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(DEPARTMENT OF NUMERICAL MATHEMATICS)

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P.W. HEMKER

ON THE COMPARISON OF LINE-GAUSS-SEIDEL AND ILU RELAXATION IN MULTIGRID ALGORITHMS

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On the comparison of LINE-GAUSS-SEIDEL and ILU relaxation in multigrid algorithms *)

by

P.W. Hemker

Abstract

In this paper we compare the efficiency of Symmetric- and ZEBRA line-Gauss Seidel relaxation and Incomplete LU-factorization relaxation in multigrid algorithms. To this end we compute convergence rates and we make an operations count for general regular 7-point discrete operators. As model problems for the numerical computation of the convergence rate we use the anisotropic diffusion and the convection diffusion equation.

KEY WORDS & PHRASES: *Multigrid method, Incomplete LU factorization, Relaxation methods*

*) This report will be submitted for publication elsewhere.

Introduction

We are interested in finding efficient variants of the multigrid algorithm that can be used for the solution of the discretized general linear 2nd order elliptic partial differential equation

$$(1) \quad -\nabla(a\nabla u) + b\nabla u + cu = f, \quad \text{on } \Omega \subset \mathbb{R}^2,$$

with variable coefficients a, b and c, on a regular rectangular discretization. In particular we are interested in the solution of regular 7-point discretizations such as they arise from the simplest finite element discretization on a regular triangulation (i.e. the FEM with continuous piecewise linear trial- and test-spaces). Further, the 7-point discretizations are the simplest discretizations that can deal with the mixed derivatives u_{xy} in (1). As a special case our discretizations include the usual 5-point finite difference discretizations, but - in this paper - we shall not consider the particular gain of efficiency that can be obtained when we tailor our methods to these cases. Neither do we treat particular shortcuts that can be made for contant coefficients.

We consider multigrid algorithms of the following type: the Multi Level Algorithm (MLA) on level (ℓ) consists of:

(1) p relaxation sweeps on level ℓ (pre-relaxation);
(2) a coarse grid correction, consisting of:
    a) computation of the residual;
    b) restriction of the residual to level (ℓ-1);
    c) approximation of the solution of the residual equation on level (ℓ-1), by either
        i) σ sweeps of the MLA on level (ℓ-1),
        ii) or, if ℓ-1 = 0, by any other reasonable method;
    d) prolongation of the correction to level ℓ;
    e) addition of the correction to the latest approximate solution;
(3) q more relaxation sweeps on level ℓ (post-relaxation).

This algorithm (called Correction Storage Cycle, by Brandt (1)) is the usual multigrid algorithm for linear equations. The strategy is determined by the fixed numbers p, σ and q. Different variants are also

possible by selecting different procedures for (i) the relaxation, (ii) the prolongation or (iii) the restriction.
Guided by our 7-point discretizations (or rather by the finite element principle behind it), we select a 7-point restriction R_7 and a 7-point prolongation P_7 in the above algorithm. The prolongation P_7 corresponds to linear interpolation on the finite element triangles. The restriction R_7 is its transposed; both P_7 and R_7 are described by the convolution star

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & 1 & \frac{1}{2} \\ & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The 7-point linear interpolation is more efficient than the usual 9-point (bi-) linear interpolation (cf. 11,12). A further advantage of P_7 and R_7, in conjunction with the finite element discretization, is that for the constant coefficient equation (1) the coarse grid FEM-discretization, A_H, yields difference stars identical with those obtained from the Galerkin approximation, i.e. the coarse grid operator

$$(2) \quad A_H = R_7 \, A_h \, P_7.$$

In this sense the FEM discretization, P_7 and R_7 are strongly related.

Most work in the Multi Level Algorithm is spent with relaxation and, therefore, the selection of an efficient relaxation procedure is of utmost importance. Three good candidates for our purpose are Symmetric and Zebra line-Gauss-Seidel relaxation and Incomplete LU-factorization (ILU)-relaxation (cf. 1, 6-11). Another good possibility, Red-Black relaxation, seems to be more appropriate for 5-point discretization and symmetric problems (cf. 3). Generally, line-Gauss-Seidel seem more efficient than point-Gauss-Seidel procedures. For a description of the relaxation procedures we refer to the literature mentioned above.

In the present context we consider the Multi Level Algorithm only as an iterative procedure to solve the discrete equation

$$(3) \quad A_h \, u_h = f_h,$$

and we neglect the fact that we probably want the solution u_h only up to an accuracy of the order of the discretization error.

The efficiency of any multigrid variant is determined by the quantity $\sqrt[W]{\rho}$ where W denotes the amount of work to obtain a reduction factor ρ in the error (or in the residual) of the approximation solution i.e.

$$\rho_k = \{\| u_h^{(k)} - u_h \|_2 / \| u_h^{(0)} - u_h \|_2\}^{1/k}$$

or

$$\bar{\rho}_k = \{\| f_h - A_h u_h^{(k)} \|_2 / \| f_h - A_h u_h^{(0)} \|_2\}^{1/k}.$$

Clearly, ρ depends on the initial estimate $u_h^{(0)}$. By Fourier two-level analysis we were able to determine spectral norms of the amplification

operators

$$\|M\|_2 = \sup|\rho_i| \quad \text{and} \quad \|\bar{M}\|_2 = \sup|\bar{\rho}_i|$$

which are different and depend on the numbers p and q (cf. 6). In this paper we consider also the spectral radii of the amplification operators, i.e. the numbers

$$\rho = \lim_{i\to\infty} \rho_i = \lim_{i\to\infty} \bar{\rho}_i.$$

This limit depends only on $\nu = p+q$ (rather than on p and q separately). For the 3-different relaxation procedures: symmetric line-Gauss-Seidel (SLGS), Zebra line-Gauss-Seidel (ZEBRA) and incomplete LU-factorization relaxation (ILU) and for different values of σ ($\sigma = 1,2$) and ν ($\nu = 1,2$), we compute the number of arithmetic operations, W, in the multi level algorithm and we determine by numerical experimentation the value of ρ. Based on these numbers we determine what variants of the MLA are more efficient.

For a similar approach we refer to (7) and (11), whose results are improved and refined in the following section.

The computational complexity

In this section we count the arithmetic operations necessary to effectuate the Multi Level Algorithm. We neglect the computation of matrix-indices (array subscripts) and other overhead costs, since these can be reduced to an unsignificant amount by skilled coding of the programs. Another reason is that no other reasonable means exist to determine the work involved in the excution of an algorithm, if we do not want to incorporate the influence of the whole computing system used and the coding capabilities of the programmer.

The amount of work involved in one cycle of the Multi Level Algorithm is

$$W^{MLA} = \{(p+q)W^{REL} + W^{CGC}\}\cdot 4/(4-\sigma),$$

plus, possibly, some preparational work W^{PREP}. Here, W^{REL} is the amount of work to effectuate a relaxation sweep, and

$$W^{CGC} = W^{RESID} + W^{RESTR} + W^{CORR}$$

is the amount of work to compute the residual, the restriction to the coarser grid, and the addition of the prolongated correction to the fine grid solution.

For the line-Gauss-Seidel relaxations, repeatedly the same sets of tridiagonal systems have to be solved. This can be done with or without the storage of the LU-decompositions of the matrices. Without this decomposition available, the solution for N unknowns takes

3N add, 3N mult and 2N div;

possibly 2N div can be replaced by 2N mult and N div, if we want to avoid (expensive) divisions. With the decompositions available it takes

2N add, 2N mult and N div

and the decomposition itself takes

N add, N mult and N div

(possibly 2N add, 3N mult resp. N add, 2N mult and N div; then to store the decompositions we need an extra space for 2N real numbers). Additional to the solution of the tridiagonal systems, the composition of the rhs takes:

for 9-point operators 6N add and 6N mult,
for 7-point operators 4N add and 4N mult,
for 5-point operators 2N add and 2N mult.

Thus, we find for our 7-point operator

i) without storage of LU-decompositions

$$W^{REL} = 7N \text{ add}, \quad 7N \text{ mult} \quad \text{and} \quad 2N \text{ div};$$

ii) with the extra storage of 2N numbers and savings for expensive divisions

$$W^{REL} = 6N \text{ add}, \quad 7N \text{ mult};$$
$$W^{PREP} = N \text{ add}, \quad 2N \text{ mult} \quad \text{and} \quad N \text{ div}.$$

A SLGS relaxation sweep, defined by $A = L - D + U$,

$$Lu^{(i+\frac{1}{2})} = f - (U-D)u^{(i)},$$
$$Uu^{(i+1)} = f - (L-D)u^{(i+\frac{1}{2})},$$

can be implemented as

$$Du^{(i+\frac{1}{2})} = \{f - (U-D)u^{(i)}\} + \{-(L-D)u^{(i+\frac{1}{2})}\},$$
$$Du^{(i+1)} = \{-(L-D)u^{(i+\frac{1}{2})}\} + \{f - (U-D)u^{(i+1)}\},$$

since (L-D) and (U-D) are strict block lower and upper triangular. After such a relaxation sweep, the residual is computed by

$$f - Au = \{f - (U-D)u^{(i+1)}\} - Lu^{(i+1)}.$$

The computation of $\{f - (U-D)u^{(i+1)}\}$ or $\{\ldots - (L-D)u^{(i+\frac{1}{2})}\}$ takes 2N add, 2N mult. Hence, one SLGS relaxation sweep takes two times the solution of a set of tridiagonal systems and an additional 5N add and 4N mult. The residual computation W^{RESID} takes 7N add, 7N mult; where part of the computation may take place before relaxation.
For the 7-point Incomplete LU-factorization (cf. also 11)

$$A_h = LU + R,$$

we determine the lower and upper-triangular matrices L and U such that $(R)_{ij} = 0$ at the 7 non-zero diagonals of A; further L and U vanish at

all zero diagonals of A and we choose $(U)_{ii} = 1$. This determines the decomposition completely. The computation of L and U from A takes

$$W^{PREP} = 7N \text{ add}, \quad 7N \text{ mult} \quad \text{and} \quad 3N \text{ div}$$

(possibly with 3N div replaced by 3N mult and N div). If we want R to be kept in storage, its computation takes an additional 2N mult and the extra storage of 2N real numbers. The relaxation sweep

$$LUu^{(i+1)} = f - Ru^{(i)}$$

takes

$$W^{REL} = 8N \text{ add}, \quad 8N \text{ mult} \quad \text{and} \quad N \text{ div},$$

(possibly N div replaced by N mult). Here in addition 2N mult are necessary if R is not kept in storage.

Thus we find for ILU relaxation

i) without storage of R

$$W^{REL} = 8N \text{ add}, \quad 10N \text{ mult}, \quad N \text{ div},$$
$$W^{PREP} = 7N \text{ add}, \quad 7N \text{ mult}, \quad 3N \text{ div};$$

ii) with storage of R and avoiding most divisions:

$$W^{REL} = 8N \text{ add}, \quad 9N \text{ mult}$$
$$W^{PREP} = 7N \text{ add}, \quad 12N \text{ mult}, \quad N \text{ div}.$$

In general for the residual computation of the 7-point operator we find

$$W^{RESID} = 7N \text{ add}, \quad 7N \text{ mult}.$$

However, after a ZEBRA relaxation, when the residual vanishes over all even (odd) lines:

$$W^{RESID.ZEBRA} = 3.5N \text{ add}, \quad 3.5N \text{ mult}.$$

After an ILU relaxation the residual computation

$$r^{(i+1)} = f_h - A_h u^{(i+1)} = R(u^{(i+1)} - u^{(i)})$$

takes only

$$W^{RESID.ILU} = 2N \text{ add}, \quad 2N \text{ mult},$$

with eventually an additional 2N mult if R is not kept in storage. For the 5-, 7-, 9-point restriction, we find per coarse gridpoint

	general	odd-even ZEBRA	even-odd
R_9:	8 add 1 mult	5 add 1 mult	3 add 1 mult
R_7:	6 add 1 mult	3 add 1 mult	2 add 1 mult
R_5:	5 add 1 mult	1 add 1 mult	3 add 1 mult

Both for the 7- and for the 9-point prolongation we find 3 add and 3 mult per coarse gridpoint. Together with the addition this yields

$$W^{CORR} = 1.75N \text{ add}, \quad 0.75 \text{ mult}.$$

Summarizing we find

Table 1

	W^{RESID}		W^{RESTR}		W^{CORR}		W^{CGC}		W^{REL}			W^{PREP}			flops		
	+	*	+	*	+	*	+	*	+	*	/	+	*	/	CGC	REL	PREP
SLGS (1)	7	7	$1\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	$10\frac{1}{4}$	8	11	10	4	-	-	-	$18\frac{1}{4}$	25	-
SLGS (2)	7	7	$1\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	$10\frac{1}{4}$	8	9	10	-	1	2	1	$18\frac{1}{4}$	19	4
ZEBRA EO(1)	$3\frac{1}{2}$	$3\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	$5\frac{3}{4}$	$4\frac{1}{2}$	7	7	2	-	-	-	$10\frac{1}{4}$	16	-
ZEBRA EO(2)	$3\frac{1}{2}$	$3\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	$5\frac{3}{4}$	$4\frac{1}{2}$	6	7	-	1	2	1	$10\frac{1}{4}$	13	4
ZEBRA OE(1)	$3\frac{1}{2}$	$3\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	6	$4\frac{1}{2}$	7	7	2	-	-	-	$10\frac{1}{2}$	16	-
ZEBRA OE(2)	$3\frac{1}{2}$	$3\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	6	$4\frac{1}{2}$	6	7	-	1	2	1	$10\frac{1}{2}$	13	4
ILU (1)	2	4	$1\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	$5\frac{1}{4}$	5	8	10	1	7	7	3	$10\frac{1}{4}$	19	17
ILU (2)	2	2	$1\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	$5\frac{1}{4}$	3	8	9	-	7	12	1	$8\frac{1}{4}$	17	20

The operation counts are given (1) with, or (2) without an additional storage requirement of 2N real numbers and avoiding divisions.

To solve the linear system (3) we also need to prepare the coarse grid discretizations. If the FEM discretization is given, the consistent coarse grid discretizations are those obtained by the Galerkin approximation (2). For each construction of a matrix

$$A_H = R_7 \, A_h \, P_7,$$

87 additions and 7 multiplications are necessary per coarse gridpoint. Thus, for all coarse grid operators together, this amounts to a total of less than

29N add and 2N mult

operations.

The efficiency of the different cycles

In this section we give the results of the computation of ρ, ρ_i, $\bar{\rho}_i$ by numerical experimentation. These results confirm those obtained theoretically in (6). Using the operations counts from the previous section, we determine the efficiency of various MG cycles. We also mention efficiency results for the first few iteration steps, where also W^{PREP} is

taken into account.

For the experiments we use as model problems the *anisotropic diffusion equation*

$$(6)\quad (\varepsilon c^2+s^2)u_{xx} + 2(\varepsilon-1)scu_{xy} + (c^2+\varepsilon s^2)u_{yy} = f,$$

and the *convection diffusion equation*

$$(7)\quad \varepsilon(u_{xx}+u_{yy}) + su_x + cu_y = f,$$

on the unit square in $\mathbb{R}^2$ and with Dirichlet boundary conditions; $s = \sin(\phi)$, $c = \cos(\phi)$.
For the discretization we use the FEM with piecewise linear functions on triangles (diagonals parallel to $y = -x$). The coarsest grid is the unit square itself. Finer levels with $h = 1/2, 1/4, 1/8, 1/16, 1/32$ are used.

In the tables (2) - (3) we give the efficiency of different MG variants for the solution of (1), when applied to the equation (6), including the preparational work after $k = 1,2,3$ steps.

$$u^{(0)} = 0; \quad u = x^2 + y^2; \quad W_k = W^{PREP} + kW^{MLA};$$
$$E_k = W_k^{0.1\log\rho_k}; \quad \bar{E}_k = W_k^{0.1\log\bar{\rho}_k};$$

i.e. the number of floating point operations to obtain a factor 0.1 reduction of the error or the residual in the first k steps; and

$$E = W^{MLA\ 0.1\log\tilde{\rho}}, \text{ with } \tilde{\rho} = \Pi_{k=1}^{3}\rho_{k+3}^{1/3},$$

i.e. the number of floating point operations to obtain a factor 0.1 reduction in the following steps.

Table 2

Problem (6) with $\varepsilon = 1$. Laplace's equation. $h = 1/32$

Relaxation	p	q	σ	E_1	E_2	E_3	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$	$\bar{E}$	$\tilde{\rho}$
ILU	0	1	1	51.0	40.7	36.9	40.7	34.1	33.5	34.2	0.103
	1	0	1	41.1	33.7	31.7	61.1	41.0	36.7	30.2	0.077
	1	1	1	44.5	39.5	37.9	34.3	33.1	33.0	34.6	0.023
	0	1	2	50.6	42.6	43.8	53.1	47.9	47.2	51.5	0.105
	1	0	2	43.4	40.4	40.8	79.4	53.6	49.2	46.5	0.082
	1	1	2	51.8	48.6	47.8	46.8	45.9	45.9	47.0	0.016
SLGS	0	1	1	69.8	70.8	70.3	65.3	65.5	66.4	69.9	0.195
	1	0	1	62.1	62.0	62.8	89.4	78.1	75.0	69.8	0.194
	1	1	1	62.0	61.2	61.2	53.8	54.6	55.4	59.6	0.055
	0	1	2	94.3	92.7	94.4	93.2	94.6	96.3	103.1	0.189
	1	0	2	63.2	73.6	79.7	135.4	108.5	103.8	100.5	0.187
	1	1	2	69.7	73.0	75.2	78.5	79.1	79.8	82.6	0.044
ZEBRA E-O	0	1	1	93.0	87.9	83.5	92.6	65.0	60.4	55.6	0.277
	1	0	1	55.6	53.5	53.3	111.4	77.7	68.1	56.1	0.280
	1	1	1	38.9	38.8	39.2	53.5	45.8	43.7	41.3	0.067
	0	1	2	58.8	56.2	60.0	126.2	87.0	81.7	74.1	0.236
	1	0	2	47.3	55.5	59.4	154.4	100.8	86.9	72.4	0.228
	1	1	2	44.5	49.7	51.7	78.2	64.4	62.3	58.4	0.057

Table 3
Problem (6) with $\varepsilon = 3/37$, $\phi = \pi/4$, h = 1/16

Relaxation	p	q	σ	E_1	E_2	E_3	$\overline{E_1}$	$\overline{E_2}$	$\overline{E_3}$	$\overline{E}$	$\tilde{\rho}$
ILU	0	1	1	41.4	34.2	31.1	38.4	32.6	32.2	37.2	0.125
	1	0	1	36.3	31.2	30.5	51.6	36.6	33.9	37.4	0.126
	1	1	1	37.3	34.6	34.8	32.4	31.9	33.2	41.3	0.043
SLGS	0	1	1	54.8	58.7	62.2	58.9	62.9	64.8	70.0	0.198
	1	0	1	50.7	56.2	59.8	73.9	71.0	70.6	71.4	0.202
	1	1	1	50.3	53.5	56.4	51.2	53.0	54.3	64.7	0.069
ZEBRA E-O	0	1	1	63.9	61.8	55.4	56.8	46.1	45.8	68.4	0.352
	1	0	1	38.1	39.7	43.8	63.3	47.7	47.2	66.8	0.343
	1	1	1	40.5	38.0	40.9	37.2	37.1	41.2	67.9	0.198

Table 4
The maximal values of E_1 and the values of E_2, E_3, E and $\tilde{\rho}$ for problem (7) with $\varepsilon = h = 1/16$ and $\phi = k\pi/8$; $k = 1,2,\ldots,8$. The multigrid strategy used is $p = q = \sigma = 1$.

	$\max_k (E_1)$	E_2, E_3, E	$\tilde{\rho}$
ILU	38	35-25	0.02 - 0.005
SLGS	60	52-44	0.03 - 0.015
ZEBRA.EO	107	80-50	0.2 - 0.1
ZEBRA.OE	185	75-50	0.2 - 0.1

Remarks and Conclusions

(1) In this paper the discrete ℓ_2-norm is used. The choice of another norm (e.g. ℓ_1 or ℓ_∞) has no essential influence on the conclusions reported.
(2) In most all cases even-odd (coarse lines first) ZEBRA relaxation appeared to be slightly more efficient than odd-even ZEBRA. (Hence we report only results for the former possibility.)
(3) W-cycles ($\sigma = 2$) have slightly smaller convergence factors than V-cycles ($\sigma = 1$), but because of the additional work they are less efficient.
(4) In the numerical experiments a clear difference between ρ_i and $\bar{\rho}_i$ is only seen for $p = 0$ or $q = 0$. As was explained in (4), a strategy with $q = 0$ yields worse reduction of the residual than for the error in the first steps of the iteration process; with $p = 0$ it yields worse convergence for the solution than for the residual.
(5) The strategy with $p = q = 1$ is the most robust.

Although particular savings can be made for Symmetric line-Gauss-Seidel relaxation, compared to ZEBRA or ordinary LGS, it is generally not competitive.

ILU_7-relaxation is more efficient than the line-Gauss-Seidel relaxations considered. Although the preparational work for the construction of the ILU-decomposition is considerable, it doesn't make the

relaxation inefficient, even in the first few iterands.

ILU_7-relaxation yields an equally robust algorithm as the line-Gauss-Seidel relaxations considered.

In short, ILU_7-relaxation, combined with the 7-point prolongation and restriction, yields an algorithm that is robust and more efficient than the other (line-Gauss-Seidel) relaxations considered.

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