane $x \le 0$ and we will show that $-\mathrm{Re}(zG(z)) \to \infty$ (uniformly) as $|z| \to \infty$.

Writing $p := -x$ and $r := |z|$ we have

$$-\mathrm{Re}(zG(z)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{-x(e^u-x)+y^2}{(e^u-x)^2+y^2} u^{\alpha-1} du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{pe^u+r^2}{(e^u+p)^2+y^2} u^{\alpha-1} du$$

$$> \frac{1}{\Gamma(\alpha)} \int_0^{\log r} \frac{r^2}{(r+r)^2+r^2} u^{\alpha-1} du = \frac{\log^\alpha r}{5\Gamma(\alpha+1)},$$

which tends to infinity as $r \to \infty$.

Now restrict $z = x + yi$ to the first quadrant such that $x > 0$ and $y \geq kx$, where k is some positive constant. We will show that also in this case $-\mathrm{Re}(zG(z)) \to \infty$ (uniformly) as $|z| \to \infty$.

As before we have

$$\Gamma(\alpha).\mathrm{Re}(zG(z)) = \int_0^\infty \frac{x(e^u-x)-y^2}{(e^u-x)^2+y^2} u^{\alpha-1} du.$$

We first show that the tail $\int_{\log(r^2/x)}^\infty$ of this integral is bounded for $r \to \infty$. Substituting $t = xe^u/r^2$ and observing that the integrand is positive on $(\log(r^2/x),\infty)$ we obtain

$$0 < \int_{\log \frac{r^2}{x}}^\infty < \int_{\log \frac{r^2}{x}}^\infty \frac{xe^u}{(e^u-x)^2+y^2} u^{\alpha-1} du$$

$$= \int_1^\infty \frac{r^2 t}{(\frac{r^2}{x}t-x)^2+y^2} \log^{\alpha-1}(\frac{r^2}{x}t)\frac{dt}{t} < \qquad \text{(note that } 0 < \alpha < 1)$$

$$< \log^{\alpha-1}(\frac{r^2}{x}) \int_1^\infty \frac{r^2}{(\frac{r^2}{x}t-x)^2+y^2} dt = \quad (\frac{r^2}{x}t - x = yu)$$

$$= \log^{\alpha-1}(\frac{r^2}{x}) \int_{\frac{\frac{r^2}{x}-x}{y}}^\infty \frac{r^2}{y^2(u^2+1)} \frac{ydu}{r^2/u} = \frac{x}{y} \log^{\alpha-1}(\frac{r^2}{x}) \int_{\frac{\frac{r^2}{x}-x}{y}}^\infty \frac{du}{u^2+1}$$

$$< \frac{x}{y} \log^{\alpha-1}(\frac{r^2}{x}) \int_{-\infty}^\infty \frac{du}{u^2+1} .$$

Since $\frac{x}{y} \le \frac{1}{k}$ and $\log^{\alpha-1}(\frac{r^2}{x}) \to 0$ as $r \to \infty$ it follows that "the tail" is bounded as $r \to \infty$. Hence, in case $x > 0$ and $y \ge kx$ we may complete the proof of our claim as follows.

$$\left| \int_0^{\log \frac{r^2}{x}} \right| = \int_0^{\log \frac{r^2}{x}} \frac{r^2 - xe^u}{(e^u - x)^2 + y^2} u^{\alpha-1} du > \int_0^{\log \frac{r^2}{x}} \frac{r^2 - xe^u}{e^{2u} + r^2} u^{\alpha-1} du$$

$$= (e^u = rt) = \int_{\frac{1}{r}}^{\frac{r}{x}} \frac{r^2 - xrt}{r^2(t^2+1)} \log^{\alpha-1}(rt)\frac{dt}{t} > \int_{\frac{1}{r}}^{1} >$$

$$> \log^{\alpha-1} r \int_{\frac{1}{r}}^{1} \frac{1-t}{t(t^2+1)} dt > \log^{\alpha-1} r \int_{\frac{1}{r}}^{\frac{1}{2}} > \frac{1}{2} \log^{\alpha} r$$

if r is large enough.

It remains to show that $|zG(z)| \to \infty$ (uniformly) as $|z| \to \infty$ with $z = x + yi$ restricted to the sector $x > 0$, $0 < y \le kx$ for some positive k. A real approach, as performed above, appears to be quite cumbersome so that we proceed by complex analytical means in this case. We first aboserve that the representation

$$\Gamma(\alpha).zG(z) = \int_0^{\infty} \frac{z}{e^u - z} u^{\alpha-1} du$$

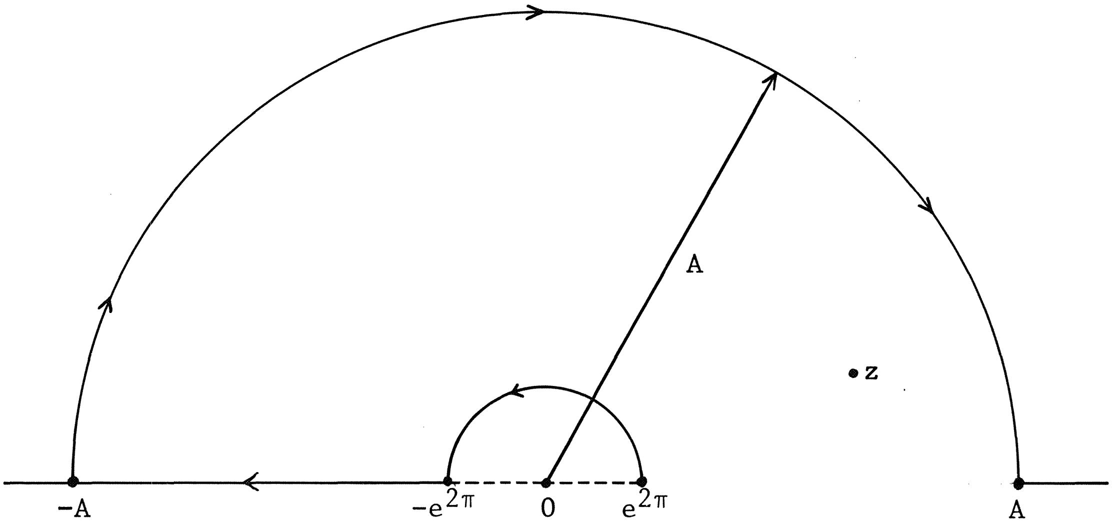
may be transformed (by $e^u = t$) into

$$\Gamma(\alpha).zG(z) = \int_1^{\infty} \frac{z}{t(t-z)} \log^{\alpha-1} t dt = \left(\int_1^{e^{2\pi}} + \int_{e^{2\pi}}^{\infty} \right),$$

where $\int_1^{e^{2\pi}} \frac{z}{t(t-z)} \log^{\alpha-1} t\, dt$ is easily seen to be bounded for $|z| \to \infty$. It remains to show that $|\int_{e^{2\pi}}^{\infty}| \to \infty$ (uniformly) as $|z| \to \infty$. Noting that

$$\int_{e^{2\pi}}^{\infty} = \lim_{A\to\infty} \int_{e^{2\pi}}^{A}$$

and replacing the integration from $e^{2\pi}$ to A (> r) along the real axis by the contour depicted below



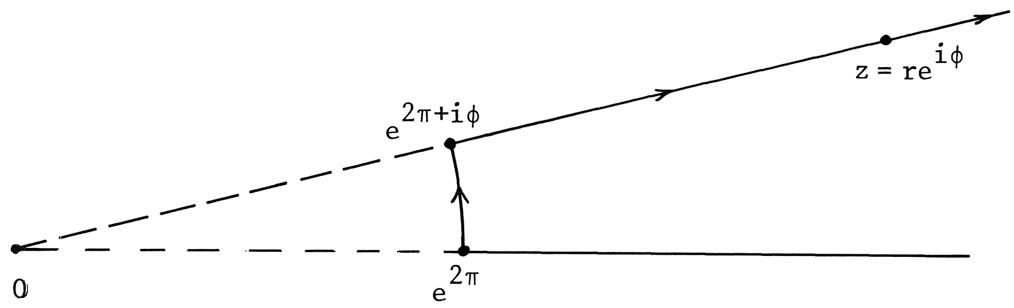
we easily obtain (by letting $A \to \infty$)

$$\int_{e^{2\pi}}^{\infty} = {\curvearrowleft}\!\!\!\int_{e^{2\pi}}^{-e^{2\pi}} + \int_{-e^{2\pi}}^{-\infty} + 2\pi i \log^{\alpha-1} z.$$

Since ${\curvearrowleft}\!\!\int_{e^{2\pi}}^{-e^{2\pi}}$ and $\log^{\alpha-1} z$ are bounded for $|z| \to \infty$ it remains to show that the integral

$$-\int_{-e^{2\pi}}^{-\infty} = \int_{e^{2\pi}}^{\infty} \frac{z}{u(u+z)} \log^{\alpha-1}(-u)\,du$$

tends uniformly to infinity as $|z| \to \infty$. Replace the contour from $e^{2\pi}$ to ∞ along the real axis by the contour depicted below.



Then the integral along the circular curve from $e^{2\pi}$ to $e^{2\pi+i\phi}$ is easily seen to be bounded as $|z| \to \infty$, whereas for the remaining integral from $e^{2\pi+i\phi}$ to $\infty . e^{i\phi}$ we have

$$\int_{e^{2\pi}}^{\infty} \frac{re^{i\phi}}{we^{i\phi}(we^{i\phi}+re^{i\phi})} \log^{\alpha-1}(-we^{i\phi})e^{i\phi}dw$$

$$= \int_{e^{2\pi}}^{\infty} \frac{r}{w(w+r)} \log^{\alpha-1}(-we^{i\phi})dw.$$

For the real part of this integral we have

$$\int_{e^{2\pi}}^{\infty} \frac{r}{w(w+r)}(\log^2 w + (\pi+\phi)^2)^{\frac{\alpha-1}{2}} \cos((1-\alpha)\frac{\phi+\pi}{\log w})dw$$

$$> (\cos((1-\alpha)\frac{\phi+\pi}{2\pi})) \int_{e^{2\pi}}^{\infty} \frac{r}{w(w+r)}(2\log^2 w)^{\frac{\alpha-1}{2}} dw$$

$$> 2^{\frac{\alpha-1}{2}} \cos(1-\alpha) \int_{e^{2\pi}}^{r} \frac{r}{w(r+r)} \log^{\alpha-1} r \; dw >> \log^{\alpha} r,$$

which tends to infinity as $r \to \infty$.

In order to complete our line of argument of Section 1 we finally prove that the (continuous) extension of the function $zG(z)$ does not assume the value 1 on the edges of the slit $(1,\infty)$. In order to see this we observe that for $x > 1$ and $y > 0$,

$$\Gamma(\alpha).\mathrm{Im}(zG(z)) = \int_0^\infty \frac{ye^u}{(e^u-x)^2+y^2}\, u^{\alpha-1}du = \int_{\frac{1-x}{y}}^\infty \frac{\log^{\alpha-1}(x+yt)}{t^2+1}\, dt$$

$$> \int_0^\infty \frac{\log^{\alpha-1}(x+yt)}{t^2+1}\, dt,$$

so that (by Lebesgue's dominated convergence theorem)

$$\liminf_{y \downarrow 0} \mathrm{Im}(zG(z)) \geq \frac{\log^{\alpha-1}x}{\Gamma(\alpha)} \frac{\pi}{2} > 0.$$

Similarly as in Section 1 we may conclude that for any $\alpha \in (0,1)$ there exists a positive constant $L = L(\alpha)$ and a positive function $f(u) = f_\alpha(u)$ for $u > 1$ such that

$$a_n = a_n(\alpha) = L + x_0^n \int_1^\infty f(u) \frac{du}{u^{n+1}},$$

and the log-convexity of $\{a_n\}_{n=0}^\infty$ follows as before.

As to the value of L one may verify that

$$L = \frac{\Lambda}{x_0} = \frac{1}{x_0} \cdot \frac{1}{G(x_0)+x_0G'(x_0)} = \frac{1}{1+x_0^2G'(x_0)},$$

where $-\Lambda$ is the residue of $\frac{1}{1-zG(z)}$ at the point $z = x_0$.

3. THE CASE $\alpha > 1$

Most of the analysis of Section 2 may easily be adapted to the case $\alpha > 1$. However, for $z \to 1$ we now have

(7) $$\lim_{z\to 1} zG(z) = \zeta(\alpha),$$

and since $\zeta(\alpha) > 1$ the function $\frac{1}{1-zG(z)}$ is bounded for $z \to 1$, so that the method of Section 1 applies here as well. In order to prove (7) we write

$$F(z) := z\mathfrak{C}(z) - \zeta(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{z}{e^u - z} - \frac{1}{e^u - 1}\right) u^{\alpha-1} du$$

$$= \frac{z-1}{\Gamma(\alpha)} \int_0^\infty \frac{e^u}{(e^u - z)(e^u - 1)} u^{\alpha-1} du = (e^u = t)$$

$$= \frac{z-1}{\Gamma(\alpha)} \int_1^\infty \frac{\log^{\alpha-1} t}{(t-z)(t-1)} dt \quad (z = 1+w,\ t = u+1)$$

$$= \frac{w}{\Gamma(\alpha)} \int_0^\infty \frac{\log^{\alpha-1}(1+u)}{(u-w)u} du.$$

Since $\alpha - 1 > 0$ it is easily seen that

$$F(z) = \frac{2\pi i}{\Gamma(\alpha)} \log^{\alpha-1}(1+w) + \frac{w}{\Gamma(\alpha)} \int_0^{-\infty . w} \frac{\log^{\alpha-1}(1+u)}{(u-w)u} du,$$

so that it remains to show that w times the last integral (I) tends to zero as $w \to 0$. By the transformation $u = -tre^{i\phi}$ $(w = re^{i\phi})$ we have

$$-wI = -w \int_0^{-\infty w} = -re^{i\phi} \int_0^\infty \frac{\log^{\alpha-1}(1-tre^{i\phi})}{(-tre^{i\phi} - re^{i\phi})(-tre^{i\phi})} (-re^{i\phi}) dt$$

$$= \int_0^\infty \frac{\log^{\alpha-1}(1-tre^{i\phi})}{(t+1)t} dt = \int_0^1 + \int_1^\infty .$$

It is easy to show that $\int_1^\infty \to 0$ as $r \to 0$. For the remaining integral $\int_0^1$ we have

$$\left|\int_0^1\right| \ll \int_0^1 \frac{(rt)^{\alpha-1}}{(t+1)t} dt < \int_0^1 \frac{(rt)^{\alpha-1}}{t} dt = r^{\alpha-1} \int_0^1 t^{\alpha-2} dt = \frac{r^{\alpha-1}}{\alpha-1},$$

which tends to zero as $r \to 0$. Hence, also in case $\alpha > 1$, the sequence a_n tends log-convex to its limit, the integral expression for a_n being formally the same as in Section 2.

4. THE CASE α COMPLEX

In this section we devote a few words to the case in which α is complex with $\mathrm{Re}(\alpha) > 0$. As before, the function

$$zG(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}, \qquad |z| < 1,\ \mathrm{Re}(\alpha) > 0,$$

has an analytic extension to $\mathbb{C}^* = \mathbb{C}\setminus[1,\infty)$ by means of the formula

$$zG(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{z}{e^u - z} u^{\alpha-1} du.$$

It is to be expected that, in general the sequence $\{a_n\}_{n=0}^{\infty}$ will be complex for complex α so that there is no obvious generalization of the log-convexity properties discussed in the previous sections. Moreover, the "main terms" in the description of the asymptotic behaviour of s_n and a_n become more complicated in case α is complex, due to the (numerically observed) fact that the equation

$$zG(z) = 1$$

can have more than one zero in $\mathbb{C}^*$. For example, writing $z = x + yi$, the above equation has *at least* two different solutions for $\alpha = \frac{1}{2} + 7i$:

$$x_1 \cong .815\ 881, \qquad y_1 \cong -.266\ 304$$

and

$$x_2 \cong -.590\ 967, \qquad y_2 \cong .781\ 403.$$

Note that both solutions even lie in the unit disc $|z| < 1$ ($|z_1| \cong .858\ 242$ and $|z_2| \cong .979\ 710$). We have not pursued this topic any further.

5. SOME RELATED PROBLEMS

<u>PROBLEM I</u>. Let $p\colon \mathbb{R}^+ \to \mathbb{R}^+$ be (Lebesgue) measurable such that $p \notin L_1(\mathbb{R}^+)$ whereas

$$\int_0^\infty p(u)e^{-u}du < \infty.$$

Defining

$$\phi(z) = \int_0^\infty \frac{z}{e^u - z}\, p(u)du, \qquad z \in \mathbb{C}^*,$$

we wonder what can be said (under suitable conditions) about the asymptotic behaviour of $\phi(z)$ as $z \to 1$, $z \in \mathbb{C}^*$, or as $|z| \to \infty$, $z \in \mathbb{C}^*$.

PROBLEM II. Does the function

$$\phi(z) := \phi_\alpha(z) := 1 - \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{z}{e^u - z}\, u^{\alpha-1} du, \qquad z \in \mathbb{C}^*,$$

always (i.e. for any fixed $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$) have at least one zero in the disc $|z| < 1$? Are *all* zeros of $\phi(z)$ simple?

ACKNOWLEDGEMENTS

I feel obliged to thank B. Dijkhuis, P.J. Hoogendoorn, H.J.J. te Riele, N.M. Temme and E. Wattel for their interest in this sort of problem and for their useful suggestions.

REFERENCES

[1] ARTIN, E., *Einfürung in die Theorie der Gammafunktion*, Teubner, 1931.

[2] FORD, W.B., *The asymptotic developments of functions defined by Maclaurin series*, Ann Arbor, University of Michigan Press, 1936.

[3] MAGNUS, W., F. OBERHETTINGER & R.P. SONI, *Formulas and theorems for the special functions of mathematical physics*, Grundlehren Math. Wiss., Band 52, Springer, 1966.

[4] TITCHMARSH, E.C., *The theory of functions*, Oxford University Press, 1960.