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(DEPARTMENT OF NUMERICAL MATHEMATICS)

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P.W. HEMKER

NUMERICAL ASPECTS OF SINGULAR PERTURBATION PROBLEMS

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Numerical aspects of singular perturbation problems *)

by

P.W. Hemker

ABSTRACT

In this paper a brief survey is given of the main problems that are encountered when singular perturbation problems are solved by numerical means. Some areas of current research are indicated.

For a two-dimensional model problem an error estimate is given for the Hughes and Brooks Streamline-Upwind Petrov-Galerkin method.

KEY WORDS & PHRASES: *singular perturbation problem; stiff boundary value problems; Petrov-Galerkin method*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Five or six years ago the field of singular perturbation problems was an almost undeveloped area of numerical analysis. Now the interest in it has grown and the number of papers devoted to the subject increases with considerable pace. Was the pioneering work by A.M. Il'in [35], Pearson [45], Dorr [21] and Kreiss et al. [1] all devoted to difference schemes for two-point boundary-value problems, now the interest is shifting to 2-dimensional problems that are discretized mostly by finite-element type methods. The academic efforts approach more and more the engineering interests of methods for solving real-life problems such as the Navier-Stokes equations with large Reynolds number.

The existence, for small values of a parameter ε, of asymptotic expansions of the solution may do away with the necessity of special attention by numerical analysis. Indeed, the solution of the reduced problem, obtained after putting $\varepsilon = 0$, can often be considered as a standard problem, as can the numerical computation of the boundary layer(s) on stretched coordinates. As soon as an asymptotic analysis is valid and a few terms in the asymptotic expansion describe the solution sufficiently accurate, one usually can rely on standard techniques to obtain numerical approximations.

However, numerical analysis of singular perturbation problems mainly concentrates on the following question: how to find a numerical approximation to the solution for small as well as for intermediate values of ε, where no short asymptotic expansion is available. Or, more general, how to construct a single numerical method that can be applied both in the case of extremely small ε and for larger values of ε, when one wouldn't consider the problem as singularly perturbed any longer.

Here two main difficulties arise from the numerical point of view. (1) In the solution of singular perturbation problems often boundary layers appear, which may be

very thin. In order to represent the solution by numerical means, a discretization of the region of definition Ω is required. E.g. a (regularly or irregularly distributed) set of points is chosen in Ω, on which the solution is approximated. To represent the solution in the boundary layer *either* enough points should be placed in this layer *or* an a priori assumption about the shape of the boundary-layer solution is necessary. If the shape of the layer - or even its location - is unknown, this can yield an awkward situation. (2) The second difficulty arises in the case of elliptic singular perturbation problems, where - for larger values of ε - all commonly used numerical methods make essential use of the ellipticity of the equation. If the reduced equation is no longer elliptic, the numerical method fails for small ε. In particular Finite Element Methods for the discretization of elliptic PDEs are well analyzed in the case of symmetric differential operators and small perturbations thereof. Many singular perturbation problems of interest, however, are given by essentially non-symmetric operators.

Numerical means to handle the first difficulty (mesh construction) are "adaptive discretization methods". Here, the proper mesh on which the differential problem is discretized, is generated during the process that computes the approximate solution. Thus, the processes of discretization and solution of the discretized equation are closely interwoven. For two-point boundary-value problems significant progress has been made in this direction [2,3,46]. Two general purpose codes for these problems, that can also be used for singularly perturbed problems, have been constructed by Lentini-Pereyra and by Ascher-Christiansen-Russell [17]. The development of numerical methods for adaptive discretization in two dimensions is only beginning. Thus far, a few attempts have been made for non-singular perturbation problems by Babuska and Rheinbold [6,7]. A program that implements an adaptive method is constructed by Bank et al. [10]. Some applications of the theory of Babuska to singular perturbation problems have been published by Reinhardt [48,49]. Apart from these results the subject seems to be completely open to research.

The emphasis in the numerical analysis of singular perturbation problems thus far lies with the 2nd difficulty: the construction and the analysis of methods for strongly non-symmetric operator equations. Here almost all research is devoted to 2nd order elliptic equations with a significant first derivative. The model problems mostly studied are: in one dimension

$$
\begin{aligned}
&-\varepsilon y'' + f y' + g y = s, \qquad \text{on } \Omega = (0,1),\\
&y(0) = \alpha, \quad y(1) = \beta, \qquad (1.1)\\
&\varepsilon > 0, \quad f,g,s \in C^0(\Omega);
\end{aligned}
$$

and in two dimensions

$$
\begin{aligned}
& -\nabla(\bar{\varepsilon}\nabla u) + \bar{b}\nabla u + cu = f && \text{on } \Omega \subset \mathbb{R}^2 \\
(1.2)\qquad & u = g && \text{on } \Gamma_D \\
& \bar{\bar{n}}\bar{\varepsilon}\nabla u = h && \text{on } \Gamma_N
\end{aligned}
$$

$\Gamma_D \cup \Gamma_N = \partial\Omega$, $\bar{\varepsilon} \in \mathbb{R}^{2\times 2}$ a positive definite matrix, $\bar{b} \in [C^1(\Omega)]^2$, with $\nabla\bar{b} = 0$ and $c,f \in C^0(\Omega)$.

As was pointed out by Brandt [15], if the more-dimensional problem is the final aim, the one-dimensional problem is often a real over-simplification. In the first place this is by the dimensionality of the problem itself. In two dimensions the shape of the boundary-layers may be much more complex than in the one-dimensional case. The second reason is directly related to the numerical approximation of the equation. If the 1-D differential equation is approximated by a difference equation

$$
(1.3)\qquad a_i y_{i-1} + b_i y_i, \quad c_i y_{i+1} = d_i,
$$

the value y_i is determined by the values y_{i-1} and y_{i+1}, as is the case when the differential problem is restricted to the interval $[x_{i-1}, x_{i+1}]$. Hence, the coefficients in (1.3) can be determined such that y_i is approximated with an arbitrary high accuracy. This situation never occurs in the case of more dimensions, where the differential solution at a gridpoint is not determined by its value at any finite number of neighbouring gridpoints.

In this paper we first consider, in section 2, simple difference schemes for a 1-dimensional problem and we show what numerical difficulties arise when the differential problem becomes singularly perturbed. Here we indicate one line of research: the search for ε-uniform methods. In section 3 we introduce global methods: collocation, Galerkin and finite elements, and in the next section we show how error estimates for the usual methods degenerate for small ε. Here we mention the line of research which, in 2-dimensional problems, applies Petrov-Galerkin methods for adapting the schemes to singular perturbations. In section 5 we describe the Hughes and Brooks Petrov-Galerkin method and in section 6 we derive an error estimate for it. Finally we give some bibliographical notes and indicate some other lines of current research.

2. FINITE DIFFERENCE METHODS

The simplest example to show what happens with the discretization of strongly non-symmetric problems by classical means is given by the equation

$$
(2.1)\qquad \begin{cases} \varepsilon y'' + y' = 0, & \text{on } (0,1) \\ y(0) = 0, \quad y(1) = 1. \end{cases}
$$

Central discretization on a regular mesh $\{0 = x_0 < x_1 < \ldots < x_N = 1 \mid x_i - x_{i-1} = h\}$ yields

$$L^c_{h,\varepsilon} y_i \equiv \varepsilon(y_{i+1} - 2y_i + y_{i-1})/h^2 + (y_{i+1} - y_{i-1})/2h = 0,$$
(2.2)
$$y_0 = 0, \quad y_N = 1.$$

Its solution reads

$$y_i = (1-r^i)/(1-r^N), \tag{2.3}$$

with $r := (2\varepsilon-h)/(2\varepsilon+h)$. The exact solution of (2.1) at x_i is given by (2.3) with $r := \exp(-h/\varepsilon)$. We see that

$$|\exp(-h/\varepsilon) - \frac{2\varepsilon-h}{2\varepsilon+h}| = O((\frac{h}{\varepsilon})^3) \quad \text{for } (\frac{h}{\varepsilon}) \to 0,$$

and hence

$$|y(x_i) - y_i| = O((\frac{h}{\varepsilon})^2) \quad \text{for } (\frac{h}{\varepsilon}) \to 0, \tag{2.4}$$

for a fixed point x_i in (0,1). The error is bounded by $C(\varepsilon)h^2$; i.e. the approximation is 2^{nd} order accurate, but the error constant $C(\varepsilon)$ may increase intolerably as $\varepsilon \to 0$. In fact we see that the discrete solution oscillates for $2\varepsilon < h$, and (for even N) we find

$$\lim_{\varepsilon\to 0} y_i = i/N \quad \text{(for even i)}$$
$$\approx \frac{h}{2\varepsilon N} \quad \text{(for odd i)},$$

whereas for the true solution $\lim_{\varepsilon\to 0} y(x_i) = 0$. The large error for small ε is clearly due to instability of the operator $L^c_{h,\varepsilon}$ since the eigenvalues of this discretized operator are

$$\lambda_i = \frac{-2\varepsilon}{h} + \frac{1}{h^2}\sqrt{4\varepsilon^2-h^2}\,\cos(\frac{i\pi}{N+1}).$$

The simplest way to overcome this instability is to use "upstream" differences instead of central differences, i.e. to take

$$L^u_{h,\varepsilon} y_i \equiv \varepsilon(y_{i+1} - 2y_i + y_{i-1})/h^2 + (y_{i+1} - y_i)/h = 0. \tag{2.5}$$

The solution of this difference equation is (2.3) with $r := \varepsilon/(\varepsilon+h)$. Here we see

$$|\exp(-h/\varepsilon) - \frac{\varepsilon}{\varepsilon+h}| = O((\frac{h}{\varepsilon})^2)$$

and hence, for a fixed $x_i \in (0,1)$,

$$(2.6) \qquad |y(x_i) - y_i| = O(\frac{h}{\varepsilon}) \quad \text{for } \frac{h}{\varepsilon} \to 0.$$

This approximation is only 1^{st} order accurate for $h \to 0$, but here we see $\lim_{\varepsilon \to 0} y_i = 0$ for $i > 0$, i.e. the asymptotic behaviour of the true solution is reflected in its discrete approximation. This success of the "upstream" difference approximation depends crucially on the choice of the right one-sided difference, which uses the "upstream" value y_{i+1} (i.e. away from the boundary layer). If we would have taken the difference approximation $(y_i - y_{i-1})/h$ for $y'(x)$, then we would not have found a good approximation to $y(x_i)$ at all.

We observe that

$$(2.7) \qquad L^u_{h,\varepsilon} = L^c_{h,\varepsilon+h/2} ,$$

i.e. the "upstream" discretization is equivalent with the central discretization if we replace the parameter ε by $\varepsilon + h/2$. The addition of the extra term $+h/2$ is called *artificial diffusion* or *artificial viscosity*.

The artificial viscosity formulation of the one-sided difference approximation can be used more generally for equation (1.1). Then the upstream discretization is equivalent with the central difference discretization, replacing ε by

$$(2.8) \qquad \varepsilon + h|f(x_i)|/2 = \varepsilon + \text{sign}(\frac{hf(x_i)}{2\varepsilon})\frac{f(x_i)h}{2} .$$

Uniform numerical methods for singular perturbation problems

Both the error bounds (2.4) and (2.6) are of the form

$$|y_i - y(x_i)| \le C(\varepsilon)h^p,$$

the error is of order p, but the error constant depends on ε. In (2.4) the error really degenerates for $\varepsilon \to 0$, but also in (2.6) we cannot improve the bound so that $C(\varepsilon)$ is independent of ε. Indeed, if we set $i = 1$ we find (Il'in [35])

$$|y_1 - y(x_1)| = \left| \frac{1 - \frac{\varepsilon}{\varepsilon+h}}{1 - (\frac{\varepsilon}{\varepsilon+h})^N} - \frac{1-\exp(-h/\varepsilon)}{1-\exp(-1/\varepsilon)} \right|$$

and

$$\lim_{\substack{h\to 0 \\ \varepsilon=h}} |y_1 - y(x_1)| = \frac{1}{2} - e^{-1} \not\to 0.$$

Thus, the error bound does not hold uniformly in ε. Il'in shows that an error bound

$$|y_i - y(x_i)| \leq Ch \quad \text{for all } x_i \in \Omega_h$$

with C independent of h or ε is obtained for the discretization of (1.1) with $g(x) = 0$, if central differences are used with an artificial diffusion:

$$+ \frac{f(x_i)h}{2} \left[\coth\left(\frac{f(x_i)h}{2\varepsilon}\right) - \frac{2\varepsilon}{f(x_i)h}\right] .$$

This corresponds to the artificial diffusion (2.8) where sign(z) is replaced by coth(z) - 1/z. The shape of this function is shown in figure 1.

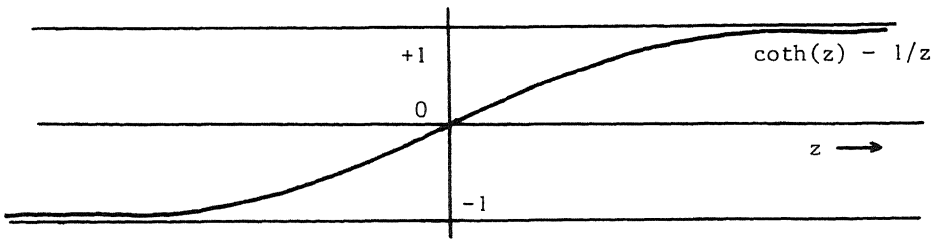


Figure 1. The exponential fitting function coth(z) - 1/z.

Miller [20,28,39] has proved that uniform bounds can be obtained only if the difference scheme is exponentially fitted, i.e. if the coefficients in the difference equations contain exponential functions.

Much effort has been spent on the construction and analysis of ε-uniform difference schemes and some results have been obtained. E.g. 2^{nd} order uniform methods have been constructed for the 1-dimensional convection diffusion equation (e.g. (1.1) with $b(x) \geq B > 0$, $c(x) \leq 0$), cf. [14,20,23,28,37]. However, fundamental difficulties arise when any of these methods are to be generalized to more dimensions and little progress has been reported so far.

3. GLOBAL METHODS

Systematic means for obtaining the discretization of a continuous equation

(3.1) $\quad Lu = f,$

L a differential operator L: $S \to V$, $u \in S$, $f \in V$, S and V Banach spaces of functions defined over a region Ω, are given by *global* or *weighted residual* methods. Here the solution u of the continuous equation is approximated by an element u_h in some finite (N-) dimensional function space S_h the *trial-space*, $S_h \subset S$. Since usually $u \notin S_h$, u_h is determined such that the residual,

$$f - Lu_h ,$$

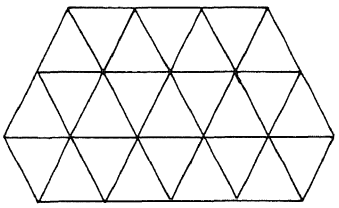
is small is some sense. Examples of weighted residual methods are *collocation* methods where

$$(f - Lu_h)(x_i) = 0$$

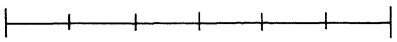
is required for N points x_i; or *Galerkin* methods where the requirement is

$$\int (f - Lu_h)\psi_i d\Omega = 0$$

for N functions ψ_i. The functions ψ_i span the finite-dimensional function space V_h, the *test-space*. If $S_h = V_h$, the method is called a *Bubnov-Galerkin* method, if $S_h \neq V_h$ it is a *Petrov-Galerkin* method. Standard *finite element* methods (FEM) are Bubnov-Galerkin methods, where the functions ψ_i have a small support in Ω. The simplest FEM is constructed by taking a triangularization T_h of the domain Ω with N vertices x_i.



T_h in $\Omega \subset \mathbb{R}^2$



T_h in $\Omega \subset \mathbb{R}^1$.

To each x_i a ψ_i is chosen such that ψ_i is linear over each triangle and $\psi_i(x_j) = \delta_{ij}$. For these ψ_i we denote

$$M^{0,1}(T_h) = \text{Span}\{\psi_i\}.$$

Before the FEM discretization is applied to an elliptic differential equation $Lu = f$, where $L: C^{k+2m}(\Omega) \to C^k(\Omega)$, this equation is reformulated to its weak form, where $L: H^m(\Omega) \to H^{-m}(\Omega)$, with $H^m(\Omega)$ and $H^{-m}(\Omega)$ the usual Sobolev spaces. Then the FEM uses $S = V = H^m(\Omega)$ and $S_h = V_h \subset H^m(\Omega)$.

EXAMPLES

The model equation (1.1) is reformulated as

$$\begin{aligned} &B(\psi,y) = \ell(\psi) \qquad \text{for all } \psi \in V \text{ with } \psi(0) = \psi(1) = 0, \\ &y(0) = \alpha, \quad y(1) = \beta, \\ &B(\psi,y) = \int_0^1 \varepsilon\psi'y' + \psi fy' + \psi gy \, dx, \\ &\ell(\psi) = \int_0^1 \psi s \, dx. \end{aligned} \tag{3.2}$$

The 2-dimensional model equation (1.2) is reformulated as

$$
\begin{aligned}
&B(\psi,u) = \ell(\psi) \qquad \text{for all } \psi \in V \text{ with } \psi = 0 \text{ on } \Gamma_D,\\
&u = g \quad \text{on } \Gamma_D,\\
&B(\psi,u) = \int \nabla\psi\bar{\varepsilon}\nabla u + \psi\bar{b}\nabla u + \psi c u \, d\Omega,\\
&\ell(\psi) = \int \psi f d\Omega + \int \psi h d\Gamma_N.
\end{aligned} \tag{3.3}
$$

For future reference we denote the latter problem also as: find $u \in S$ with $u = g$ on Γ_D and

$$B(\psi,u) = \ell(\psi) \quad \text{for all } \psi \in V_B ,$$

$$B(\psi,u) = (\nabla\psi,\bar{\varepsilon}\nabla u) + (\psi,\bar{b}\nabla u) + (\psi,cu),$$

$$\ell(\psi) = (\psi,f) + \langle\psi,h\rangle_N ,$$

$(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_N$ denote the $L^2(\Omega)$ and the $L^2(\Gamma_N)$ innerproducts.

4. THE ONE-DIMENSIONAL PROBLEM

If we discretize equation (2.1), or rather its weak formulation

$$\int_0^1 -\varepsilon\psi'(x)y'(x) + \psi(x)y'(x)dx = 0, \tag{4.1}$$

by the FEM with $S_h = V_h = M^{0,1}(T_h)$ on a regular partition $T_h = \{0 = x_0 < x_1 < \ldots < x_N = 1\}$ of $[0,1]$, then the discrete equation

$$
\begin{aligned}
&\sum_{j=0}^{N} \int_0^1 -\varepsilon\psi_i'\psi_j' + \psi_i\psi_j' \, dx \, u_h(x_j) = 0,\\
&u_h(0) = 0, \quad u_h(1) = 1,
\end{aligned} \tag{4.2}
$$

is completely equivalent with eq. (2.2). Thus, we see that this FEM suffers from the same lack of stability as the central difference discretization.

In order to see this effect appear in a degenerating error estimate for a Bubnov-Galerkin method, when combined with the strongly asymmetric bilinear form $B(v,u)$, we consider equation (1.1) with constant coefficients ε, f, $g \geq 0$ and homogeneous boundary conditions in its variational formulation (3.2). The solution satisfies

$$B(v,u) = \ell(v) \qquad \forall v \in V_B;$$

the approximate solution u_h satisfies

$$B(v_h,u_h) = \ell(v_h) \qquad \forall v_h \in V_h \subset V_B.$$

Hence, the error $e = u_h - u$ satisfies

$$B(v_h,e) = 0 \qquad \forall v_h \in V_h.$$

We easily derive

$$\varepsilon\|e'\|^2 + (e,ge) = B(e,e) = B(u-u_h,e) = B(u-v_h,e) \qquad \forall v_h \in V_h = S_h. \tag{4.3}$$

Further we find for $\eta = u - v_h^*$, v_h^* the best approximant in S_h to u,

$$\begin{aligned} B(\eta,e) &= (\eta',\varepsilon e') + (\eta,fe') + (\eta,ge) \\ &\le \frac{\varepsilon}{2}\|\eta'\|^2 + \frac{\varepsilon}{2}\|e'\|^2 + \frac{1}{2}g\|\eta\|^2 + \frac{1}{2}g\|e\|^2 \\ &\quad + \frac{f}{2p}\|\eta\| + \frac{pf}{2}\|e'\| \qquad \text{for all } p > 0, \end{aligned} \tag{4.4}$$

whence

$$(\varepsilon-pf)\|e'\|^2 + g\|e\|^2 \le \varepsilon\|\eta'\|^2 + (g+\frac{f}{p})\|\eta\|^2.$$

To obtain a positive coefficient for $\|e'\|$ we have to select p such that $pf/\varepsilon = 1/c < 1$. We get

$$\varepsilon(1-\tfrac{1}{c})\|e'\|^2 + g\|e\|^2 \le \varepsilon\|\eta'\|^2 + (g + f\,\frac{cf}{\varepsilon})\|\eta\|, \tag{4.5}$$

where η is the error of the best approximant of u in S_h. For large ratios $\frac{f}{\varepsilon}$ this cannot be a useful error estimate since the rhs is $\mathcal{O}(\varepsilon^{-1})$ for $\varepsilon \to 0$.

Several remedies have been proposed to obtain better global methods:

(i) to use artificial diffusion in the FEM discretization, i.e. to solve the problem for a value ε which is increased up to $\mathcal{O}(h)$;

(ii) to adapt the space S_h, such that the (rapidly changing) solution can be better approximated by elements from S_h [41];

(iii) to adapt the space V_h such that a stable discretization is found [30].

The first possibility is the simplest to apply, but it has the severe disadvantage that such a method can be only 1^{st} order accurate. The loss of accuracy is seen over the whole region, but becomes particularly apparent in the boundary layers which show up as overly diffusive. Hence, this approach was severely criticised for practical applications [16,34].

To obtain higher order methods that can be applied to problem (1.1), the adaptation of the trial- and test-space has been studied by several authors [11,12,13,16, 18,24,25,26,29,30,36,41,42,43]. Almost all the analysis is made for the case where no turning points are present, i.e. $|f(x)| \geq C > 0$. In this case a boundary layer appears only at the end of the interval and the boundary layer is of exponential type. Therefore, in order to fit the solution by an element of the trial space, the inclusion of exponential trial-functions in S_h is a natural procedure [13,41].

In order to obtain good pointwise approximations at the nodal points, it is advantageous to adapt the test-space. For the bounded bilinear form $B(v,u)$ and the Green's function $G(\cdot,\cdot)$ for problem (1.1), we find

$$B(v_h, u-u_h) = 0 \qquad \text{for all } v_h \in V_h;$$

$$|(u-u_h)(x)| = |B(G(x,\cdot),\ u-u_h)|$$

$$= |B(G(x,\cdot) - v_h,\ u-u_h)| \leq C\|G(x_i) - v_h\|_V \|u-u_h\|_S;$$

$$|(u-u_h)(x_i)| \leq C\|u-u_h\|_S \inf_{v_h \in V_h} \|G(x_i,\cdot) - v_h\|_V.$$

Hence, it is clear that pointwise errors are essentially smaller than the error in the $\|\cdot\|_S$-norm if the Green's function $G(x_i,\cdot)$ can be well approximated by elements in V_h.

The Green's functions $G(x_i,\cdot)$, being solutions of the adjoint of the original problem, show exponential boundary layers themselves (with the direction reversed). In practical methods, bases in S_h and in V_h are used of which all elements have a limited support. A typical basis of exponentially fitted spaces S_h and V_h is shown in figure 2. To obtain higher order approximations, these spaces can be supplemented with piecewise polynomials [25,26,30].

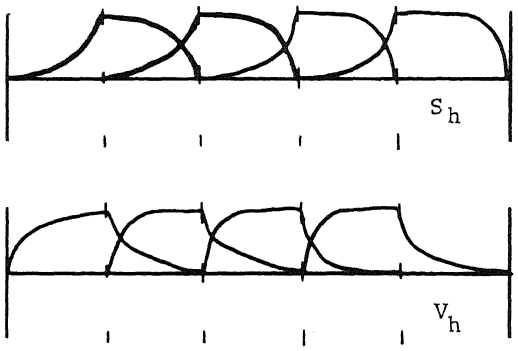


Figure 2. A basis of exponentially fitted functions in S_h and in V_h for the equation $-\varepsilon y'' + y' = f$.

Both pointwise and global error-estimates, in the norm $\|e\|_\varepsilon = \sqrt{\varepsilon\|e'\|^2 + \|e\|^2}$, can be found in de Groen [25,26]. A further analysis based on the Generalized Lax-Milgram theorem

is found in [8].

Other adaptations of the space of test-functions V_h are introduced by Christie et al. [18], who use

$$V_h = \{v \in H_0^1(a,b) \mid v = \sum_{j=1}^{N-1} v_j \phi_j^\alpha(x)\},$$

$$\phi_j^\alpha(x) = \begin{cases} (x-x_{j-1})/h_j + 3\alpha(x-x_{j-1})(x_j-x)/h_j^2, & x \in (x_{j-1},x_j), \\ (x_{j+1}-x)/h_{j+1} - 3\alpha(x-x_j)(x_{j+1}-x)/h_{j+1}^2, & x \in (x_j,x_{j+1}), \end{cases}$$

i.e. the piecewise linear functions are perturbed with piecewise quadratics. The parameter α is adjusted such that stable approximations are obtained. The selection of an optimal α is studied in [24].

Another Petrov-Galerkin method for the problem (1.1) is studied by Barrett and Morton [11,12]. They consider a test-space $V_h = N_\varepsilon S_h$, where N_ε is an operator which approximately symmetrizes the bilinear form B, i.e. N_ε is an approximation to the operator $N: H_0^1 \to H_0^1$ such that the operator $B(Nw_1,w_2)$ is a symmetric bilinear form.

5. THE TWO-DIMENSIONAL PROBLEM

We consider equation (1.2) and we study a Petrov-Galerkin method as was proposed by Hughes and Brooks [16]. For the trial-space we take $S_h = M^{0,1}(T_h)$ and for the test-space we use $V_h = \tilde{M}S_h$, where $\tilde{M}: H^1(\Omega) \to L_2(\Omega)$ is the operator defined by

$$\tilde{M}u = u + k\bar{b}\nabla u, \quad k \in \mathbb{R}, \ k \geq 0, \tag{5.1}$$

with $k = k(h,\varepsilon,\ldots)$. For the special case $k = 0$ the method is identical with the classical FEM and the method fails for singular perturbation problems. However, we search for a proper choice of k and derive error bounds that make sense also for small ε. We see that for $S_h = M^{0,1}(T_h)$ the range $V_h = \tilde{M}S_h$ consists of functions that are the sum of a continuous, piecewise linear function ψ and a discontinuous, piecewise constant function $k\bar{b}\nabla\psi$.

Since $V_h \not\subset H^1(\Omega)$ we have first to reformulate our original equation to another variational form than (3.3), before we can discretize it with this V_h. We construct an intermediate formulation between the classical form (1.2) and the weak form (3.3). By application of Green's formula over the domain Ω, it is clear that any solution $u \in C^2(\Omega)$ of (1.2) is also a solution of (3.3); but, in general, if (3.3) admits a solution, (1.2) does not. Classical regularity theory, however, shows that for data smooth enough the solution of (3.3) is a solution of (1.2) indeed. In short, as soon as the existence of a solution of (1.2) and the uniqueness of the solution of (3.3) is established, the two are different formulations of one and the same problem.

For our formulation, let us consider an arbitrary partition of Ω in open

subdomains $\{\Omega_e\}$ such that

$$\bar{\Omega} = \bigcup_e \bar{\Omega}_e \quad \text{and} \quad \Omega_{e_i} \cap \Omega_{e_j} = \emptyset \quad \text{if } i \neq j, \tag{5.2}$$

and the boundaries of Ω_e are piecewise smooth for each Ω_e. We distinguish the following (disjunct) boundaries: the Dirichlet boundary Γ_D, the Neumann boundary Γ_N and the internal boundary Γ_{int}, such that

$$\Gamma_D \cup \Gamma_N \cup \Gamma_{int} = \bigcup_e \partial\Omega_e. \tag{5.3}$$

Let us now consider solutions $u \in H^1(\Omega) \cap C^0(\bar{\Omega})$ of (3.3) such that $u|_{\Omega e} \in H^2(\Omega_e)$ for all Ω_e. Possibly $u \notin C^2(\Omega)$ and hence equation (1.2) has no classical meaning on Γ_{int}. In order to interprete the weak problem in a classical sense, we formulate a condition over the inter-element boundaries that is satisfied by u. By definition, the solution u of (3.2) satisfies

$$(\nabla v, \bar{\varepsilon}\nabla u) + (v, \bar{b}\nabla u) + (v, cu) = (v, f) + \langle v, h\rangle$$
$$\text{for all } v \in H_B^1(\Omega) = \{u \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\} \tag{5.4}$$

Since $u|_{\Omega e} \in H^2(\Omega_e)$, we know for all $v \in C_B^1(\Omega) = \{v \in C^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$,

$$\sum_e \int_{\Omega e} v(-\nabla(\bar{\varepsilon}\nabla u) + \bar{b}\nabla u + cu)\,d\Omega_e = \sum_e \int_{\Omega e} (\nabla v \bar{\varepsilon}\nabla u + v\bar{b}\nabla u + vcu)\,d\Omega_e$$
$$- \int_{\Gamma_N} v\bar{\bar{n}}\bar{\varepsilon}\nabla u\, d\Gamma + \int_{\Gamma_{int}} \mathrm{jmp}(v\bar{\bar{n}}\bar{\varepsilon}\nabla u)\,d\Gamma, \tag{5.5}$$

where $\bar{n}$ is the outward normal on $\partial\Omega_e$, and on the boundary between Ω_{e_i} and Ω_{e_j} (i.e. for $x \in \bar{\Omega}_{e_i} \cap \bar{\Omega}_{e_j}$) we use the notation

$$\mathrm{jmp}(v\bar{\bar{n}}\bar{\varepsilon}\nabla u)(x) = -\sum_{k=i,j} \lim_{\substack{h\to 0 \\ x+h\in\Omega_{e_k}}} v_{e_k} \bar{n}_{e_k} \bar{\varepsilon}\nabla u(x+h). \tag{5.6}$$

Equations (5.4) and (5.5) together with the boundary conditions on Γ_D and Γ_N yield

$$\int_{\Gamma_{int}} \mathrm{jmp}(v\bar{n}\,\bar{\varepsilon}\nabla u)\,d\Gamma = 0 \quad \text{for all } v \in C_B^1(\Omega),$$

and hence $\bar{\bar{n}}\bar{\varepsilon}\nabla u$ is continuous over inter-element boundaries.

Now we reformulate the problem (1.2) as:
find $u \in S = \{u \mid u \in H^1(\Omega) \cap C^0(\bar{\Omega});\ u|_{\Omega_e} \in H^2(\Omega_e)\}$ such that

$$
(5.7)\qquad
\begin{aligned}
&\text{(i)} && -\nabla(\varepsilon\nabla u) + \bar{b}\nabla u + cu = f \text{ on each } \Omega_e;\\
&\text{(ii)} && \bar{\bar{n}}\bar{\varepsilon}\nabla u \text{ is continuous over inter-element boundaries}\\
&\text{(iii)} && u = g \quad \text{on } \Gamma_D;\\
&\text{(iv)} && \bar{\bar{n}}\bar{\varepsilon}\nabla u = h \quad \text{on } \Gamma_N.
\end{aligned}
$$

It is immediately clear that a solution of (1.2) also satisfies (5.7) and that a solution of (5.7) is also a solution of (3.3).

The problem (5.7) can be written in a variational formulation as: find $u \in S$ such that $u = g$ on Γ_D and

$$
(5.8)\qquad
\begin{aligned}
&\sum_e \int_{\Omega_e} (w+v)(-\nabla(\bar{\varepsilon}\nabla u) + \bar{b}\nabla u + cu - f)\,d\Omega + \int_{\Gamma_{int}} v\bar{\bar{n}}\bar{\varepsilon}\nabla u\,d\Gamma +\\
&+ \int_{\Gamma_N} v(\bar{\bar{n}}\bar{\varepsilon}\nabla u - h)\,d\Gamma = 0 \qquad \text{for all } w \in L^2(\Omega) \text{ and all } v \in H^1_B(\Omega).
\end{aligned}
$$

Taking $w = Mv$, $M: H^1_B(\Omega) \to L^2(\Omega)$ an injection, we can write this variational formulation as: find $u \in S$ such that for all $v \in H^1_B(\Omega)$

$$\tilde{B}(v,u) = \tilde{\ell}(v),$$

where

$$\tilde{B}(v,u) = (Mv, -\nabla(\bar{\varepsilon}\nabla u)) + (\nabla v\bar{\varepsilon}\nabla u) + (v+Mv, \bar{b}\nabla u + cu),$$

$$\tilde{\ell}(u) = (v + Mv, f) + \langle v,h\rangle_N.$$

This variational form is discretized to obtain the Hughes and Brooks Petrov-Galerkin method: the trial-space is $S_h = M^{0,1}(T_h) \subset S$ and the test space is $V_h = S_h \cap H^1_B(\Omega)$. Further $M: H^1_B(\Omega) \to L^2(\Omega)$ is chosen to be

$$Mv = k\bar{b}\nabla v \qquad \text{for all } v \in H^1_B(\Omega).$$

6. AN ERROR ESTIMATE FOR THE PETROV-GALERKIN METHOD

We consider the problem (5.7) or (5.8) with the additional assumptions

$$
(6.1)\qquad
\begin{aligned}
&\bar{\varepsilon} \in \mathbb{R}^{2\times 2} && ; \ \bar{\varepsilon} \text{ positive definite};\\
&\bar{b} \in [H^1(\Omega)]^2 && ; \ \nabla\bar{b} = 0;\\
&c \in C^0(\Omega) && ; \ 0 \le c(x);\\
&f \in L^2(\Omega) && ;\\
&g \text{ such that a } \tilde{g} \in H^2(\Omega) \text{ exists for which } \tilde{g}|_{\Gamma_D} = g;\\
&h \in L^2(\Gamma_N) && ; \ \bar{\bar{n}}\bar{b} \ge 0 \text{ on } \Gamma_N.
\end{aligned}
$$

However, for simplicity in the formulations to follow, we restrict ourselves further to constant c and k. By C we denote a generic constant which is independent of h or ε.

To derive an error estimate for the Hughes- and Brooks Petrov-Galerkin method, we denote the error as $e = u_h - u$, where u and u_h satisfy

$$\tilde{B}(v,u) = \tilde{\ell}(v) \quad \text{for all } v \in H_B^1(\Omega),$$

and

$$\tilde{B}(v_h,u_h) = \tilde{\ell}(v_h) \quad \text{for all } v_h \in V_h \subset H_B^1(L).$$

Hence

$$\tilde{B}(v_h,e) = 0 \text{ for all } v_h \in V_h.$$

Analogous to (4.3) - (4.4) we find

$$\tilde{B}(e,e) = \tilde{B}(u_h - u + v_h,e) = \tilde{B}(v_h - u,e) = \tilde{B}(\eta,e). \tag{6.2}$$

With the norms $\|u\| = \sqrt{(u,u)}$ (the $L^2(\Omega)$ norm) and $\|\nabla u\|_E = \sqrt{(\nabla u,\bar{\varepsilon}\nabla u)}$, we get, for any $t,p > 0$.

$$\begin{aligned}\tilde{B}(\eta,e) &= -(k\bar{b}\nabla\eta,\nabla(\bar{\varepsilon}\nabla u)) + (\nabla\eta\bar{\varepsilon}\nabla u) + (k\bar{b}\nabla\eta + \eta,\bar{b}\nabla e + ce)\\ &= -(k\bar{b}\nabla\eta,\nabla(\bar{\varepsilon}\nabla u)) + t[\|\nabla e\|_E^2 + \frac{k+p}{2}\|\bar{b}\nabla e\|^2 + c\|e\|^2]\\ &\quad + \frac{1}{2t}[\frac{1}{2}\|\nabla\eta\|_E^2 + k(1+kc)\|\bar{b}\nabla\eta\|^2 + (c+\frac{1}{p})\|\eta\|^2].\end{aligned} \tag{6.3}$$

Similarly we derive

$$\tilde{B}(e,e) = -(k\bar{b}\nabla e,\nabla(\bar{\varepsilon}\nabla e)) + \|\nabla e\|_E^2 + k\|\bar{b}\nabla e\|^2 + c\|e\|^2 + (1+kc)(e,\bar{b}\nabla e). \tag{6.4}$$

Using $e = 0$ on Γ_D and $\bar{n}\bar{b} \geq 0$ on Γ_N, we see

$$(1+kc)(e,\bar{b}\nabla e) = \frac{1+kc}{2}\langle\bar{n}\bar{b}e^2\rangle \geq 0. \tag{6.5}$$

Combination of (6.2) - (6.5) with $t = \frac{1}{2}$ yields

$$\begin{aligned}&\frac{1}{2}(\|\nabla e\|_E^2 + c\|e\|^2 + (1+kc)\langle\bar{n}\bar{b}e^2\rangle) + \frac{1}{4}(3k-p)\|\bar{b}\nabla e\|^2\\ &= (k\bar{b}\nabla e,\nabla(\bar{\varepsilon}\nabla u)) - (k\bar{b}\nabla\eta,\nabla(\bar{\varepsilon}\nabla u)) + \frac{1}{2}\|\nabla\eta\|_E^2 + k(1+kc)\|\bar{b}\nabla\eta\|^2 + (c+\frac{1}{p})\|\eta\|^2.\end{aligned}$$

Now we use the well-known approximation result [19]

$$\inf_{u_h \in S_h} \|u-u_h\| + h\|\nabla u-\nabla u_h\| \le Ch^2\|u\|_2 ,$$

with $\|\cdot\|_2$, the norm in $H^2(\Omega)$, to obtain

$$|||e|||^2 := \|\nabla e\|_E^2 + c\|e\|^2 + (1+kc)\langle \bar{\bar{n}}\bar{b}e^2\rangle + \frac{1}{2}(3k-p)\|\bar{b}\nabla e\|$$

(6.6)

$$\le 2k\varepsilon_m\|\bar{b}\nabla e\|\|u\|_2 + C\|u\|_2^2\{kb_m\varepsilon_m h + h^2(\varepsilon_m + kb_m + (c+\frac{1}{p})k^2)\} ,$$

where ε_m is the greater eigenvalue of $\bar{\varepsilon}$,

$$b_m = \max(\|b_1\|_{L^\infty(\Omega)}, \|b_2\|_{L^\infty(\Omega)}).$$

From (6.6), for $\bar{b} = 0$, we derive the error estimate for the symmetric case:

$$\|\nabla e\|_E^2 + c\|e\|^2 \le Ch^2(\varepsilon_m + h^2)\|u\|_2^2. \tag{6.7}$$

If $\bar{b} \neq 0$ we find the estimate for the asymmetric case. With $k \neq 0$ we recover the estimate for the ordinary FEM:

$$\|\nabla e\|_E^2 + c\|e\|^2 - \frac{p}{2}\|\bar{b}\nabla e\|^2 + \langle \bar{\bar{n}}\bar{b}e^2\rangle \le Ch^2[\varepsilon_m + (c+p^{-1})/h^2]\|u\|_2^2. \tag{6.8}$$

In order that $C\|\nabla e\|_E^2 < \|\nabla e\|_E^2 - \frac{p}{2}\|\bar{b}\nabla e\|^2$ we have to select p such that

$$\frac{p}{2} < \frac{\varepsilon_{min}}{b_m^2} ;$$

ε_{min} the smaller eigenvalue of ε. Hence, the error bound is proportional to ε_{min}^{-1} for small values of ε_{min}.

However, if $k \neq 0$ we may set $p = k$ and obtain

$$|||e|||^2 = \|\nabla e\|_E^2 + k\|\bar{b}\nabla e\|^2 + c\|e\|^2 + (1+kc)\langle \bar{\bar{n}}\bar{b}e^2\rangle$$

$$\le 2\sqrt{k}\,\varepsilon_{max}|||e|||\,\|u\|_2 + C\{kb_m\varepsilon_m h + h^2[\varepsilon_m + kb_m + (c+\tfrac{1}{k})h^2]\}\|u\|_2^2 ,$$

from which we derive (assuming $b_m < C$)

$$|||e||| = \sqrt{k}\,\varepsilon\|u\|_2 + \|u\|_2\sqrt{k\varepsilon_m^2 + C\{\varepsilon_m kh + h^2\varepsilon_m + h^2k + h^4c + h^4/k\}}.$$

For $\varepsilon_m \le h$ we find

$$|||e||| \le C\|u\|_2\sqrt{kh^2 + h^3 + h^4c + h^4/k} .$$

Hence, for small $\varepsilon_m \le O(h)$, an optimal bound is found for $k = O(h)$; then we find

$$|||e||| \leq Ch^{3/2} \|u\|_2 .$$

For $\varepsilon_m \approx 1$ we find

$$|||e||| \leq C\|u\|_2 \sqrt{k + h^2 + h^4/k} \leq Ch\|u\|_2 ,$$

for the optimal $k = O(h^2)$. For this ε_m and k, the errorbound is as sharp as in the classical case with b = 0 (cf. eqn. (6.7)). For $\varepsilon_m \approx 1$, if we assume $C_1 h^2 \leq k \leq C_2 h$, we find

$$|||e||| \leq C\|u\|_2 \sqrt{k\varepsilon_m^2 + h^2\varepsilon + h^4/k} \leq Ch\|u\|_2 ,$$

for the optimal $k = h^2/\varepsilon_m$.

We conclude that the optimal value of k is given by

$$k = O(h) \qquad \text{if } \varepsilon_m \leq Ch ,$$

$$k = O(h^2/\varepsilon_m) \quad \text{if } \varepsilon_m \geq Ch .$$

Here we notice a correspondence between the optimal value of k and the optimal artificial diffusion as found e.g. in Il'in's method (section 2).

Another analogy between the additional terms for $k \neq 0$ and artificial diffusion for the FEM was already mentioned by Hughes and Brooks, who noticed that the 2nd order difference terms in the discrete operator (5.8) $(\nabla v_h, \bar{\varepsilon}\nabla u_h)$ and $(k\bar{b}\nabla v_h, \bar{b}\nabla u_h)$ can be combined to

$$(\nabla v_h, (\bar{\varepsilon} + k\bar{b}^T\bar{b})\nabla u_h) .$$

Thus, to the original diffusion tensor $\bar{\varepsilon}$ an artificial term $k\bar{b}^T\bar{b}$ is added. This diffusion acts only in the direction of $\bar{b}$, whence Hughes and Brooks call their method a Streamline-Upwind Petrov-Galerkin method. (For the relation between upwinding and artificial diffusion see section 2.)

Combining the estimates for $\varepsilon \leq h$ and $\varepsilon \geq h$, we find with the optimal

$$k \approx \frac{h}{\varepsilon} \cdot \min(h, \varepsilon_m)$$

the following error estimate

$$\sqrt{\|\nabla e\|_E^2 + h\|\bar{b}\nabla e\|^2 + c\|e\|^2} < C\|u\|_2 \, h\sqrt{\max(h, \varepsilon_m)} .$$

7. FURTHER DEVELOPMENTS AND BIBLIOGRAPHICAL NOTES

In the previous sections, for the 2-dimensional singular perturbation problem we only considered the Streamline-Upwind Petrov-Galerkin method by Hughes and Brooks. Further developments and applications of this method, including time dependent problems are considered by Johnson and Nävert [36,43].

Two other approaches to 2-dimensional problems should be mentioned at least in a short survey, although an extensive discussion would be somewhat off the road. The first is the finite difference method as devised by S. Osher [38,44]. This method is based on one-sided schemes approximating a scalar conservation law

$$\frac{\partial u}{\partial t} = (f(u))_x,$$

and can be applied to non-linear problems of the form

$$\varepsilon y'' + a(y)y' + b(x,y) = f(x)$$

and its more-dimensional analogues. It is particularly suited for shock modelling with strong shocks (interior layers and hyperbolic reduced equations).

The other method is related to multigrid methods and is studied in [15,31,32]. Here unstable accurate and stable inaccurate discretizations (e.g. simple central - and artificial diffusion discretization) are combined in an iterative process, to obtain stable and accurate results. The advantage of this method over the above-mentioned Petrov-Galerkin and upwind methods is that there is no directional bias in the discretization (the discretization scheme is independent of the flow-direction $\bar{b}$). However, the method is not easy to implement and further research is needed before its value is proved in practice.

Bibliographical notes

In the last few years a number of books have appeared in which the numerical solution of singular perturbation problems was the main subject. We mention [5,20,30, 33,34,40]. The book [20] contains a bibliography of about 200 papers. A sequel to [40], the proceedings of the BAIL II conference, will appear in 1982. Research in the field is going on. A special issue of the journal Computer Methods in Applied Mechanics and Engineering, devoted to the topic "Optimal Finite Element Methods for Fluid Mechanics and Nonsymmetric Operator Problems" is in preparation. Furthermore, in 1981 a number of conferences was held at which the numerical treatment of singular perturbation problems was not the central issue, but in which several contributions were devoted to the subject. The proceedings of these conferences [9,22,27] will appear in the near future.

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