H.A. Wilbrink & A.E. Brouwer

A Characterization of Two Classes of Semi Partial Geometries by Their Parameters

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A characterization of two classes of semi partial geometries by their parameters *)

by

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ABSTRACT

We show that, under mild restrictions on the parameters, semi-partial geometries with \( \mu = \alpha^2 \) or \( \mu = \alpha(\alpha+1) \) are determined by their parameters.

KEY WORDS & PHRASES: Semi-partial geometry, partial geometry, strongly regular graph

*) This report will be submitted for publication elsewhere.
Let $X$ be a (finite) nonempty set and $L$ a set of subsets of $X$. Elements of $X$ are called points, elements of $L$ are called lines. The pair $(X,L)$ is called a partial linear space if any two distinct points are on at most one line.

Two distinct points $x$ and $y$ are called collinear if there exists $L \in L$ such that $x,y \in L$, noncollinear otherwise. Two distinct lines $L$ and $M$ are called concurrent if $|L \cap M| = 1$.

We write $x \sim y$ ($x \not\sim y$) to denote that $x$ and $y$ are collinear (noncollinear). Similarly $L \sim M$ ($L \not\sim M$) means $|L \cap M| = 1(|L \cap M| = 0)$.

If $x \sim y$ ($L \sim M$) we denote by $xy$ ($LM$) the line (point) incident with $x$ and $y$ ($L$ and $M$).

For a nonincident point-line pair $(x,L)$ we define:

$$[L,x] := \{ y \in X | y \in L, y \sim x \},$$

$$[x,L] := \{ M \in L | x \in M, L \sim M \}.$$

Given positive integers $s,t,\alpha,\mu$, the partial linear space $(X,L)$ is called a semi-partial geometry (s.p.g) with parameters $s,t,\alpha,\mu$ if:

(i) every line contains $s+1$ points,
(ii) every point is on $t+1$ lines,
(iii) for all $x \in X, L \in L$, $x \not\in L$ we have $|[x,L]| \in \{0,\alpha\}$,
(iv) for all $x,y \in X$ with $x \not\sim y$ the number of points $z$ such that $x \sim z \sim y$ equals $\mu$.

A semi-partial geometry which satisfies $|[x,L]| = \alpha$ for all $x \in X, L \in L$ with $x \not\in L$, or equivalently which satisfies $\mu = \alpha(t+1)$, is also called a partial geometry (p.g).

The point-graph of the partial linear space $(X,L)$ is the graph with vertex set $X$, two distinct vertices $x$ and $y$ being adjacent iff $x \sim y$. The point-graph of a semi-partial geometry is easily seen to be strongly regular. Let $(X,L)$ be a semi-partial geometry.

For $x,y \in X, x \not\sim y$ we define
\[ [x,y] := \{ L \in L | x \in L, \|[L,y]\| = \alpha \}. \]

It is easy to see that \( \alpha = s+1 \) iff any two distinct points are collinear iff \((X,L)\) is a Steiner system \(S(2,s+1,|X|)\). We shall always assume \( s \geq \alpha \), hence noncollinear points exist.

Let \( x,y \in X, x \neq y \). Then \( \mu = |[x,y]_\alpha| \) and \( |[x,y]_\alpha| \geq |[x,L]| = \alpha \) if \( L \in [y,x] \). Hence, \( \mu \geq \alpha^2 \) and

\[ (*) \quad \mu = \alpha^2 \iff \forall K \in [x,y], L \in [y,x] : K \sim L, \]

\[ (* *) \quad \mu = \alpha(\alpha+1) \iff \text{every line } K \in [x,y] \text{ intersect every line } L \in [y,x] \]

but one.

This is the basic observation we use in showing that, under mild restrictions on the parameters, semi partial geometries with \( \mu = \alpha^2 \) or \( \mu = \alpha(\alpha+1) \) satisfy the Diagonal Axiom (D).

(D) \: Let \( x_1, x_2, x_3, x_4 \) be four distinct points no three on a line, such that \( x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3 \).
Then also \( x_2 \sim x_4 \).

From DEBROUY [1], it then follows that such a semi-partial geometry is known.

2. SEMI-PARTIAL GEOMETRIES WITH \( \mu = \alpha^2 \).

Our first theorem deals with the case \( \alpha = 1, \mu = 1 \).

**Theorem 1.** Every strongly regular graph with parameters \((n,k,\lambda,\mu = 1)\) is the point-graph of a s.p.g. with \( s = \lambda+1, t = \frac{k}{\lambda+1} - 1, \alpha = 1, \mu = 1 \).

**Proof.** Let \((X,E)\) be a strongly regular graph with \( \mu = 1 \), and let \( x \in X \).
Since two nonadjacent points in \( \Gamma(x) \) cannot have a common neighbour in \( \Gamma(x) \),
the induced subgraph on \( \Gamma(x) \) in the union of cliques. This induced subgraph
has valency \( \lambda \), so it is the union of \( \frac{k}{\lambda+1} \) cliques of size \( \lambda+1 \). \( \Box \)

Next we deal with the case \( \alpha = 2, \mu = 4 \).
THEOREM 2. Let \((X, L)\) be a s.p.g. with parameters \(s, t, \alpha = 2, \mu = 4\). Then \((X, L)\) satisfies \((D)\).

PROOF. Let \(x_1, x_2, x_3, x_4\) be four distinct points no three on a line, such that \(x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1\). If \(x_2 \neq x_4\), then we can apply \((*)\) to the points \(x_2\) and \(x_4\). Since \(x_1x_4 \in [x_4, x_2]\) and \(x_2x_3 \in [x_2, x_4]\), \(x_1x_4\) and \(x_2x_3\) intersect in a point \(\neq x_2, x_3\). Now \(3 \leq |[x_1, x_2, x_3]| \leq \alpha = 2\), a contradiction. \(\square\)

Let \(U\) be a set containing \(t+3\) elements. Then we denote by \(U_{2, 3}\) the s.p.g. which has as points the 2-subsets of \(U\), as lines the 3-subsets of \(U\) together with the natural incidence.

The parameters are \(s=2, t, \alpha=2, \mu=4\).

DEBROEY [1] showed that a s.p.g. with \(t \geq 1, \alpha=2, \mu=4\) satisfying \((D)\) is isomorphic to a \(U_{2, 3}\). Hence we have the following theorem.

THEOREM 3. A s.p.g. with \(t \geq 1, \alpha=2, \mu=4\) is isomorphic to a \(U_{2, 3}\). \(\square\)

REMARK. A s.p.g. with \(t=1, \alpha=2, \mu=4\) is isomorphic to the geometry of edges and vertices of the complete graph \(K_{s+2}\).

We now consider the case \(\alpha \geq 2\). For the remainder of this section let \((X, L)\) be a s.p.g with \(\alpha \geq 2\) and \(\mu = \alpha^2\).

LEMMA 1. Let \(x \in X, L \in L\) such that \([L, x] = \{z_1, \ldots, z_\alpha\}\). Let \(M\) be a line through \(z_1\) intersecting \(xz_2\) in a point \(u \neq x, z_2\). Suppose there exists \(y \in L, y \neq z_1, \ldots, z_\alpha\) with \(u \neq y\). Then \(M\) intersects \(xz_i\) for all \(i = 1, \ldots, \alpha\) (see Figure 1).

![Figure 1](image.png)
PROOF. By (*) applied to x and y, the α lines \( L = L_1, L_2, \ldots, L_\alpha \) of \([y,x]\) intersect the \( \alpha \) lines \( \alpha x_1, \ldots, \alpha x_\alpha \) of \([x,y]\). In particular \( L_1, \ldots, L_\alpha \) intersect \( \alpha x_2 \). Hence \( [y,u] = [y,x] = \{L_1, \ldots, L_\alpha\} \).

Since \( M \in [u,y] \), M intersects \( L_1, \ldots, L_\alpha \) in points \( v_1 = z_1, v_2, \ldots, v_\alpha \) respectively. If \( x \sim v_i \) for all \( i \), then the \( \alpha+1 \) points \( u, v_1, v_2, \ldots, v_\alpha \) on \( M \) are all collinear with \( x \), a contradiction. Hence \( x \not\sim v_i \) for some \( i \). Since \( L_1 \) intersects \( \alpha x_1, \ldots, \alpha x_\alpha \) it follows that \( [x,v_i] = [x,y] = \{x_1, \ldots, x_\alpha\} \).

Since \( M \in [v_i,x] \), M intersects all lines in \([x,v_i]\). □

**Lemma 2.** Let \( x \in X, L \in L, x \not\in L \) such that \([L,x] = \{z_1, \ldots, z_\alpha\}\). Let \( M \) be a line through \( z_1 \) intersecting \( \alpha x_2 \) in a point \( u \not\in x_2 \). If \( s > \alpha \), then \( M \) intersects \( \alpha x_1 \) for all \( i = 1, \ldots, \alpha \).

**Proof.** Assume that \( M \) intersects \( \alpha x_1 \), \( i = 1, \ldots, \beta \) \( (2 \leq \beta < \alpha) \) in points \( v_1 = z_1, \ldots, v_\beta \) respectively and does not intersect \( \alpha x_\beta+1, \ldots, x_\alpha \). Take \( y \in L, y \not\sim z_1, \ldots, z_\alpha \). By lemma 1 \( y \sim u_1, i = 1, \ldots, \beta \).

Since \([M,x] = \alpha \), there is a \( v \in M \) such that \( v \sim x, v \not\in u_1, \ldots, u_\beta \).

Also \( v \sim z \) for all \( z \in \bigcup_{i=1}^{\beta} [yu_1, x] \), for if \( v \not\sim z \) for some \( z \in [yu_1, x] \), then \( v \in [v,z] \) and \( yu_1 \in [z,v] \). Hence \( v \sim yu_1 \) and so \( yu_1 \) intersects the \( \alpha+1 \) lines \( x_1, x_1, \ldots, x_\alpha \) through \( x \), a contradiction. The points of \( \bigcup_{i=1}^{\beta} [yu_1, x] \) are therefore on the \( \alpha \) lines \( M = v_1, v_2, \ldots, v_\alpha \) of \([v,y]\).

Since \( s > \alpha \) we can take \( y' \in L \) such that \( y' \not\sim y, z_1, \ldots, z_\alpha \).

Now if \( z \in [yu_2, x] \), then \( z \sim y' \). Indeed, as shown \( z \) is on some \( vz_1 \) and since \( vz_1 \) intersects at most \( \alpha-1 \) of the lines \( x_1, \ldots, x_\alpha \), it follows from Lemma 1 that every point of intersection of \( vz_1 \) and a line \( x_1 \), so in particular \( z \), is collinear with \( y' \).

But now we have \([yu_2, y'] \geq [yu_2, x] \cup \{y\} = \alpha+1 \), a contradiction. □

**Lemma 3.** Let \( x \in X, L \in L, x \not\in L \) such that \([L,x] = \{z_1, \ldots, z_\alpha\}\). If \( s > \alpha \), then every line \( M \) not through \( x \) which intersects two lines of \([x,L] = \{xz_1, \ldots, xz_\alpha\} \) also intersects \( L \) and all lines of \([x,L]\).

**Proof.** The number of pairs \((u,v) \not\in (z_1, z_2)\) such that \( u \in xz_1, v \in xz_2 \), \( u,v \not\in x, u \sim v \) equals \( s(\alpha-1)-1 \). Every line \( M \not\in xz_1, \ldots, xz_\alpha \) which intersects \( L \) and \( xz_1, \ldots, xz_\alpha \) gives rise to such a pair \((u,v)\). By (*) and Lemma 2 the number of these lines equals \( s(s-1)(\alpha-1) + \alpha(\alpha-1) = s(s-1)-1 \). □
Let $L_1, L_2 \in L$ intersect in a point $x$. If $L$ is any line intersecting $L_1$ and $L_2$ not in $x$, we let $L_3, L_4, \ldots, L_\alpha$ be the other lines in $[x, L]$. By lemma 3, $L_3, L_4, \ldots, L_\alpha$ are independent of the choice of $L$. Put

$$L(L_1, L_2) := \{ L_1, L_2, \ldots, L_\alpha \} \cup \{ L \in L | L \sim L_1, L_2, LL_1 \neq x \neq LL_2 \},$$

$$X(L_1, L_2) := \cup_{L \in L(L_1, L_2)} L.$$

**Lemma 4.** Let $L_1, L_2 \in L$, $L_1 \sim L_2$. If $s > \alpha$, then $<L_1, L_2> := (X(L_1, L_2), L(L_1, L_2))$ is a partial geometry (in fact a dual design) with parameters $\tilde{s} = s$, $\tilde{t} = \alpha$, $\tilde{\alpha} = \alpha$.

**Proof.** Clearly two points are on at most one line and each line contains $s+1$ points. Using (*) and Lemma 3 it follows immediately that every point $x \in X(L_1, L_2)$ is on $\alpha$ lines of $L(L_1, L_2)$ so $\tilde{t} + 1 = \alpha$. It also follows immediately that any two lines of $L(L_1, L_2)$ intersect, hence $\tilde{\alpha} = \tilde{t} + 1 = \alpha$. □

Notice that for $M_1, M_2 \in L(L_1, L_2)$, $M_1 \neq M_2$, $M_1 \sim M_2$ we have $<M_1, M_2> = <L_1, L_2>$. Notice also that for any two noncollinear points $x$ and $y$ of $<L_1, L_2>$ there are $\tilde{u} = \tilde{a}(\tilde{t} + 1) = \alpha^2 = \mu$ points $z \in X(L_1, L_2)$ collinear with both $x$ and $y$, i.e. the common neighbours of $x$ and $y$ in $(X, L)$ are the common neighbours of $x$ and $y$ in $<L_1, L_2>$.

**Theorem 4.** Let $(X, L)$ be a s.p.g. with parameters $s, t, \alpha (\geq 2)$, $\mu = \alpha^2$. If $s > \alpha$ and $t \geq \alpha$, then $(X, L)$ satisfies (D).

**Proof.** Let $x_1, x_2, x_3, x_4$ be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.

Suppose $x_2 \neq x_4$. Since $x_2 \sim x_1 \sim x_4$ it follows that

$$x_1 \in <x_4, x_3, x_2, x_3> \quad (\dagger)$$

In $(X, L)$ there are $\lambda = s-1 + (\alpha-1)t$ points collinear with both $x_1$ and $x_3$. In $<x_4, x_3, x_2, x_3>$ there are $\tilde{\lambda} = \tilde{s}-1 + (\tilde{\alpha}-1)\tilde{t} = (s-1) + (\alpha-1)^2$ points collinear with both $x_1$ and $x_3$. Since $\tilde{t} \geq \alpha = \tilde{t} + 1$ it follows that $\tilde{\lambda} < \lambda$ and so there exists $x_5 \in X \setminus X(x_4, x_3, x_2, x_3)$ such that $x_1 \sim x_5 \sim x_3$. Now application of
DEBROEY [1] showed that a s.p.g. with parameters \( s, t, \alpha > 2 \), \( \mu = \alpha^2 \) satisfying (D) is of the following type: the "points" are the lines of \( \text{PG}(d, q) \), the "lines" are the planes in \( \text{PG}(d, q) \) for some prime power \( q \) and \( d \in \mathbb{N} \), \( d \geq 4 \). In this case \( s = q(q+1) \), \( t = (q-1)^{-1}(q^{d-1}-1)-1 \), \( \alpha = q+1 \), \( \mu = (q+1)^2 \).

**THEOREM 5.** Let \((X, L)\) be a s.p.g. with parameters \( s, t, \alpha > 2 \), \( \mu = \alpha^2 \). If \( s > \alpha \) and \( t \geq \alpha \), then \((X, L)\) is isomorphic to the s.p.g. consisting of the lines and planes in \( \text{PG}(d, q) \). In particular \( s = q(q+1) \), \( t = (q-1)^{-1}(q^{d-1}-1)-1 \), \( \alpha = q+1 \), \( \mu = (q+1)^2 \).

The only interesting case remaining is \( s = \alpha \). Now if \((X, E)\) is a Moore graph of valency \( r \), i.e. a strongly regular graph with \( \lambda = 0 \), \( \mu = 1 \), then \((X, \Gamma(x) \mid x \in X)\) is easily seen to be a s.p.g. with parameters \( s = t = \alpha = r-1 \), \( \mu = (r-1)^2 \) (here \( \Gamma(x) = \{ y \in X \mid (x, y) \in E \} \)). The point graph of this s.p.g. is the complement of \((X, E)\). Such a s.p.g. does not satisfy (D) for \( r > 2 \). From the following theorem follows immediately that a s.p.g. with \( \mu = \alpha^2 \), \( s = \alpha \) is necessarily of this type.

**THEOREM 6.** Let \((X, L)\) be a s.p.g. with \( t \geq \alpha \), \( \mu = \alpha^2 \) and \( s = \alpha \). Then \( t = \alpha \).

**PROOF.** Let \( x, y \in X \), \( x \neq y \). Let \( [x, y] = \{ L_1, \ldots, L_{\alpha} \} \), \( [y, u] = \{ M_1, \ldots, M_{\alpha} \} \) and put \( z_{ij} = L_i \cap M_j \), \( i, j = 1, \ldots, \alpha \) (see figure 2).

![Figure 2](image-url)
The number of \((z_{ij}, z_{k\ell})\) with \(i \neq k, j \neq \ell, z_{ij} \sim z_{k\ell}\) equals \(a^2 \cdot (a-1)(a-2)\). Now let \(K\) be a line through \(x, K \neq L_1, \ldots, L_a\), and let \(u\) be a point on \(K, u \neq x\).

Then \(u\) is collinear with \((a-1)\) of the \(a\) points \(z_{i,1}, \ldots, z_{i,a}\), for \(i = 1, \ldots, a\). Since \(u \neq y\), \(u\) is collinear with all of \(z_{i,j}, \ldots, z_{a,j}\) or with none, for \(j = 1, \ldots, a\).

It follows that there are \(a\) lines through \(u\) intersecting \((a-1)\) of the \(a\) lines \(M_1, \ldots, M_a\). Hence each point \(u \neq x\) on \(K\) gives rise to \(a(a-1)(a-2)\) pairs \((z_{ij}, z_{k\ell})\) as described, so \(K\) gives rise to all \(a^2(a-1)(a-2)\) pairs \((z_{ij}, z_{k\ell})\).

Suppose \(t > a\), then we can find two such lines \(K\) and \(K'\). It follows that for \(u \in K\), the \(a\) lines through \(u\) intersecting \((a-1)\) of the \(a\) lines \(M_1, \ldots, M_a\) also intersect \(K'\). But now \([u, K'] = a+1\), a contradiction. \(\square\)

3. SEMI-PARTIAL GEOMETRIES WITH \(\mu = \alpha(a+1)\).

In this section \((X, L)\) is a semi-partial geometry with parameters \(s, t, \alpha\) and \(\mu = \alpha(a+1)\).

If \(x, y \in X, x \neq y\) we shall always denote the \(a+1\) lines in \([x, y]\) by \(K_1, \ldots, K_{a+1}\), and the \((a+1)\) lines in \([y, x]\) by \(L_1, \ldots, L_{a+1}\). By (**) we can number these lines in such a way that \(K_i \cap L_i = \emptyset, i = 1, \ldots, a+1\) and \(K_i \cap L_j \neq \emptyset, i, j = 1, \ldots, a+1, i \neq j\) (see figure 3).

![Figure 3](image-url)

Figure 3.

Again our aim will be to show that the diagonal axiom (D) holds. We first
deal with the case $\alpha = 2$.

**Lemma 3.** If $\alpha = 2$ and $t > s$, then a set of 3 collinear points not on one line can be extended to a set of 4 collinear points no 3 on a line.

**Proof.** Let $x$, $a$ and $b$ be three distinct collinear points not on one line. There are $t - 1$ lines $\neq xa, ab$ through $a$ and on each of those lines there is a point $y_i \sim b$, $y_i \neq a$, $i = 1, \ldots, t - 1$. Suppose $y_i \not\in x$ for all $i = 1, \ldots, t - 1$. Now for each $i = 1, \ldots, t - 1$, $ay_i \neq xb$ (for otherwise $|[a, xb]| \geq 3$) and $by_i \neq xa$. Also $xa, xb \in [x, y_1]$ and $ay_i, by_i \in [y_i, x]$. Hence, by (**) there is a third line through $y_i$ intersecting $xa$ and $xb$ in points $u_i$ and $v_i$ respectively. Clearly $u_i \neq u_j$ if $i \neq j$, for $u_i = u_j$ implies $x, v_i, v_j \in [u_i, xb]$. Thus $xa$ contains $t + 1 > s + 1$ points (namely $x, a, u_1, \ldots, u_{t - 1}$), a contradiction. 

**Lemma 6.** Suppose $\alpha = 2$. If $x_1, x_2, x_3, x_4$ are four distinct collinear points, no three on a line, then no point can be collinear with exactly three of these four points.

**Proof.** Suppose $x_5$ is collinear with $x_2, x_3, x_4$ and $x_1 \not\in x_5$. Clearly $x_5 \not\in x_2x_3, x_2x_4, x_3x_4$. Hence $\{x_1x_2, x_1x_3, x_1x_4\} = [x_1, x_5]$ and $\{x_5x_2, x_5x_3, x_5x_4\} = [x_5, x_1]$ so $x_5x_2$ has to intersect $x_1x_3$ or $x_1x_4$ by (**). But then $|[x_2, x_1x_3]|$ or $|[x_2, x_1x_4]| > 2$, a contradiction.

**Lemma 7.** Same hypothesis as in lemma 6. Then the only points collinear with exactly two points of $\{x_1, x_2, x_3, x_4\}$ are the points on the lines $x_1x_j$, $i \neq j$.

**Proof.** Suppose $x_5 \sim x_1, x_4$ and $x_5 \not\in x_2x_3$, $x_5 \not\in x_1x_4$ (see figure 4).

![Figure 4](image-url)
Apply (**) to $x_3$ and $x_5$ to get a line $ab$ through $x_3$ with $a \in x_5x_4$, $b \in x_5x_1$.
Similarly (**) applied to $x_5$ and $x_2$ gives us a line $cd$ through $x_2$ with $c \in x_5x_4$, $d \in x_5x_1$. Clearly $b \neq c$ so we can apply (**) to $b$ and $c$. It follows that $ab \cap cd = \emptyset$. Also $x_2 \neq a$ and (**) applied to $x_2$ and $a$ yields:
$ab \cap cd \neq \emptyset$ or $ab \cap x_2x_4 \neq \emptyset$. Hence $ab \cap x_2x_4 \neq \emptyset$, a contradiction since
$\{x_2, x_4\} = [x_2x_4, x_3]$. []

**THEOREM 7.** If $(X,L)$ is a s.p.g with parameters $s,t,\alpha = 2$, $\mu = 6$ and $t > s$,
then $(X,L)$ satisfies (D).

**PROOF.** Let $x_1, x_2, x_3$ and $x_4$ be four distinct points no three on a line such
that $x_4 \sim x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_2$. By Lemma 5 there exists $x_5 \sim x_2, x_3, x_4$.

By Lemmas 6 and 7 $x_1 \sim x_3, x_5$. []

**REMARK.** If $(X,L)$ is a s.p.g but not a partial geometry, then $t \geq s$ (see
DEBROEY & THAS [2]). Using the integrality conditions for the multiplicities of the eigenvalues of a strongly regular graph it follows that a s.p.g with $s=t$, $\alpha=2$ and $\mu=6$ satisfies $(8(s^2-24s+25)(8(s+1)(2s^3-9s^2+19s-30))^2$.

From this one easily deduces an upper bound for $s$. The remaining cases were
checked by computer and only $s=t=28$ survived. Thus, every s.p.g which is
not a partial geometry satisfies (D) or has $s=t=28$ (and 103125 points).

We now turn to the case $\alpha \geq 3$. We shall make two additional assumptions
in this case. The first assumption is $\alpha \neq 3$, the second assumption is
$s \geq f(\alpha)$ where $f$ is defined in Lemma 9. Notice that this bound on $s$ is used
only in the proof of Lemma 9.

**LEMMA 8.** Let $x, y \in X$, $x \neq y$ and suppose $[x, y] = [K_1, \ldots, K_{\alpha+1}]$, $[y, x] =
[L_1, \ldots, L_{\alpha+1}]$ such that $K_i \cap L_i = \emptyset$, $i = 1, \ldots, \alpha+1$. If $M$ is a line inter-
secting $\sigma \geq 1$ lines of $[x, y]$, $\tau \geq 1$ lines of $[y, x]$ and $\sigma < \tau$, then $\sigma = \alpha-1$ and $\tau = \alpha$.

**PROOF.** Since $\sigma < \tau$, there exists a point of intersection $u$ of $M$ with a line
$L_1 \in [y, x]$ such that $u$ is not on one of the lines of $[x, y]$. Then $w \neq x$ and
so, applying (**) to $u$ and $x$, it follows that $M \in [u, x]$ intersects $\alpha-1$
of the $\alpha$ lines $K_1, K_2, \ldots, K_{\alpha-1}, K_{\alpha+1} \in [x, u]$. Thus $\alpha-1 \leq \sigma < \tau \leq \alpha$,
which proves our claim. []
**Lemma 9.** Let \( x \in X \) and \( L \in L \) such that \( x \notin L \) and \( x \) is collinear with a points \( z_2, z_3, \ldots, z_{\alpha+1} \) on \( L \). Let \( M \) be a line through \( z_{\alpha+1} \) meeting \( xx_\alpha \) in a point \( u \neq x, z_\alpha \). Suppose \( s \geq f(\alpha) \) where \( f(4) = 12 \), \( f(5) = 16 \), \( f(6) = f(7) = 17 \), \( f(8) = 18 \), \( f(9) = 19 \), \( f(10) = 21 \), \( f(11) = 23 \), \( f(\alpha) = 2\alpha \) (\( \alpha \geq 12 \)). Then \( M \) intersects at least \( \alpha-1 \) lines of \([x,L]\).

**Proof.** Suppose \( M \) does not meet at least two lines of \([x,L]\), \( xz_2 \) and \( xz_3 \), say. Since \( s \geq 2\alpha \) we can find \( y \in L \) such that \( x/y/f/u \). Let \([x,y] = \{K_1, K_2 = xz_2, \ldots, K_{\alpha+1} = xz_{\alpha+1}\} \) and \([y,x] = \{L_1 = L, L_2, L_3, \ldots, L_{\alpha+1}\} \) with \( K_1 \cap L_1 = \emptyset \).

Looking at \( u \) and \( y \) we find that \( M \) intersects \( \alpha-1 \) of the \( \alpha \) lines \( L_i \), \( \alpha \neq i \). Every point \( L_i \) which is collinear with \( x \) is on a line \( L_j \), \( j \neq \alpha \). If \( L_i \cap M \neq \emptyset \) for these \( \alpha-1 \) \( i \)’s, we find that \( M \) meets at least \( \alpha-1 \) of the \( \alpha \) lines \( K_1, \ldots, K_{\alpha+1} \), hence at least \( \alpha-1 \) of the lines \( K_2, \ldots, K_{\alpha+1} \), a contradiction.

Let \( t = L_i \cap M \) be a point not collinear with \( x \). Considering \( x/t/f \) we see that \( M \) intersects \( \alpha-1 \) of the \( \alpha \) lines in \([x,y] \backslash \{K_1\} \). This shows that \( i = 2 \) or \( 3 \), so there are at most two such points \( t \), and that \( M \) meets \( K_1, K_4, K_5, \ldots, K_{\alpha+1} \).

Let \( V = \{K_4, K_5, M, \ldots, K_{\alpha+1}\} \) and count pairs \((y, v), y \in L, y/f/x, v \in V, v \neq y\).

The number of such pairs is at least \((s-\alpha+1)(\alpha-5)\) (first choose \( y, s-\alpha+1 \) possibilities, then given \( y \) we can find \( \alpha-3 \) points \( L_i \cap M \) as above, possibly one on \( K_1(y) \), and one is \( z_{\alpha+1} \), and at most \((\alpha-3)(\alpha-2)\) (first choose \( v \), then \( y \)). It follows that for \( \alpha > 5 \), \( s \leq 2\alpha-1 + \frac{6}{\alpha-5} \). Let \( W = V \cup \{q, q'\} = \{w \in M|w/x\} \) and count pairs \((y, w), y \in L, y/f/x, w \in W, w \neq y\). This yields \((s-\alpha+1)(\alpha-4) \leq (\alpha-3)(\alpha-2) + 2(\alpha-1)\), hence \( s \leq 2\alpha + \frac{8}{\alpha-4} \) if \( \alpha > 4 \). Above we saw that for any \( y \in L \) with \( x/f/y/u \), \( K_1 = K_1(y) \) meets \( M \). But if \( s+1 > \alpha + (\alpha-2) + 2(\alpha-1) = 4\alpha-4 \), we can find \( y \in L \) such that \( y/f/x \), \( u,q \) and \( q' \), a contradiction. Therefore we have \( s < 4\alpha-4 \). We now have obtained a contradiction for all \( \alpha \geq 4 \) and the lemma is proved. \( \square \)

**Lemma 10.** Some hypotheses as in Lemma 9. Then \( M \) intersects exactly \( \alpha-1 \) lines of \([x,L]\).

**Proof.** Take \( y \in L, y/f/x \) and let \( K_1 \) and \( L_i \) be defined as before. Put \( K := K_{\alpha+1} \) and let \( A(x,L) \) be the set of lines \( \neq K \) through \( z_{\alpha+1} \) intersecting at least \( \alpha-1 \) lines of \([x,L]\), \( A(y,K) \) the set of lines \( \neq K \) through \( z_{\alpha+1} \) intersecting at least \( \alpha-1 \) lines of \([y,K]\). Suppose a lines of \( A(x,L) \) intersects \( \alpha-1 \) lines of \([x,L]\) and \( b \) lines of \( A(x,L) \) intersect \( \alpha \) lines of \([x,L]\). Counting
the points $u \sim z_{a+1}$ on $K_2, K_3, \ldots, K_a$, such that $u \neq x, z_2, \ldots, z_a$ yields $a(a-2) + b(a-1) = (a-1)(a-2)$. Hence $a = 0$ and $b = a-2$ or $a = a-1$ and $b = 0$. Thus $|A(x, L)| = a-2$ or $a-1$ according as every line in $A(x, L)$ intersects all lines or all but one line in $[x, L]$. A similar result holds for $A(y, K)$.

Now $A(x, L) = A(y, K)$ for suppose $N \in A(x, L)$ then by Lemma 8, $N$ intersects at least $a-1$ lines of $[y, x]$, so at least $a-2 \geq 2$ lines of $[y, K]$. Hence $N \in A(y, K)$ by Lemma 9. Similarly, $N \in A(y, K)$ implies $N \in A(x, L)$. Suppose $|A(x, L)| = a-2$, i.e. there are $a-2$ lines through $z_{a+1}$ intersecting all lines of $[x, L] \cup [y, K]$. It follows that $K_2 \not\subset z_{a+1}$ so we can apply (**) to $K_2$. This shows that $L_{a+1} \subset [K_2 \setminus z_{a+1}]$ intersects all $N \in A(y, K) \subset [z_{a+1}, K_2 \setminus z_{a+1}]$, a contradiction, for $L_{a+1} \subset N$ implies $|[y, N]| \geq a+1$. □

**Lemma 11.** Let $x \in X$, $L \in L$ such that $x$ is collinear with a point $z_2, \ldots, z_{a+1}$ on $L$. Let $M$ be a line through $z_{a+1}$ intersecting $a-1$ lines of $[x, L]$ and let $y \in L$, $y \neq x$. Then, if $[x, y] = \{K_1(y), K_2 = xz_2, \ldots, K_{a+1} = xz_{a+1}\}$, $M$ intersects $K_1(y)$.

**Proof.** Suppose $M$ does not intersect $K_2$, say. As shown in Lemma 10, $M$ also intersects $a-1$ lines of $[y, K_{a+1}] = \{L_1 = L, L_2, \ldots, L_a\}$. So $M$ intersects at least one of $L_{a-1}$ and $L_a$ and since $a \geq 4$, $L_2 \neq L_{a-1}, L_a$. Suppose $M$ intersects $L_{a-1}(L_a)$ in a point $v$. If $v \neq x$ then apply (**) to $v$ and $x$. It follows that $M \in [v, x]$ intersects $K_1(y) \in [x, v]$ for $M$ misses $K_2 \in [x, v]$. If $v = x$ then $v = L_{a-1}K_i (v = L_{a-1}K_i)$ for some $i$. By Lemma 10 applied to $x$ and $L_{a-1}(L_a)$ it follows that $M$ intersects $K_i(y) \subset [x, L_{a-1}] (K_i(y) \subset [x, L_a])$, for $M$ does not intersect $K_2 \in [x, L_{a-1}] (K_2 \in [x, L_a])$. □

**Corollary.** The line $K_1(y)$ is the same for all $y \in L$, $y \neq x$.

**Lemma 12.** Let $x \in X$, $L \in L$ such that $x$ is collinear with a point $z_2, z_3, \ldots, z_{a+1}$ on $L$. Put $K_i = xz_i$, $i = 2, \ldots, a+1$ and let $K_1$ be defined by $\{K_1, K_2, \ldots, K_{a+1}\} = [x, y]$ for any $y \in L$, $y \neq x$. Then every line which intersects $K_1$ and a $K_i(\#)$ not in $x$, intersects $L$ and therefore exactly a lines of $\{K_1, \ldots, K_{a+1}\}$.

**Proof.** Fix $i \in \{2, \ldots, a+1\}$. The number of pairs $(u, v)$ such that
If in Lemma 12 we replace $L = L_1$ by a line $L_j$ missing $K_j$, then it follows that every line intersecting two lines of $\{K_1, \ldots, K_{a+1}\}$ not in $x$, intersects exactly $a$ lines of $\{K_1, \ldots, K_{a+1}\}$. Using this result and the foregoing lemmas we can now proceed as in the case $\alpha = a^2$. For any two intersecting lines $L_1, L_2$ we can define in an obvious way a partial geometry $(L_1, L_2) = (X(L_1, L_2), l(L_1, L_2))$, now with parameters $\bar{s} = s, \bar{t} = a, \bar{a} = a$ (so $(L_1, L_2)$ is an $(\alpha+1)$-net of order $s+1$). Again $\bar{\mu} = \bar{a}(\bar{t}+1) = a(\alpha+1) = \mu$, so with the same proof as the proof of Theorem 4 we have the following theorem.

**Theorem 8.** Let $(X, L)$ be a s.p.g. with parameters $s, t, a, \mu = a(\alpha+1)$. If $\alpha \geq 4$, $s \geq f(\alpha)$ (as in Lemma 9) and $t \geq a+1$ (i.e. if $(X, L)$ is not a p.g.), then $(X, L)$ satisfies (D).

Fix a $(d-2)$-dimensional subspace $S$ of $\text{PG}(d, q)$, $q$ a prime power, $d \in \mathbb{N}$. Then with the lines of $\text{PG}(d, q)$ which have no point with 3 in common as "points" and with the planes of $\text{PG}(d, q)$ intersecting $S$ in exactly one point as "lines" and with the natural incidence relation, one obtains a s.p.g. with parameters $s = q^2-1, t = (q-1)(q^{d-1}-1)$, $\alpha = q, \mu = q(q+1)$.

DEBROEY [1] showed that a s.p.g. with parameters $s, t, a \geq 2, \mu = a(\alpha+1)$ and satisfying (D) is of this type. Combining this result with Theorems 7 and 8 we arrive at the following theorem.

**Theorem 9.** Let $(X, L)$ be a s.p.g. with parameters $s, t, a, \mu = a(\alpha+1)$ which is not a p.g.. If $\alpha = 2$ and not $s = t = 28$ or if $\alpha \geq 4$ and $s \geq f(\alpha)$, then $(X, L)$ is isomorphic to a s.p.g. consisting of the lines in $\text{PG}(d, q)$ missing a given $(d-2)$-dimensional subspace of $\text{PG}(d, q)$ and the planes intel-
secting this subspace in one point. In particular $s = q^2 - 1$,
$t = (q-1)^{-1}(q^{d-1} - 1)^{-1}$, $a = q$, $u = q(q+1)$ for some prime power $q$ and $d \in \mathbb{N}$
and any s.p.g. with these parameters with $q \neq 3$ and $d \geq 4$ is of this type.

REFERENCES

[1] DEBROEY, I., *Semi partial geometries satisfying the diagonal axiom*,

(A) 25 (1978) 242-250.