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REDUCIBILITY OF ALGEBRAICALLY STABLE GENERAL LINEAR METHODS

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Reducibility of algebraically stable general linear methods *)

by

K. Dekker

ABSTRACT

An equivalence relation on the class of general linear methods is defined and it is shown that the property of algebraic stability is invariant for this relation. Methods which are algebraically stable for a singular matrix G turn out to be reducible. On the other hand, algebraically stable reducible methods are algebraically stable for a singular G. Each algebraically stable irreducible method is shown to be equivalent with a method which is stable for the identity matrix.

KEY WORDS & PHRASES: *Numerical analysis, Ordinary differential equations, General linear methods, Algebraic stability*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In considering the applicability of a numerical method for the solution of stiff non-linear ordinary differential equations, much attention has been paid to the class of dissipative equations. DAHLQUIST [5] introduced the concept of G-stability for linear multistep methods and BUTCHER [3] the idea of B-stability in the case of Runge-Kutta methods. In 1979, BURRAGE and BUTCHER [1], and independently CROUZEIX [4], showed that B-stability is equivalent to a new stability property, so called algebraic stability, under very mild restrictions (see also HUNSDORFER and SPIJKER [8]). In a more recent paper [2], they generalised the concept of algebraic stability for general linear methods.

The analysis and construction of algebraically stable general linear methods seems to be rather cumbersome [6]. The aim of this paper is to introduce some simplifications in order to facilitate the analysis.

In section 2 we present the class of general linear methods and the definitions of algebraic stability and consistency.

In section 3 we introduce an equivalence relation between general linear methods and show that algebraic stability is a property of equivalence classes. We define reducibility of a method and prove in subsequent lemmata a relationship between reducibility and singularity of certain matrices G and D which appear in the definition of algebraic stability. As far as the matrix D is involved, these results might be regarded as a generalization of a similar property of implicit Runge-Kutta methods (see HAIRER [7]). As our main result we have that algebraic stability of an irreducible method implies that both G and D are positive. Moreover, any stable irreducible method is equivalent with a method which is algebraically stable for G being the identity matrix.

In section 4 we show that consistency is a property of the equivalence classes, and that each reducible consistent method can be reduced to an irreducible consistent method.

Finally we present an example in section 5 in order to illustrate the equivalence relation and the process by which a reducible method can be reduced.

2. SURVEY ON GENERAL LINEAR METHODS

In the solution of the initial value problem

$$(2.1) \qquad y'(x) = f(y(x)), \qquad y(x_0) = y_0, \qquad f\colon \mathbb{R}^m \to \mathbb{R}^m,$$

we consider the step by step method with r external and s internal stages

$$(2.2) \qquad \begin{aligned} Y_i^{(n)} &= \sum_{j=1}^{r} a_{ij}^{(1)} y_j^{(n-1)} + h \sum_{j=1}^{s} b_{ij}^{(1)} f(Y_j^{(n)}), \qquad i = 1,2,\dots,s, \\ y_i^{(n)} &= \sum_{j=1}^{r} a_{ij}^{(2)} y_j^{(n-1)} + h \sum_{j=1}^{s} b_{ij}^{(2)} f(Y_j^{(n)}), \qquad i = 1,2,\dots,r. \end{aligned}$$

The partitioning of such a linear method was first proposed by BURRAGE and BUTCHER [2] and a representation can be given by the partitioned matrix $\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$. In the text we will use the shorthand notation (A,B).

BURRAGE and BUTCHER [2] analyse the stability behaviour of these methods for *monotonic* problems, i.e. equations satisfying for all $u \in \mathbb{R}^m$

$$(2.3) \qquad \langle u, f(u)\rangle \le 0,$$

where $\langle , \rangle$ is some real pseudo inner product. They define a pseudo inner product for sequences of r vectors from $\mathbb{R}^m$, say U and V, based on the inner product on $\mathbb{R}^m$ and a symmetric non-negative definite matrix G:

$$\langle U,V\rangle_G = \sum_{i,j=1}^{r} g_{ij} \langle u_i, v_j\rangle.$$

<u>DEFINITION 2.1.</u> A method (2.2) is said to be *monotonic* if for any monotonic problem (2.1) there exists a non-zero non-negative symmetric matrix G, such that the computed results satisfy $\|y^{(n)}\|_G \le \|y^{(n-1)}\|_G$.

<u>DEFINITION 2.2.</u> The general linear method (2.2) is *algebraically stable* for a given matrix G if a non-negative diagonal matrix D exists such that the matrix

$$\text{(2.4)} \qquad M = \begin{bmatrix} G - A_2^T G A_2 & A_1^T D - A_2^T G B_2 \\ \\ D A_1 - B_2^T G A_2 & B_1^T D + D B_1 - B_2^T G B_2 \end{bmatrix}$$

is non-negative definite.
Throughout this paper we will assume that G is symmetric and non-negative. It has been proved [2] that algebraic stability for G implies monotonicity in the norm induced by G. Application of scheme (2.2) to the equation $y' = 0$ for $y \in I \subset \mathbb{R}$ shows that any sensible method should satisfy

DEFINITION 2.3. A general linear method is *pre-consistent* if there exists a vector $u \in \mathbb{R}^r$, known as the pre-consistency vector, such that

$$\text{(2.5)} \qquad A_1 u = e, \qquad e = [1,1,\ldots,1]^T \in \mathbb{R}^s,$$
$$A_2 u = u.$$

In addition, a method is *consistent* if a vector v exists, such that

$$\text{(2.6)} \qquad A_2 v + B_2 e = u + v.$$

3. REDUCIBILITY

The class of general linear methods, defined by (2.2), contains a variety of schemes, many of which bear a close resemblance. For example, when we multiply the solution vectors $y^{(n)}$, computed with some scheme, with a non-singular matrix, we obtain a different scheme, which behaves similarly. In this section we will try to give a standardised formulation of (2.2).

DEFINITION 3.1. Two general linear methods (A,B) and $(\tilde{A},\tilde{B})$ are said to be *equivalent* iff there exists a non-singular matrix T and a permutation matrix P, such that

$$\text{(3.1)} \qquad \begin{bmatrix} \tilde{A}_1 & \tilde{B}_1 \\ \tilde{A}_2 & \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} P^T A_1 T & P^T B_1 P \\ T^{-1} A_2 T & T^{-1} B_2 P \end{bmatrix}.$$

LEMMA 3.1. *Let* (A,B) *be algebraically stable for* G *and assume that* (A,B) *and* $(\tilde{A},\tilde{B})$ *are equivalent. Then* $(\tilde{A},\tilde{B})$ *is algebraically stable for* $T^T GT$.

PROOF. The algebraic stability is a direct consequence of the non-negativity of $\begin{bmatrix} T^T & 0 \\ 0 & P^T \end{bmatrix} M \begin{bmatrix} T & 0 \\ 0 & P \end{bmatrix}$. □

Given a linear method, we can form new schemes by inserting irrelevant internal stages, i.e. not affecting the final result; alternatively, we may combine two methods into formulation (2.2) or add some external stages. In our analysis we wish to avoid these methods and consider irreducible methods only.

DEFINITION 3.2. A method is called *reducible* if it is equivalent with a method $(\tilde{A},\tilde{B})$, such that the following equalities hold for $s_1 < s$ or $1 \le r_1 < r$

$$
(3.2)\quad
\begin{array}{llll}
\text{(a)} & \tilde{a}^{(1)}_{ij} = 0, & i = 1,2,\ldots,s_1, & j = r_1+1,\ldots,r, \\
\text{(b)} & \tilde{a}^{(2)}_{ij} = 0, & i = 1,2,\ldots,r_1, & j = r_1+1,\ldots,r, \\
\text{(c)} & \tilde{b}^{(1)}_{ij} = 0, & i = 1,2,\ldots,s_1, & j = s_1+1,\ldots,s, \\
\text{(d)} & \tilde{b}^{(2)}_{ij} = 0, & i = 1,2,\ldots,r_1, & j = s_1+1,\ldots,s.
\end{array}
$$

In particular, we call the method s-*reducible* if $s_1 < s$ and $r_1 = r$ and r-*reducible* if $r_1 < r$ and $s_1 = s$.

Because of the equivalence relation we may assume that a reducible method can be written as

$$
(3.3)\quad
\begin{array}{ll}
A_1 = \left[\begin{array}{c|c} A_{1,1} & 0 \\ \hline A_{1,2} & A_{1,3} \end{array}\right], & B_1 = \left[\begin{array}{c|c} B_{1,1} & 0 \\ \hline B_{1,2} & B_{1,3} \end{array}\right], \\[2ex]
A_2 = \left[\begin{array}{c|c} A_{2,1} & 0 \\ \hline A_{2,2} & A_{2,3} \end{array}\right], & B_2 = \left[\begin{array}{c|c} B_{2,1} & 0 \\ \hline B_{2,2} & B_{2,3} \end{array}\right],
\end{array}
$$

where $A_{1,1} \in \mathbb{R}^{s_1 \times r_1}$, $A_{2,1} \in \mathbb{R}^{r_1 \times r_1}$, $B_{1,1} \in \mathbb{R}^{s_1 \times s_1}$; $B_{2,1} \in \mathbb{R}^{r_1 \times s_1}$

DEFINITION 3.3. The general linear method $\begin{bmatrix} A_{1,1} & B_{1,1} \\ A_{2,1} & B_{2,1} \end{bmatrix}$, defined by (3.3), is called the *reduced* method of (A,B).

In the following lemmata we will establish a relationship between reducibility and singularity of G and D; moreover, we show that reduction preserves algebraic stability. At first, we state a useful property of symmetric non-negative matrices.

LEMMA 3.2. *Let* M *be symmetric and non-negative. If* v *is a vector such that* $v^TMv = 0$ *then it is an eigenvector of* M *with* $Mv = 0$.

PROOF. For any vector w and arbitrary constant ε non-negativity implies that

$$(v+\varepsilon w)^T M(v+\varepsilon w) \geq 0.$$

However, the term independent of ε in this form vanishes, so that the coefficient of ε should be zero too. Since this holds for all w, the result follows. □

LEMMA 3.3. *Let* (A,B) *be algebraically stable for a non-singular matrix* G. *Assume that the diagonal matrix* D *from* (2.4) *is singular. Then* (A,B) *is* s*-reducible.*

PROOF. Let v be a vector from the null space of D. Then,
$(0\ v^T)M\binom{0}{v} = v^T(B_1^TD+DB_1-B_2^TGB_2)v = -v^TB_2^TGB_2v = 0$ because M is non-negative and G is positive. Thus, according to lemma 3.2, $(B_1^TD+DB_1-B_2^TGB_2)v = 0$ and $GB_2v = 0$, so that $DB_1v = 0$ and $B_2v = 0$, as G is non-singular.
Now, as D is diagonal, the null space of D is given by linear combinations of the basis vectors e_j, $j = \tilde{s}+1,\ldots,s$ (possibly after renumbering).
$DB_1e_j = 0$ and $B_2e_j = 0$ imply (3.2c) and (3.2d) with $r_1 = r$ and $s_1 = \tilde{s}$ so the method is s-reducible. □

LEMMA 3.4. *Let* (A,B) *be algebraically stable for a singular matrix* G *and assume that* D *is non-singular. Then* (A,B) *is* r*-reducible.*

PROOF. Let w be a vector from the null space of G. Then, considering the quadratic form $(w^T 0)M\binom{w}{0}$ and using lemma 3.2, we find that $DA_1 w = 0$ and $GA_2 w = 0$. Let T be a non-singular matrix, such that the last columns of T $\tilde{r}+1,\dots,r$ span the null space of G. Then, $DA_1 Te_j = 0$ for $j = \tilde{r}+1,\dots,r$ and the non-singularity of D implies that (3.2a) holds. Moreover, $GT\,T^{-1}A_2 Te_j = 0$, $j = \tilde{r}+1,\dots,r$, implies that $T(T^{-1}A_2 T)e_j$ lies in the null space of G; thus $e_i^T(T^{-1}A_2 T)e_j = 0$, $i = 1,2,\dots,\tilde{r}$ and we conclude that (3.2) holds with $r_1 = \tilde{r}$ and $s_1 = s$. □

Combining these lemmata we obtain

LEMMA 3.5. *Let* (A,B) *be algebraically stable for a matrix* G *and assume that at least one of the matrices* G *and* D *is singular. Let* $r^* = \text{rank}(G)$ *and* $s^* = \text{rank}(D)$. *Then* (A,B) *is reducible with* $s_1 = s^*$ *and* $r_1 = r^*$.

PROOF. Let v be a vector from the null space of D and w a vector from the nullspace of G. According to the proofs of the lemmata 3.3 and 3.4 we have $GB_2 v = 0$, $DB_1 v = 0$, $DA_1 w = 0$ and $GA_2 w = 0$. Thus (3.2) with $r_1 = r^*$, $s_1 = s^*$ holds after a suitable transformation. □

We remark that the converse statement of this lemma is not true, because we can construct reducible methods which are algebraically stable for a non-singular G whereas D is positive, too. For example, the scheme consisting of two algebraically stable irreducible methods which are computationally independent of each other. However, we will show that any reducible algebraically stable method can be reduced to a method which is algebraically stable for a positive matrix G and a positive matrix D. For simplicity we will assume in the sequel that a reducible method is already of the form (3.3).

LEMMA 3.6. *Let* (A,B) *be algebraically stable for* G *and assume that* (A,B) *is* s*-reducible. Then the algebraic stability condition is satisfied for a diagonal matrix* $\tilde{D}$, *with* $\text{rank}(\tilde{D}) \le s_1$.

PROOF. Suppose D is the diagonal matrix from (2.4). Let I_{s_1} be the diagonal matrix with ones on the first s_1 diagonal positions and zeros elsewhere. According to the non-negativity of M we have

$$\begin{bmatrix} I & 0 \\ 0 & I_{s_1} \end{bmatrix} M \begin{bmatrix} I & 0 \\ 0 & I_{s_1} \end{bmatrix} \geq 0.$$

Defining $\tilde{D} = DI_{s_1}$ and using $B_2 I_{s_1} = B_2$, $I_{s_1} B_1 I_{s_1} = I_{s_1} B_1$, we obtain from the expression

$$\begin{bmatrix} G - A_2^T G A_2 & A_1^T \tilde{D} - A_2^T G B_2 \\ \tilde{D} A_1 - B_2^T G A_2 & B_1^T \tilde{D} + \tilde{D} B_1^T - B_2^T G B_2 \end{bmatrix} \geq 0;$$

thus, in (2.4) we can replace D by $\tilde{D}$. Moreover, $\operatorname{rank}(\tilde{D}) \leq \operatorname{rank}(I_{s_1}) = s_1$. □

LEMMA 3.7. *Let* (A,B) *be algebraically stable for a positive* G. *Then* (A,B) *is equivalent with a method which is algebraically stable for* I.

PROOF. Let $L^T L$ (L lower triangular) be a Cholesky decomposition of G. Then the matrix

$$\begin{bmatrix} L^{-T} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} L^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I - L^{-T} A_2^T L^T L A_2 L^{-1} & L^{-T} A_1^T D - L^{-T} A_2^T L^T L B_2 \\ D A_1 L^{-1} - B_2^T L^T L A_2 L^{-1} & B_1^T D_1 + D B_1^T - B_2^T L^T L B_2 \end{bmatrix}$$

is non-negative and therefore the method $(\tilde{A},\tilde{B}) = \begin{bmatrix} A_1 L^{-1} & B_1 \\ L A_2 L^{-1} & L B_2 \end{bmatrix}$ algebraically stable for I. □

LEMMA 3.8. *Let* (A,B) *be algebraically stable for* G = I *and assume that* (A,B) *is* r*-reducible. Then* (A,B) *is algebraically stable for* $\tilde{G} = I_{r_1}$, *where* I_{r_1} *denotes the square matrix of dimension* r *with ones on the first* r_1 *diagonal positions and zeros elsewhere.*

PROOF. Using $I_{r_1} A_2 I_{r_1} = I_{r_1} A_2$ and $A_1 I_{r_1} = A_1$ we obtain

$$\begin{bmatrix} I_{r_1} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} I_{r_1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I_{r_1} - A_2^T I_{r_1} A_2 & A_1^T D - A_2^T I_{r_1} B_2 \\ D A_1 - B_2^T I_{r_1} A_2 & D B_1 + B_1^T D - B_2^T I_{r_1} B_2 \end{bmatrix} -$$

$$- [I_{r_1} A_2^T \quad B_2^T] \, (I - I_{r_1}) \begin{bmatrix} A_2 I_{r_1} \\ B_2 \end{bmatrix}.$$

The second term on the right hand side is obviously non-negative; therefore, non-negativity of M implies non-negativity of the first matrix on the right hand side, so we conclude that (A,B) is algebraically stable for I_{r_1}. □

We remark that we have chosen a peculiar decomposition of G in Lemma 3.7 in order that the transformations LA_2L^{-1}, LB_2 and A_1L^{-1} preserve the form of these matrices if they are of the reducible form (3.3). Combining our previous Lemmata yields our main result.

THEOREM 3.9. *Let* (A,B) *be a general linear method which is algebraically stable for* G *and let* D *be the diagonal matrix in* (2.4). *Then one of the following statements holds.*

(i) (A,B) *is irreducible and both* G *and* D *are positive.*

(ii) (A,B) *is algebraically stable for* $\tilde{G}$ *and* D *such that* $1 \le \mathrm{rank}(\tilde{G}) < \mathrm{rank}(G)$.

(iii) (A,B) *is algebraically stable for* G *and a diagonal* $\tilde{D}$ *such that* $\mathrm{rank}(\tilde{D}) < \mathrm{rank}(D)$.

(iv) (A,B) *can be reduced to a method which is algebraically stable for a positive* $\tilde{G}$, *and either* $\tilde{D}$ *is positive or* $\tilde{D}$ *is empty.*

PROOF. At first we observe that irreducibility implies (i) as a consequence of lemma 3.5. Secondly, if at least one of the matrices G and D is singular, the method is reducible according to this lemma. When we choose the transformation matrix T in such a way that the last columns span the null space of G and define the positive square matrices $\tilde{G}$ and $\tilde{D}$ by omitting the zero rows and columns from T^TGT and P^TDP, it is obvious that the reduced method is algebraically stable for $\tilde{G}$ and $\tilde{D}$. We note that $\tilde{D}$ might be empty, if D is a zero matrix. Next, assume that G and D are positive and (A,B) is s-reducible. Then there exists a $\tilde{D}$ such that $\mathrm{rank}(\tilde{D}) \le s_1 < s = \mathrm{rank}(D)$ according to lemma 3.6. Finally, let G and D be positive and suppose (A,B) is transformed into the form (3.3) with $r_1 < r$. Let L^TL be a decomposition of G. Application of the transformation (3.1) with $T = L^{-1}$ yields a method which is algebraically stable for I, according to lemma 3.7. Moreover, the transformation preserves the special form (3.3), so the transformed method is still reducible with $r_1 < r$. Thus we can apply lemma 3.8 and find that the transformed method is algebraically stable for I_{r_1}. Therefore, the original method is stable for $L^T I_{r_1} L$, according to lemma (3.1). □

DEFINITION 3.4. A general linear method (A,B) is said to contain $(\tilde{A},\tilde{B})$ if either $(\tilde{A},\tilde{B})$ is equivalent with (A,B) or $(\tilde{A},\tilde{B})$ can be reduced from a method which is equivalent with (A,B).

COROLLARY 3.10. *Any algebraically stable method contains an irreducible method which is algebraically stable for a positive matrix* G *and a positive diagonal matrix* D.

LEMMA 3.11. *Let* (A,B) *be irreducible and algebraically stable for* G. *Then the dimension of the null space of* A_2-I *is at most* s.

PROOF. Suppose that the dimension of the null space of A_2-I is larger than s. Then we have at least s+1 independent vectors $v_1,v_2,\ldots,v_{s+1}$ such that $A_2v_j = v_j$. Thus, there exists a non trivial linear combination $v = \sum_{j=1}^{s+1} \alpha_j v_j$ satisfying $DA_1v = 0$. Moreover, $A_2v = v$. However, these relations imply that the method is reducible, which is a contradiction. Thus the assumption is not valid and we conclude that the dimension is at most equal to s. □

4. CONSISTENCY

In this section we show that pre-consistency and consistency properties of a general linear method are preserved by the transformation and reduction processes of the previous section. Moreover, we prove that any pre-consistent method is equivalent with a very special method.

LEMMA 4.1. *Let* (A,B) *and* $(\tilde{A},\tilde{B})$ *be equivalent. Then* (A,B) *is consistent iff* $(\tilde{A},\tilde{B})$ *is consistent.*

PROOF. Assume (A,B) is consistent with pre-consistency vector u and the additional vector v. Assume that the equivalence relation is determined by the transformation matrices T and P. Then we have $\tilde{A}_2T^{-1}u = T^{-1}A_2TT^{-1}u = T^{-1}u$ and $\tilde{A}_1T^{-1}u = P^TA_1TT^{-1}u = P^Te = e$, so $T^{-1}u$ is a pre-consistency vector of $(\tilde{A},\tilde{B})$. Moreover, $\tilde{A}_2T^{-1}v+\tilde{B}_2e = T^{-1}A_2TT^{-1}v+T^{-1}B_2Pe = T^{-1}(A_2v+B_2e) = T^{-1}(u+v)$, according to (2.6). Thus $(\tilde{A},\tilde{B})$ is consistent. □

Choosing a transformation T with the pre-consistency vector as its first column, we can transform a method into a standardized form:

COROLLARY 4.2. *A pre-consistent method is equivalent with a method having* $e_1 = [1,0,\dots,0]^T$ *as pre-consistency vector.*

In the sequel we will implicitly assume that a method is consistent and we will call a method stable if it is algebraically stable for some non-zero matrix G. Observing that the matrix A_2 has an eigenvalue equal to one and that stability implies that A_2 is power bounded, we arrive at (see [6]).

LEMMA 4.3. *A stable general linear method* (A,B) *is equivalent with a method* $(\tilde{A},\tilde{B})$ *such that* $e_i^T\tilde{A}_2e_j = e_j^T\tilde{A}_2e_i = \delta_{ij}$, $i = 1,2,\dots,\tilde{r}$, *for some* $\tilde{r}$, $1 \le \tilde{r} \le r$, *where* δ_{ij} *denotes the Kronecker delta.*

LEMMA 4.4. *Let* (A,B) *be algebraically stable for* G *and let* u *be the pre-consistency vector. Then* Gu = 0 *implies that* (A,B) *can be reduced to a method without internal stages.*

PROOF. Using $A_2u = u$ we find that $(G-A_2^TGA_2)u = 0$. Thus, algebraic stability and lemma 3.2 implies $M\binom{u}{0} = 0$, which yields $DA_1u-B_2^TGA_2u = De-B_2^TGu = De = 0$. Consequently, the diagonal matrix D contains zeros only and according to lemma 3.5 (A,B) can be transformed to form (3.3) with $s_1 = \text{rank}(D) = 0$. □

LEMMA 4.5. *Let* (A,B) *be consistent and* $(\tilde{A},\tilde{B})$ *a reduced method with at least one internal stage. Then* $(\tilde{A},\tilde{B})$ *is consistent.*

PROOF. Assume (A,B) is of form (3.3) and u is the pre-consistency vector. Then, $I_{r_1}A_2I_{r_1}^2u = I_{r_1}A_2u = I_{r_1}u$ and $I_{s_1}A_1I_{r_1}^2u = I_{s_1}A_1u = I_{s_1}e$. Because $s_1 \ge 1$, $I_{r_1}u$ is a pre-consistency vector for the reduced method. Moreover, let v be the additional vector, then we deduce from (2.6) that $I_{r_1}A_2I_{r_1}^2v+I_{r_1}B_2I_{s_1}^2e = I_{r_1}(A_2v+B_2e) = I_{r_1}(u+v)$, so $I_{r_1}v$ is the additional vector of the reduced method satisfying (2.6). □

Now, let (A,B) be irreducible, stable for G with e_1 as pre-consistency vector. Let R^TR be the Cholesky decomposition of G, and $r = e_1^TRe_1$. Then one easily verifies that the transformed method (3.1) with $T = rR^{-1}$ and $P = I$ is algebraically stable for r^2I. But it is obvious from the definition of algebraic stability that a method stable for a G is also stable for αG if $\alpha > 0$. Using corollary 4.2 we obtain

LEMMA 4.6. *Any irreducible, pre-consistent stable method is equivalent with a method which is stable for the identity matrix and has* e_1 *as pre-consistency vector.*

5. EXAMPLE

In this section we present a simple example in order to illustrate the transformations (3.1) and the reduction process. The example shows that the reduction need not be uniquely determined.
Consider the method with $r = 4$, $s = 3$ given by

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 3/2 & -1/2 & 1/2 & -3/2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 7/8 & -1/8 & 1/8 & 1/8 \\ -3/8 & 5/8 & 3/8 & 3/8 \\ 3/8 & 3/8 & 5/8 & -3/8 \\ 1/8 & 1/8 & -1/8 & 7/8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1/4 & 1/4 & 3/8 \\ 1/4 & -1/4 & 1/8 \\ 1/4 & -1/4 & -1/8 \\ 1/4 & 1/4 & -3/8 \end{bmatrix}.$$

After some calculations we find out that the method is algebraically stable for $G = p(1,-1,-1,1)^T(1,-1,-1,1)+q(3,-1,1,-3)^T(3,-1,1,-3)$, with $p > 0$ and $q > 0$, whereas the diagonal matrix D is equal to $\mathrm{diag}(0,p,4q)$. Thus, according to lemma 3.5, the method is reducible and it can be transformed with $r_1 = s_1 = 2$.
As the vectors $v_1 = (1,1,1,1)^T$ and $v_2 = (-1,-3,3,1)^T$ span the null space of G, we might choose the transformation matrix $T_1 = (e_1,e_2,v_1,v_2)$. Let the permutation matrix P interchange the first and third column. Then the transformed method $(\tilde{A},\tilde{B})$ is given by

$$\tilde{A}_1 = \begin{bmatrix} 3/2 & -1/2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/8 & 1/8 & 0 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & -3/2 & 0 \\ -1/2 & 1/2 & 1/4 \\ 1/8 & -1/4 & 0 \end{bmatrix}.$$

One easily verifies that the transformed method is algebraically stable for $\tilde{G} = T_1^T G T_1 = p(1,-1,0,0)^T(1,-1,0,0)+q(3,-1,0,0)^T(3,-1,0,0)$ and that $\tilde{D} = P^T DP = \mathrm{diag}(4q,p,0)$.
Consequently, the reduced method is stable for $G_1 = p(1,-1)^T(1,-1)+q(3,-1)^T(3,-1)$. However, this method which is obtained by taking the upper left hand matrices of the transformed method, is reducible too.
In fact, when we denote the upper left hand matrix in $\tilde{A}_1$ by $A_{1,1}$ the transformation $T_2 = A_{1,1}^{-1}$ transforms the reduced method to

$$\bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

this method is algebraically stable for $\bar{G} = \mathrm{diag}(4q,p)$. Following lemma 3.8, we find that the method $(\bar{A},\bar{B})$ is stable for both $\bar{G}_1 = \mathrm{diag}(0,p)$ and $\bar{G}_2 = \mathrm{diag}(4q,0)$. Actually, $(\bar{A},\bar{B})$ reduces to two independent methods, Backward Euler and the implicit midpoint rule, which are both algebraically stable.
Finally, we remark that the pre-consistency vectors of (A,B), $(\tilde{A},\tilde{B})$ and $(\bar{A},\bar{B})$ are given by $u = \frac{1}{4}(3,-1,1,1)^T$, $T_1^{-1}u = \frac{1}{4}(2,-2,1,0)$ and $T_2^{-1}I_2T_1^{-1}u = \frac{1}{2}(1,-1)$ respectively.

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REFERENCES

[1] BURRAGE, K. and J.C. BUTCHER, *Stability criteria for implicit Runge-Kutta methods*, SIAM J. Num. Anal. 16, No. 1, 1979, pp.46-57.

[2] BURRAGE, K. and J.C. BUTCHER, *Non-linear stability of a general class of differential equation methods*, BIT 20, 1980, pp.185-203.

[3] BUTCHER, J.C., *A stability property of implicit Runge-Kutta methods*, BIT 15, 1975, pp.358-361.

[4] CROUZEIX, M., *Sur la B-stabilité des méthodes de Runge-Kutta*, Numer. Math. 32, 1979, pp.75-82.

[5] DAHLQUIST, G., *On stability and error analysis for stiff non-linear problems I*, Dept. Information Processing, Royal Institute of Technology, Stockholm, Report NA 75-08.

[6] DEKKER, K., *Algebraic stability of general linear methods*, Auckland Computer Science Report No. 25, 1981.

[7] HAIRER, E., *Highest possible order of algebraically stable diagonally implicit Runge-Kutta methods*, BIT 20, 1980, pp.254-256.

[8] HUNSDORFER, W.H. and M.N. SPIJKER, *A note on B-stability of Runge-Kutta methods*, Numer. Math. 36, 1981, pp.319-331.

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