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A CHARACTERIZATION OF TWO CLASSES OF SEMI PARTIAL GEOMETRIES BY THEIR PARAMETERS

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A characterization of two classes of semi partial geometries by their parameters *)

by

H.A. Wilbrink & A.E. Brouwer

ABSTRACT

We show that, under mild restrictions on the parameters, semi-partial geometries with $\mu = \alpha^2$ or $\mu = \alpha(\alpha+1)$ are determined by their parameters.

KEY WORDS & PHRASES: *Semi-partial geometry, partial geometry, strongly regular graph*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let X be a (finite) nonempty set and L a set of subsets of X. Elements of X are called *points*, elements of L are called *lines*. The pair (X,L) is called a *partial linear space* if any two distinct points are on at most one line.

Two distinct points x and y are called *collinear* if there exists $L \in L$ such that $x,y \in L$, *noncollinear* otherwise. Two distinct lines L and M are called *concurrent* if $|L \cap M| = 1$.

We write $x \sim y$ ($x \not\sim y$) to denote that x and y are collinear (noncollinear). Similarly $L \sim M$ ($L \not\sim M$) means $|L \cap M| = 1$ ($|L \cap M| = 0$).

If $x \sim y$ ($L \sim M$) we denote by xy (LM) the line (point) incident with x and y (L and M).

For a nonincident point-line pair (x,L) we define:

$$[L,x] := \{y \in X | y \in L,\ y \sim x\},$$

$$[x,L] := \{M \in L | x \in M,\ L \sim M\}.$$

Given positive integers s,t,α,μ, the partial linear space (X,L) is called a *semi-partial geometry* (s.p.g) with parameters s,t,α,μ if:

(i) every line contains s+1 points,

(ii) every point is on t+1 lines,

(iii) for all $x \in X$, $L \in L$, $x \notin L$ we have $|[x,L]| \in \{0,\alpha\}$,

(iv) for all $x,y \in X$ with $x \not\sim y$ the number of points z such that $x \sim z \sim y$ equals μ.

A semi-partial geometry which satisfies $|[x,L]| = \alpha$ for all $x \in X$, $L \in L$ with $x \notin L$, or equivalently which satisfies $\mu = \alpha(t+1)$, is also called a *partial geometry* (p.g).

The *point-graph* of the partial linear space (X,L) is the graph with vertex set X, two distinct vertices x and y being adjacent iff $x \sim y$. The point-graph of a semi-partial geometry is easily seen to be strongly regular. Let (X,L) be a semi-partial geometry.

For $x,y \in X$, $x \not\sim y$ we define

$$[x,y] := \{L \in \mathcal{L} \mid x \in L,\ |[L,y]| = \alpha\}.$$

It is easy to see that $\alpha = s+1$ iff any two distinct points are collinear iff $(X,\mathcal{L})$ is a Steiner system $S(2,s+1,|X|)$. We shall always assume $s \geq \alpha$, hence noncollinear points exist.

Let $x,y \in X$, $x \not\sim y$. Then $\mu = |[x,y]|\alpha$ and $|[x,y]| \geq |[x,L]| = \alpha$ if $L \in [y,x]$. Hence, $\mu \geq \alpha^2$ and

(*) $\qquad \mu = \alpha^2 \Longleftrightarrow \forall K \in [x,y],\ L \in [y,x]:\ K \sim L,$

(* *) $\qquad \mu = \alpha(\alpha+1) \Longleftrightarrow$ every line $K \in [x,y]$ intersect every line $L \in [y,x]$ but one.

This is the basic observation we use in showing that, under mild restrictions on the parameters, semi partial geometries with $\mu = \alpha^2$ or $\mu = \alpha(\alpha+1)$ satisfy the Diagonal Axiom (D).

(D) : Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.
Then also $x_2 \sim x_4$.

From DEBROEY [1], it then follows that such a semi-partial geometry is known.

2. SEMI-PARTIAL GEOMETRIES WITH $\mu = \alpha^2$.

Our first theorem deals with the case $\alpha = 1$, $\mu = 1$.

<u>THEOREM 1</u>. *Every strongly regular graph with parameters* $(n,k,\lambda,\mu = 1)$ *is the point-graph of a s.p.g. with* $s = \lambda+1$, $t = \frac{k}{\lambda+1} - 1$, $\alpha=1$, $\mu=1$.

<u>PROOF</u>. Let (X,E) be a strongly regular graph with $\mu = 1$, and let $x \in X$. Since two nonadjacent points in $\Gamma(x)$ cannot have a common neighbour in $\Gamma(x)$, the induced subgraph on $\Gamma(x)$ in the union of cliques. This induced subgraph has valency λ, so it is the union of $\frac{k}{\lambda+1}$ cliques of size $\lambda+1$. □

Next we deal with the case $\alpha = 2$, $\mu = 4$.

THEOREM 2. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* s,t, $\alpha = 2$, $\mu = 4$. *Then* $(X,\mathcal{L})$ *satisfies* (D).

PROOF. Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$. If $x_2 \not\sim x_4$, then we can apply (*) to the points x_2 and x_4. Since $x_1x_4 \in [x_4,x_2]$ and $x_2x_3 \in [x_2,x_4]$, x_1x_4 and x_2x_3 intersect in a point $\neq x_2,x_3$. Now $3 \leq |[x_1,x_2x_3]| \leq \alpha=2$, a contradiction. □

Let U be a set containing t+3 elements. Then we denote by $U_{2,3}$ the s.p.g. which has as points the 2-subsets of U, as lines the 3-subsets of U together with the natural incidence.

The parameters are $s=2$, t, $\alpha=2$, $\mu=4$.

DEBROEY [1] showed that a s.p.g. with $t>1$, $\alpha=2$, $\mu=4$ satisfying (D) is isomorphic to a $U_{2,3}$. Hence we have the following theorem.

THEOREM 3. *A s.p.g. with* $t>1$, $\alpha=2$, $\mu=4$ *is isomorphic to a* $U_{2,3}$. □

REMARK. A s.p.g. with $t=1$, $\alpha=2$, $\mu=4$ is isomorphic to the geometry of edges and vertices of the complete graph K_{s+2}.

We now consider the case $\alpha>2$. For the remainder of this section let $(X,\mathcal{L})$ be a s.p.g with $\alpha>2$ and $\mu = \alpha^2$.

LEMMA 1. *Let* $x \in X$, $L \in \mathcal{L}$, $x \notin L$ *such that* $[L,x] = \{z_1,\dots,z_\alpha\}$. *Let* M *be a line through* z_1 *intersecting* xz_2 *in a point* $u \neq x,z_2$. *Suppose there exists* $y \in L$, $y \neq z_1,\dots,z_\alpha$ *with* $u \not\sim y$. *Then* M *intersects* xz_i *for all* $i = 1,\dots,\alpha$ (*see figure* 1).

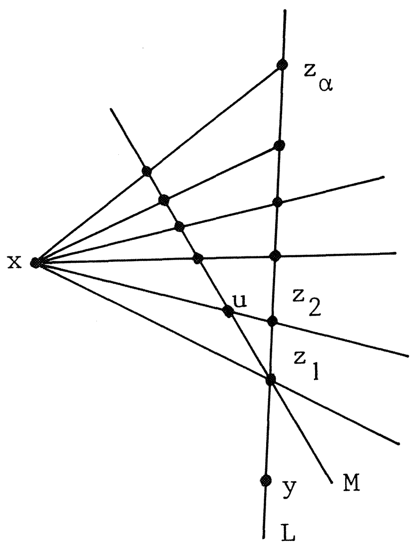

Figure 1.

PROOF. By (*) applied to x and y, the α lines $L = L_1, L_2, \dots, L_\alpha$ of $[y,x]$ intersect the α lines $xz_1, \dots, xz_\alpha$ of $[x,y]$. In particular $L_1, \dots, L_\alpha$ intersect xz_2. Hence $[y,u] = [y,x] = \{L_1, \dots, L_\alpha\}$.

Since $M \in [u,y]$, M intersects $L_1, \dots, L_\alpha$ in points $v_1 = z_1, v_2, \dots, v_\alpha$ respectively. If $x \sim v_i$ for all i, then the $\alpha+1$ points $u, v_1 v_2, \dots, v_\alpha$ on M are all collinear with x, a contradiction. Hence $x \not\sim v_i$ for some i. Since L_i intersects $xz_1, \dots, xz_\alpha$ it follows that $[x,v_i] = [x,y] = \{xz_1, \dots, xz_\alpha\}$. Since $M \in [v_i, x]$, M intersects all lines in $[x, v_i]$. □

LEMMA 2. *Let* $x \in X$, $L \in \mathcal{L}$, $x \notin L$ *such that* $[L,x] = \{z_1, \dots, z_\alpha\}$. *Let* M *be a line through* z_1 *intersecting* xz_2 *in a point* $u \neq x, z_2$. *If* $s > \alpha$, *then* M *intersects* xz_i *for all* $i = 1, \dots, \alpha$.

PROOF. Assume that M intersects xz_i, $i = 1, \dots, \beta$ $(2 \le \beta < \alpha)$ in points $u_1 = z_1$, $u_2 = u, \dots, u_\beta$ respectively and does not intersect $xz_{\beta+1}, \dots, xz_\alpha$. Take $y \in L$, $y \neq z_1, \dots, z_\alpha$. By lemma 1 $y \sim u_i$, $i = 1, \dots, \beta$.

Since $|[M,x]| = \alpha$, there is a $v \in M$ such that $v \sim x$, $v \neq u_1, \dots, u_\beta$. Also $v \sim z$ for all $z \in \bigcup_{i=1}^{\beta} [yu_i, x]$, for if $v \not\sim z$ for some $z \in [yu_i, x]$, then $vx \in [v,z]$ and $yu_i \in [z,v]$. Hence $vx \sim yu_i$ and so yu_i intersects the $\alpha+1$ lines $xv, xz_1, \dots, xz_\alpha$ through x, a contradiction. The points of $\bigcup_{i=1}^{\beta} [yu_i, x]$ are therefore on the α lines $M = vz_1, vz_2, \dots, vz_\alpha$ of $[v,y]$.

Since $s > \alpha$ we can take $y' \in L$ such that $y' \neq y, z_1, \dots, z_\alpha$.

Now if $z \in [yu_2, x]$, then $z \sim y'$. Indeed, as shown z is on some vz_i and since vz_i intersects at most $\alpha-1$ of the lines $xz_1, \dots, xz_\alpha$, it follows from Lemma 1 that every point of intersection of vz_i and a line xz_j, so in particular z, is collinear with y'.

But now we have $|[yu_2, y']| \geq |[yu_2, x] \cup \{y\}| = \alpha+1$, a contradiction. □

LEMMA 3. *Let* $x \in X$, $L \in \mathcal{L}$, $x \notin L$ *such that* $[L,x] = \{z_1, \dots, z_\alpha\}$. *If* $s > \alpha$, *then every line* M *not through* x *which intersects two lines of* $[x,L] = \{xz_1, \dots, xz_\alpha\}$ *also intersects* L *and all lines of* $[x,L]$.

PROOF. The number of pairs $(u,v) \neq (z_1, z_2)$ such that $u \in xz_1$, $v \in xz_2$, $u,v \neq x$, $u \sim v$ equals $s(\alpha-1)-1$. Every line $M \neq xz_1, \dots, xz_\alpha$ which intersects L and $xz_1, \dots, xz_\alpha$ gives rise to such a pair (u,v). By (*) and lemma 2 the number of these lines equals $(s+1-\alpha)(\alpha-1) + \alpha(\alpha-2) = s(\alpha-1)-1$. □

Let $L_1,L_2 \in \mathcal{L}$ intersect in a point x. If L is any line intersecting L_1 and L_2 not in x, we let $L_3,L_4,\ldots,L_\alpha$ be the other lines in [x,L]. By lemma 3, $L_3,L_4,\ldots,L_\alpha$ are independent of the choice of L. Put

$$\mathcal{L}(L_1,L_2) := \{L_1,L_2,\ldots,L_\alpha\} \cup \{L \in \mathcal{L} \mid L \sim L_1,L_2, LL_1 \neq x \neq LL_2\},$$

$$X(L_1,L_2) := \bigcup_{L \in \mathcal{L}(L_1,L_2)} L$$

LEMMA 4. *Let* $L_1,L_2 \in \mathcal{L}$, $L_1 \sim L_2$. *If* $s > \alpha$, *then* $\langle L_1,L_2\rangle := (X(L_1,L_2),\mathcal{L}(L_1,L_2))$ *is a partial geometry (in fact a dual design) with parameters* $\tilde{s} = s$, $\tilde{t} = \alpha-1$, $\tilde{\alpha} = \alpha$.

PROOF. Clearly two points are on at most one line and each line contains s+1 points. Using (*) and Lemma 3 it follows immediately that every point $x \in X(L_1,L_2)$ is on α lines of $\mathcal{L}(L_1,L_2)$ so $\tilde{t}+1 = \alpha$. It also follows immediately that any two lines of $\mathcal{L}(L_1,L_2)$ intersect, hence $\tilde{\alpha} = \tilde{t}+1 = \alpha$. □

Notice that for $M_1,M_2 \in \mathcal{L}(L_1,L_2)$, $M_1 \neq M_2$, $M_1 \sim M_2$ we have $\langle M_1,M_2\rangle = \langle L_1,L_2\rangle$. Notice also that for any two noncollinear points x and y of $\langle L_1,L_2\rangle$ there are $\tilde{\mu} = \tilde{\alpha}(\tilde{t}+1) = \alpha^2 = \mu$ points $z \in X(L_1,L_2)$ collinear with both x and y, i.e. the common neighbours of x and y in $(X,\mathcal{L})$ are the common neighbours of x and y in $\langle L_1,L_2\rangle$.

THEOREM 4. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* $s,t,\alpha(>2)$, $\mu = \alpha^2$. *If* $s > \alpha$ *and* $t \geq \alpha$, *then* $(X,\mathcal{L})$ *satisfies* (D).

PROOF. Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.

Suppose $x_2 \not\sim x_4$. Since $x_2 \sim x_1 \sim x_4$ it follows that

$$x_1 \in \langle x_4x_3, x_2x_3\rangle \qquad (\dagger)$$

In $(X,\mathcal{L})$ there are $\lambda = s-1 + (\alpha-1)t$ points collinear with both x_1 and x_3. In $\langle x_4x_3,x_2x_3\rangle$ there are $\tilde{\lambda} = \tilde{s}-1 + (\tilde{\alpha}-1)\,\tilde{t} = (s-1) + (\alpha-1)^2$ points collinear with both x_1 and x_3. Since $t \geq \alpha = \tilde{t} + 1$ it follows that $\tilde{\lambda} < \lambda$ and so there exists $x_5 \in X \backslash X(x_4x_3,x_2x_3)$ such that $x_1 \sim x_5 \sim x_3$. Now application of

(†)

to x_1,x_5,x_3,x_4 yields $x_5 \sim x_4$,
to x_1,x_2,x_3,x_5 yields $x_5 \sim x_2$,
to x_4,x_1,x_2,x_5 yields $x_2 \sim x_4$. □

DEBROEY [1] showed that a s.p.g. with parameters $s,t,\alpha(>2)$, $\mu = \alpha^2$ satisfying (D) is of the following type: the "points" are the lines of PG(d,q), the "lines" are the planes in PG(d,q) for some prime power q and $d \in \mathbb{N}$, $d \geq 4$. In this case $s = q(q+1)$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q+1$, $\mu = (q+1)^2$.

THEOREM 5. *Let* (X,L) *be a s.p.g. with parameters* $s,t,\alpha(>2)$, $\mu = \alpha^2$. *If* $s > \alpha$ *and* $t \geq \alpha$, *then* (X,L) *is isomorphic to the s.p.g. consisting of the lines and planes in* PG(d,q). *In particular* $s = q(q+1)$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q+1$, $\mu = (q+1)^2$.

The only interesting case remaining is $s = \alpha$. Now if (X,E) is a Moore graph of valency r, i.e. a strongly regular graph with $\lambda = 0$, $\mu = 1$, then $(X,\{\Gamma(x) \mid x \in X\})$ is easily seen to be a s.p.g. with parameters $s = t = \alpha =$ $= r-1$, $\mu = (r-1)^2$ (here $\Gamma(x) = \{y \in X \mid (x,y) \in E\}$). The point graph of this s.p.g. is the complement of (X,E). Such a s.p.g. does not satisfy (D) for $r > 2$. From the following theorem follows immediately that a s.p.g. with $\mu = \alpha^2$, $s = \alpha$ is necessarily of this type.

THEOREM 6. *Let* (X,L) *be a s.p.g. with* $t \geq \alpha$, $\mu = \alpha^2$ *and* $s = \alpha$. *Then* $t = \alpha$.

PROOF. Let $x,y \in X$, $x \not\sim y$. Let $[x,y] = \{L_1,\dots,L_\alpha\}$, $[y,u] = \{M_1,\dots,M_\alpha]$ and put $z_{ij} = L_i M_j$, $i,j = 1,\dots,\alpha$ (see figure 2).

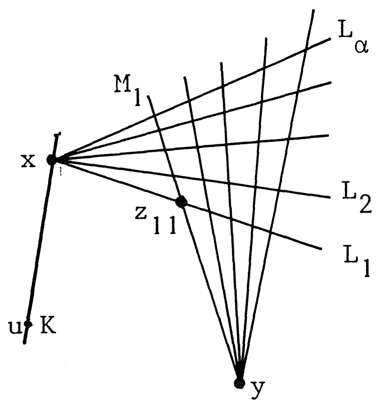

Figure 2.

The number of $(z_{ij},z_{k\ell})$ with $i \neq k$, $j \neq \ell$, $z_{ij} \sim z_{k\ell}$ equals $\alpha^2 \cdot (\alpha-1)(\alpha-2)$. Now let K be a line through x, $K \neq L_1,\ldots,L_\alpha$, and let u be a point on K, $u \neq x$.

Then u is collinear with $(\alpha-1)$ of the α points $z_{i,1},\ldots,z_{i,\alpha}$, for $i = 1,\ldots,\alpha$. Since $u \not\sim y$, u is collinear with all of $z_{1,j},\ldots,z_{\alpha,j}$ or with none, for $j = 1,\ldots,\alpha$.

It follows that there are α lines through u intersecting $(\alpha-1)$ of the α lines $M_1,\ldots,M_\alpha$. Hence each point $u \neq x$ on K gives rise to $\alpha(\alpha-1)(\alpha-2)$ pairs $(z_{ij},z_{k\ell})$ as described, so K gives rise to all $\alpha^2(\alpha-1)(\alpha-2)$ pairs $(z_{ij},z_{k\ell})$.

Suppose $t > \alpha$, then we can find two such lines K and K'. It follows that for $u \in K$, the α lines through u intersecting $(\alpha-1)$ of the α lines $M_1,\ldots,M_\alpha$ also intersect K'. But now $|[u,K']| = \alpha+1$, a contradiction. □

3. SEMI-PARTIAL GEOMETRIES WITH $\mu = \alpha(\alpha+1)$.

In this section (X,L) is a semi-partial geometry with parameters s,t,α and $\mu = \alpha(\alpha+1)$.

If $x,y \in X$, $x \not\sim y$ we shall always denote the $\alpha+1$ lines in $[x,y]$ by $K_1,\ldots,K_{\alpha+1}$, and the $(\alpha+1)$ lines in $[y,x]$ by $L_1,\ldots,L_{\alpha+1}$. By (**) we can number these lines in such a way that $K_i \cap L_i = \emptyset$, $i = 1,\ldots,\alpha+1$ and $K_i \cap L_j \neq \emptyset$, $i,j = 1,\ldots,\alpha+1$, $i \neq j$ (see figure 3).

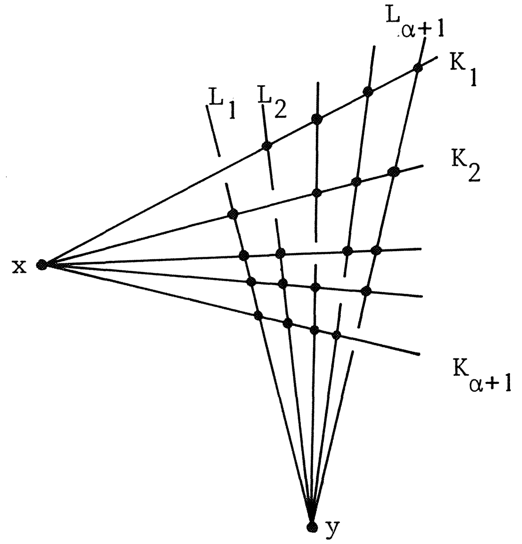

Figure 3.

Again our aim will be to show that the diagonal axiom (D) holds. We first deal with the case $\alpha = 2$.

LEMMA 5. *If* $\alpha = 2$ *and* $t > s$, *then a set of* 3 *collinear points not on one line can be extended to a set of* 4 *collinear points no 3 on a line.*

PROOF. Let x, a and b be three distinct collinear points not on one line. There are t-1 lines $\neq$ xa,ab through a and on each of those lines there is a point $y_i \sim b$, $y_i \neq a$, $i = 1,\dots,t-1$. Suppose $y_i \not\sim x$ for all $i = 1,\dots,t-1$. Now for each $i = 1,\dots,t-1$, $ay_i \not\sim xb$ (for otherwise $|[a,xb]| \geq 3$) and $by_i \not\sim xa$. Also $xa, xb \in [x,y_i]$ and $ay_i, by_i \in [y_i,x]$. Hence, by (**) there is a third line through y_i intersecting xa and xb in points u_i and v_i respectively. Clearly $u_i \neq u_j$ if $i \neq j$, for $u_i = u_j$ implies $x, v_i, v_j \in [u_i, xb]$. Thus xa contains $t+1 > s+1$ points (namely $x, a, u_1, \dots, u_{t-1}$), a contradiction. □

LEMMA 6. *Suppose* $\alpha = 2$. *If* x_1, x_2, x_3, x_4 *are four distinct collinear points, no three on a line, then no point can be collinear with exactly three of these four points.*

PROOF. Suppose x_5 is collinear with x_2, x_3, x_4 and $x_1 \not\sim x_5$. Clearly $x_5 \notin x_2x_3, x_2x_4, x_3x_4$. Hence $\{x_1x_2, x_1x_3, x_1x_4\} = [x_1, x_5]$ and $\{x_5x_2, x_5x_3, x_5x_4\} = [x_5, x_1]$ so x_5x_2 has to intersect x_1x_3 or x_1x_4 by (**). But then $|[x_2, x_1x_3]|$ or $|[x_2, x_1x_4]| > 2$, a contradiction. □

LEMMA 7. *Same hypothesis as in lemma 6. Then the only points collinear with exactly two points of* $\{x_1, x_2, x_3, x_4\}$ *are the points on the lines* x_ix_j, $i \neq j$.

PROOF. Suppose $x_5 \sim x_1, x_4$ and $x_5 \not\sim x_2, x_3$, $x_5 \notin x_1x_4$ (see figure 4).

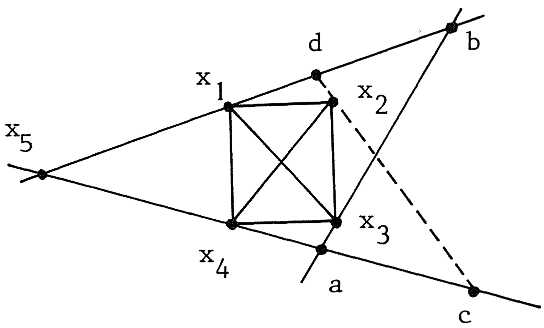

Figure 4.

Apply (**) to x_3 and x_5 to get a line ab through x_3 with $a \in x_5x_4$, $b \in x_5x_1$. Similarly (**) applied to x_5 and x_2 gives us a line cd through x_2 with $c \in x_5x_4$, $d \in x_5x_1$. Clearly $b \not\sim c$ so we can apply (**) to b and c. It follows that $ab \cap cd = \emptyset$. Also $x_2 \not\sim a$ and (**) applied to x_2 and a yields: $ab \cap cd \neq \emptyset$ or $ab \cap x_2x_4 \neq \emptyset$. Hence $ab \cap x_2x_4 \neq \emptyset$, a contradiction since $\{x_2,x_4\} = [x_2x_4,x_3]$. □

THEOREM 7. *If* (X,L) *is a s.p.g with parameters* $s,t,\alpha = 2$, $\mu = 6$ and $t>s$, *then* (X,L) *satisfies* (D).

PROOF. Let x_1,x_2,x_3 and x_4 be four distinct points no three on a line such that $x_4 \sim x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_2$. By Lemma 5 there exists $x_5 \sim x_2,x_3,x_4$.

By Lemmas 6 and 7 $x_1 \sim x_3,x_5$. □

REMARK. If (X,L) is a s.p.g but not a partial geometry, then $t \geq s$ (see DEBROEY & THAS [2]). Using the integrality conditions for the multiplicities of the eigenvalues of a strongly regular graph it follows that a s.p.g with $s=t$, $\alpha=2$ and $\mu=6$ satisfies $(8s^2-24s+25)\,|\,\{8(s+1)(2s^3-9s^2+19s-30)\}^2$. From this one easily deduces an upper bound for s. The remaining cases were checked by computer and only $s=t=28$ survived. Thus, every s.p.g which is not a partial geometry satisfies (D) or has $s=t=28$ (and 103125 points).

We now turn to the case $\alpha \geq 3$. We shall make two additional assumptions in this case. The first assumption is $\alpha \neq 3$, the second assumption is $s \geq f(\alpha)$ where f is defined in Lemma 9. Notice that this bound on s is used only in the proof of Lemma 9.

LEMMA 8. *Let* $x,y \in X$, $x \not\sim y$ *and suppose* $[x,y] = [K_1,\ldots,K_{\alpha+1}]$, $[y,x] = [L_1,\ldots,L_{\alpha+1}]$ *such that* $K_i \cap L_i = \emptyset$, $i = 1,\ldots,\alpha+1$. *If* M *is a line intersecting* $\sigma \geq 1$ *lines of* $[x,y]$, $\tau \geq 1$ *lines of* $[y,x]$ *and* $\sigma < \tau$, *then* $\sigma = \alpha-1$ *and* $\tau = \alpha$.

PROOF. Since $\sigma < \tau$, there exists a point of intersection u of M with a line $L_i \in [y,x]$ such that u is not on one of the lines of $[x,y]$. Then $u \not\sim x$ and so, applying (**) to u and x, it follows that $M \in [u,x]$ intersects $\alpha-1$ of the α lines $K_1,K_2,\ldots,K_{i-1},K_{i+1},\ldots,K_{\alpha+1} \in [x,u]$. Thus $\alpha-1 \leq \sigma < \tau \leq \alpha$, which proves our claim. □

LEMMA 9. *Let* $x \in X$ *and* $L \in \mathcal{L}$ *such that* $x \notin L$ *and* x *is collinear with* α *points* $z_2, z_3, \ldots, z_{\alpha+1}$ *on* L. *Let* M *be a line through* $z_{\alpha+1}$ *meeting* xz_α *in a point* $u \neq x, z_\alpha$. *Suppose* $s \geq f(\alpha)$ *where* $f(4) = 12$, $f(5) = 16$, $f(6) = f(7) = 17$, $f(8) = 18$, $f(9) = 19$, $f(10) = 21$, $f(11) = 23$, $f(\alpha) = 2\alpha$ $(\alpha \geq 12)$. *Then* M *intersects at least* $\alpha-1$ *lines of* $[x,L]$.

PROOF. Suppose M does not meet at least two lines of $[x,L]$, xz_2 and xz_3, say. Since $s \geq 2\alpha$ we can find $y \in L$ such that $x \not\sim y \not\sim u$. Let $[x,y] = \{K_1, K_2 = xz_2, \ldots, K_{\alpha+1} = xZ_{\alpha+1}\}$ and $[y,x] = \{L_1 = L, L_2, L_3, \ldots, L_{\alpha+1}\}$ with $K_i \cap L_i = \emptyset$.

Looking at u and y we find that M intersects $\alpha-1$ of the α lines L_i, $i \neq \alpha$. Every point L_iM which is collinear with x is on a line K_j, $j \neq \alpha$. If $L_iM \sim x$ for these $\alpha-1$ i's, we find that M meets at least α of the lines $K_1, \ldots, K_{\alpha+1}$, hence at least $\alpha-1$ of the lines $K_2, \ldots, K_{\alpha+1}$, a contradiction. Let $t = L_iM$ be a point not collinear with x. Considering $x \not\sim t$ we see that M intersects $\alpha-1$ of the α lines in $[x,y] \backslash \{K_i\}$. This shows that $i = 2$ or 3, so there are at most two such points t, and that M meets $K_1, K_4, K_5, \ldots, K_{\alpha+1}$. Let $V = \{K_4M, K_5M, \ldots, K_\alpha M\}$ and count pairs (y,v), $y \in L$, $y \not\sim x$, $v \in V$, $v \sim y$. The number of such pairs is at least $(s-\alpha+1)(\alpha-5)$ (first choose y, $s-\alpha+1$ possibilities, then given y we can find $\alpha-3$ points $L_iM \sim x$ as above, possibly one on $K_1(y)$, and one is $z_{\alpha+1}$), and at most $(\alpha-3)(\alpha-2)$ (first choose v, then y). It follows that for $\alpha > 5$, $s \leq 2\alpha - 1 + \lfloor \frac{6}{\alpha-5} \rfloor$. Let $W = V \cup \{q,q'\} = \{w \in M | w \sim x\}$ and count pairs (y,w), $y \in L$, $y \not\sim x$, $w \in W$, $w \sim y$. This yields $(s-\alpha+1)(\alpha-4) \leq (\alpha-3)(\alpha-2) + 2(\alpha-1)$, hence $s \leq 2\alpha + \lfloor \frac{8}{\alpha-4} \rfloor$ if $\alpha > 4$. Above we saw that for any $y \in L$ with $x \not\sim y \not\sim u$, $K_1 = K_1(y)$ meets M. But if $s+1 > \alpha + (\alpha-2) + 2(\alpha-1) = 4\alpha-4$, we can find $y \in L$ such that $y \not\sim x$, u,q and q', a contradiction. Therefore we have $s < 4\alpha-4$. We now have obtained a contradiction for all $\alpha \geq 4$ and the lemma is proved. □

LEMMA 10. *Some hypotheses as in Lemma 9. Then* M *intersects exactly* $\alpha-1$ *lines of* $[x,L]$.

PROOF. Take $y \in L$, $y \not\sim x$ and let K_i and L_i be defined as before. Put $K := K_{\alpha+1}$ and let $A(x,L)$ be the set of lines $\neq K,L$ through $z_{\alpha+1}$ intersecting at least $\alpha-1$ lines of $[x,L]$, $A(y,K)$ the set of lines $\neq K,L$ through $z_{\alpha+1}$ intersecting at least $\alpha-1$ lines of $[y,K]$. Suppose a lines of $A(x,L)$ intersects $\alpha-1$ lines of $[x,L]$ and b lines of $A(x,L)$ intersect α lines of $[x,L]$. Counting the points $u\sim z_{\alpha+1}$ on $K_2, K_3, \ldots, K_\alpha$, such that $u \neq x, z_2, \ldots, z_\alpha$ yields $a(\alpha-2) + b(\alpha-1) = (\alpha-1)(\alpha-2)$. Hence $a = 0$ and $b = \alpha-2$ or $a = \alpha-1$ and $b = 0$. Thus $|A(x,L)| = \alpha-2$ or $\alpha-1$ according as every line in $A(x,L)$ intersects all lines or all but one line in $[x,L]$. A similar result holds for $A(y,K)$. Now $A(x,L) = A(y,K)$, for suppose $N \in A(x,L)$ then by Lemma 8, N intersects at least $\alpha-1$ lines of $[y,x]$, so at least $\alpha-2 \geq 2$ lines of $[y,K]$. Hence $N \in A(y,K)$ by Lemma 9. Similarly, $N \in A(y,k)$ implies $N \in A(x,L)$. Suppose $|A(x,L)| = \alpha-2$, i.e. there are $\alpha-2$ lines through $z_{\alpha+1}$ intersecting all lines of $[x,L] \cup [y,K]$. It follows that $K_2L_{\alpha+1} \not\sim z_{\alpha+1}$ so we can apply (**) to $K_2L_{\alpha+1}$ and $z_{\alpha+1}$. This shows that $L_{\alpha+1} \in [K_2L_{\alpha+1}, z_{\alpha+1}]$ intersects all $N \in A(y,K) \subseteq [z_{\alpha+1}, K_2L_{\alpha+1}]$, a contradiction, for $L_{\alpha+1}\sim N$ implies $|[y,N]| \geq$ $\geq \alpha+1$. □

LEMMA 11. *Let* $x \in X$, $L \in L$ *such that* x *is collinear with* α *points* $z_2, \ldots, z_{\alpha+1}$ *on* L. *Let* M *be a line through* $z_{\alpha+1}$ *intersecting* $\alpha-1$ *lines of* $[x,L]$ *and let* $y \in L$, $y\not\sim x$. *Then, if* $[x,y] = \{K_1(y), K_2=xz_2, \ldots, K_{\alpha+1}=xz_{\alpha+1}\}$, M *intersects* $K_1(y)$.

PROOF. Suppose M does not intersect K_2, say. As shown in Lemma 10, M also intersects $\alpha-1$ lines of $[y,K_{\alpha+1}] = \{L_1=L, L_2, \ldots, L_\alpha\}$. So M intersects at least one of $L_{\alpha-1}$ and L_α and since $\alpha \geq 4$, $L_2 \neq L_{\alpha-1}, L_\alpha$. Suppose M intersects $L_{\alpha-1}$ (L_α) in a point v. If $v\not\sim x$ then apply (**) to v and x. It follows that $M \in [v,x]$ intersects $K_1(y) \in [x,v]$ for M misses $K_2 \in [x,v]$. If $v\sim x$ then $v = L_{\alpha-1}K_i$ ($v=L_\alpha K_i$) for some i. By Lemma 10 applied to x and $L_{\alpha-1}$ (L_α) it follows that M intersects $K_1(y) \in [x,L_{\alpha-1}]$ ($K_1(y) \in [x,L_\alpha]$), for M does not intersect $K_2 \in [x,L_{\alpha-1}]$ ($K_2 \in [x,L_\alpha]$). □

COROLLARY. *The line* $K_1(y)$ *is the same for all* $y \in L$, $y \not\sim x$.

LEMMA 12. *Let* $x \in X$, $L \in L$ *such that* x *is collinear with* α *points* $z_2, z_3, \ldots, z_{\alpha+1}$ *on* L. *Put* $K_i=xz_i$, $i=2,\ldots,\alpha+1$ *and let* K_1 *be defined by* $\{K_1, K_2, \ldots, K_{\alpha+1}\} = [x,y]$ *for any* $y \in L$, $y \not\sim x$. *Then every line which intersects* K_1 *and a* $K_i (i\neq 1)$ *not in* x, *intersects* L *and therefore exactly* α *lines of* $\{K_1, \ldots, K_{\alpha+1}\}$.

PROOF. Fix $i \in \{2,\ldots,\alpha+1\}$. The number of pairs (u,v) such that

$u \in K_1\setminus\{x\}$, $v \in K_i\setminus\{x\}$, $u \sim v$ equals $s(\alpha-1)$. If $y \in L$, $y \not\sim x$ and $[y,x] = \{L_1=L,L_2,\ldots,L_{\alpha+1}\}$, then each of the $\alpha-1$ lines $L_2,L_3,\ldots,L_{i-1},L_{i+1},\ldots,L_{\alpha+1}$ gives rise to such a pair (u,v). Each point z_j, $j = 2,3,\ldots,i-1,i+1,\ldots,\alpha+1$ is on $\alpha-1$ lines $\neq K_j,L$ which intersect α lines of $\{K_1,\ldots,K_{\alpha+1}\}$. They all intersect K_1 by Lemma 11 and no two miss the same K_k since otherwise some K_ℓ would be hit $\alpha+1$ times. Thus each point z_j, $j=2,3,\ldots,i-1,i+1,\ldots,\alpha+1$ gives rise to $(\alpha-2)$ pairs (u,v). Finally there are $(\alpha-1)$ pairs (u,v) with $v = z_i$. In all, the lines intersecting L contain $(s+1-\alpha)(\alpha-1) + (\alpha-1)(\alpha-2) + (\alpha-1) = s(\alpha-1)$, i.e. all, pairs (u,v). □

If in Lemma 12 we replace $L = L_1$ by a line L_j missing K_j, then it follows that every line intersecting two lines of $\{K_1,\ldots,K_{\alpha+1}\}$ not in x, intersects exactly α lines of $\{K_1,\ldots,K_{\alpha+1}\}$. Using this result and the foregoing lemmas we can now proceed as in the case $\mu = \alpha^2$. For any two intersecting lines L_1,L_2 we can define in an obvious way a partial geometry $\langle L_1,L_2\rangle = (X(L_1,L_2),\mathcal{L}(L_1,L_2))$, now with parameters $\tilde{s} = s$, $\tilde{t} = \alpha$, $\tilde{\alpha} = \alpha$ (so $\langle L_1,L_2\rangle$ is an $(\alpha+1)$-net of order $s+1$). Again $\tilde{\mu} = \tilde{\alpha}(\tilde{t}+1) = \alpha(\alpha+1) = \mu$, so with the same proof as the proof of Theorem 4 we have the following theorem.

THEOREM 8. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* $s,t,\alpha,\mu = \alpha(\alpha+1)$. *If* $\alpha \geq 4$, $s \geq f(\alpha)$ (f as in Lemma 9) *and* $t \geq \alpha+1$ (i.e. *if* $(X,\mathcal{L})$ *is not a p.g.*), *then* $(X,\mathcal{L})$ *satisfies* (D).

Fix a $(d-2)$-dimensional subspace S of $PG(d,q)$, q a prime power, $d \in \mathbb{N}$. Then with the lines of $PG(d,q)$ which have no point with S in common as "points" and with the planes of $PG(d,q)$ intersecting S in exactly one point as "lines" and with the natural incidence relation, one obtains a s.p.g. with parameters $s = q^2-1$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q$, $\mu = q(q+1)$.

DEBROEY [1] showed that a s.p.g. with parameters $s,t,\alpha \geq 2$, $\mu = \alpha(\alpha+1)$ and satisfying (D) is of this type. Combining this result with Theorems 7 and 8 we arrive at the following theorem.

THEOREM 9. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* $s,t,\alpha,\mu = \alpha(\alpha+1)$ *which is not a p.g.. If* $\alpha = 2$ *and not* $s = t = 28$ *or if* $\alpha \geq 4$ *and* $s \geq f(\alpha)$, *then* $(X,\mathcal{L})$ *is isomorphic to a s.p.g. consisting of the lines in* $PG(d,q)$ *missing a given* $(d-2)$*-dimensional subspace of* $PG(d,q)$ *and the planes inter-*

secting this subspace in one point. In particular $s = q^2-1$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q$, $\mu = q(q+1)$ *for some prime power* q *and* $d \in \mathbb{N}$ *and any s.p.g. with these parameters with* $q \neq 3$ *and* $d \geq 4$ *is of this type.*

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