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AFDELING TOEGEPASTE WISKUNDE TW 225/82 SEPTEMBER
(DEPARTMENT OF APPLIED MATHEMATICS)

S.A. VAN GILS

ON A FORMULA FOR THE DIRECTION OF HOPF BIFURCATION

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 34C25, 34K99, 34C35, 34C45, 45D05, 45G10, 45M99

On a formula for the direction of Hopf bifurcation *)

by

S.A. van Gils

ABSTRACT

In this paper we derive a formula determining generically the direction of Hopf bifurcation for a large class of evolution equations including o.d.e.'s, retarded functional differential equations, Volterra integral equations and parabolic differential equations. The result is an algorithm which in the case of o.d.e.'s and retarded functional differential equations resolves into the inversion of two matrices.

KEY WORDS & PHRASES: *retarded functional differential equations, dynamical system, variation-of-constants formula, center manifold, Hopf bifurcation*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In recent years several methods to compute the direction of Hopf bifurcation have been developed The main motivation for all this labour is given by the "Principle of Exchange of Stability", which implies that the stability of the bifurcating solutions is determined by the direction. Chow and Mallet-Paret [3] deal with the method of averaging. Hassard and Wan [11] use the center manifold and a reduction to the Poincaré normal form. This yields explicit formulae which however do not have a very clear structure. See also [4,12,13,14,16,18]. Iooss and Joseph [15] advocate a method based on the Fredholm alternative which enormously shortens the length of the computations. Unfortunately, the use of the Fredholm alternative may be difficult, if not impossible, for equations in abstract spaces.

Here we like to point out that one can combine the center manifold theory and the economic use of the Fredholm alternative in $\mathbb{R}^n$ to arrive, without too many computations, at a bifurcation formula with a clear structure and applicable to many evolution equations. It is in fact and algorithm. In the case of o.d.e.'s in $\mathbb{R}^n$, retarded functional differential equations (r.f.d.e.'s) and Volterra integral equations two matrices have to be inverted. In [17] Stech derives this formula in the case of r.f.d.e's with infinite delay using center manifold theory and averaging techniques. The crucial quantity for the direction, c, involves derivatives of the nonlinearity N at the origin, the eigenvector p (p^*) of the infinitesimal generator A (A^*) at $i\omega_o$ ($-i\omega_o$), the point where an eigenvalue crosses the imaginary axis, the resolvent of A at 0 and $2i\omega_o$ and the projection operator P_o on $N(A-i\omega_o)$.

$$\begin{aligned} c = \tfrac{1}{2} &< P_o\, N_{xxx}(0)\ (p^2,\bar{p}),\ p^* > + \\ &< P_o\, N_{xx}(0)\ (-A^{-1}N_{xx}(0)(p,\bar{p}),p),\ p^* > + \\ \tfrac{1}{2} &< P_o\, N_{xx}(0)\ ((2i\omega_o-A)^{-1}N_{xx}(0)p^2,\bar{p}),\ p^* >. \end{aligned}$$

To deal with an equation in a Banach Space it is sometimes, like for instance in the case of Volterra integral equations [8] or r.f.d.e.'s, more convenient to work with the variation-of-constants formula rather than with the equation itself. In the above mentioned cases one needs to extend the action of the semigroups involved to a large space.

In section 2 we start from a quite general variation-of-constants formula and impose conditions which are sufficient to guarantee the existence of a smooth center manifold, which we actually construct. Our approach differs somewhat from the usual one (see [2] and the references given there). On the center manifold the flow is governed by an o.d.e. We apply the results for Hopf bifurcation in the finite dimensional case. Initially this yields formulae, depending on the approximation of the center manifold, which can be further simplified.

In section 3 we apply our result to the case of a r.f.d.e.

$$\begin{aligned} c(\text{r.f.d.e.}) = \tfrac{1}{2} &< r_{xxx}(0)\ (p^2,\bar{p}),\ p^*(0+) > + \\ &< r_{xx}(0)\ (\Delta(0,0)^{-1} r_{xx}(0)(p,\bar{p}),p),\ p^*(0+) > + \\ \tfrac{1}{2} &< r_{xx}(0)\ (e^{2i\omega_o\cdot}\Delta(0,2i\omega_o)^{-1} r_{xx}(0)p^2,\bar{p}),\ p^*(0+) >. \end{aligned}$$

See section 3 for the notation. In [7] the application to Vollterra integral equations has already been given.

NOTATION

$\mathbb{C}^n$ — n dimensional complex vector space with inner product

$$<a,b> = \sum_{i=1}^{n} a_i.\overline{b_i}$$

$NBV([o,r];\mathbb{R}^n)$ — Banach space of suitably normalized functions of bounded variation of the interval $[o,r]$ into $\mathbb{R}^n$.

$BC^\eta(\mathbb{R};Z)$ — Banach space of continuous mappings f of $\mathbb{R}$ into the Banach space Z with the norm $\|\ \|^\eta$

$\|\ \|^\eta$ — $\| F \|^\eta = \sup\{e^{-\eta|s|}\|F(s)\|_Z \mid s \in \mathbb{R}\}$

$<\ ,\ >$ — Duality pairing, spaces should be clear from the context

T — Transpose of a matrix

Ω — Open subset of $\mathbb{R}$ which contains zero

2. THE BIFURCATION FORMULA

In this section we derive the bifurcation formula starting from the variation-of-constants formula (2.1) below. First of all we make the assumptions which guarantee the existence of a finite dimensional invariant manifold on which we are allowed to differentiate the flow.

H_1: $\{T(s)\}$ forms a strongly continuous semigroup of bounded linear operators on the Banach space X with infinitesimal generator A.

H_2: A has compact resolvent for $\lambda \in \rho(A)$. Only a finite number of eigenvalues lie in a right half plane containing the imaginary axis. For any $\varepsilon > o$ there exists a $K > o$ such that

$$\|T(s)x\| \leq Ke^{(\gamma_+ - \varepsilon)s} \|x\| \text{ for } s \leq o \text{ and } x \in X_+$$

$$\|T(s)x\| \leq Ke^{\varepsilon|s|} \|x\| \text{ for } -\infty < s < \infty \text{ and } x \in X_o$$

$$\|T(s)x\| \quad Ke^{(\gamma_- + \varepsilon)s} \|x\| \text{ for } s \geq o \text{ and } x \in X_-$$

where

$$\gamma_+ = \inf\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\,\lambda > 0\}$$

$$\gamma_- = \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\,\lambda > 0\}$$

$$X_o = \bigcup_{\lambda \in \Lambda_o} N(A-\lambda I)^{k(\lambda)},\quad \Lambda_o = \{\lambda \in \sigma(A) \mid \mathrm{Re}\,\lambda = 0\},$$

$$X_+ = \bigcup_{\lambda \in \Lambda_+} N(A-\lambda I)^{k(\lambda)},\quad \Lambda_+ = \{\lambda \in \sigma(A) \mid \mathrm{Re}\,\lambda > 0\},$$

$$X_- = \bigcap_{\lambda \in \Lambda_o \cup \Lambda_+} R(A-\lambda I)^{k(\lambda)},$$

$k(\lambda)$ is the Riesz index corresponding to λ.

H_3: X is embedded into the Banach space Y to which $T(s)$ can be extended. The projection operators P_-, P_o, P_+ (projectiong X onto X_-, X_o, X_+ respectively) can be extended to Y, summing again to identity, such that

(i) P_o, P_+ map Y into X_o, X_+,

(ii) if f is a continuous mapping of the interval $[\sigma, s]$ into Y then $\int_\sigma^s T(s-\tau)f(\tau)d\tau$ is an element of X and

$P \int_\sigma^s T(s-\tau)f(\tau)d\tau = \int_\sigma^s T(s-\tau)Pf(\tau)d\tau$ when P is either P_-, P_o or P_+,

(iii) $\| P_- \int_\sigma^s T(s-\tau)f(\tau)d\tau \|_X \le K \int_\sigma^s {}^{(\gamma_-+\varepsilon)(s-\tau)} \| P_- f(\tau)\|_Y d\tau$.

H_4: L and N are C^k, $k \ge 3$, smooth mappings of $\Omega \times X$ into Y. The k-th derivative is uniformly continuous. All derivaties are globally bounded. Moreover, N and its first derivative with respect to x are bounded by some constant ν that we choose to satisfy a suitable bound later on. Finally for all $x \in X$, $\mu \in \Omega$: $L(0,x) = N(\mu,0) = 0$ and $N_x(0,0) = 0$.

REMARK 1. The global boundedness of L and N is not a serious requirement since we are interested in local results. It can be a chieved by modifying L and N outside a neighbourhood of the origin.

REMARK 2. At places where it does not lead to confusion we will write $L(0), N_x(0)$ etc. instead of $L(0,0)$, $N_x(0,0)$.

$$x(s) = T(s-\sigma)x(\sigma) + \int_\sigma^s T(s-\tau)\{L(\mu,x(\tau)) + N(\mu,x(\tau))\}d\tau, \quad -\infty < \sigma \le s < \infty. \tag{2.1}$$

The assumptions are strong enough to state

THEOREM 2.1. *The exists a unique* C^k*-function* $C = C(\mu,\phi)$ *of* $\Omega_1 \times X_o$ *into* X (Ω_1 *is a neighbourhood of zero contained in* Ω) *such that*

(i) $C(\mu,\phi) = X^*(\mu,\phi)(0)$, *where* $X^*(\mu,\phi)(s)$ *is the unique solution of* (2.1) *such that*

a) $P_o X^*(\mu,\phi)(0) = \phi$,

b) $X^*(\mu,\phi) \in BC^\eta(\mathbb{R};X)$, $\eta \in (0,\min\{-\gamma_-,\gamma_+\})$

For this function the following identities hold

(ii) $C(\mu,0) = 0$,

(iii) $C_\phi(\mu,0)\,\psi = \psi$, $\psi \in X_o$ (tangency property),

(iv) $C_{\phi\phi}(0,0)(\psi_1,\psi_2) = \int_0^\infty T(\tau)P_- N_{xx}(0,0)(T(-\tau)\psi_1, T(-\tau)\psi_2)d\tau + \int_0^{-\infty} T(\tau)P_+ N_{xx}(0,0)(T(-\tau)\psi_1, T(-\tau)\psi_2)d\tau$,

(v) $\quad C(\mu,P_o x^*(\mu,\phi)(s)) = x^*(\mu,\phi)(s)$ (invariance).

PROOF. Our intention is to apply a contraction argument. Therefore we first study the inhomogeneous linear equation. For an arbitrary element $\eta \in (0,\min\{-\gamma_-,\gamma_+\})$ the expression

$$(Kh)(s) = P_- \int_{-\infty}^{s} T(s-\tau)h(\tau)d\tau + P_o \int_{0}^{s} T(s-\tau)h(\tau)d\tau + P_+ \int_{s}^{\infty} T(s-\tau)h(\tau)d\tau$$

defines a continuous linear operator K from $BC^\eta(\mathbb{R};Y)$ into $BC^\eta(\mathbb{R};X)$. We show this on basis of the first term. From H_3 we have the estimate

$$\| P_- \int_{p}^{q} T(\tau)h(s-\tau)d\tau\| \leq K\|h\|^\eta \, e^{\eta|s|} \int_{p}^{q} e^{(\gamma_-+\varepsilon+\eta)} \, d\tau$$

which guarantees the existence of the improper integral since $\gamma_-+\varepsilon+\eta < 0$. To see the continuity in s choose $\xi \in (\eta,\min\{-\gamma_-,\gamma_+\})$. Then

$$\| P_- \int_{0}^{\infty} T(\tau)(h(s_1-\tau)-h(s_2-\tau))d\tau\| \leq$$
$$K \int_{0}^{\infty} e^{(\gamma_-+\varepsilon+\xi)\tau}(e^{-\xi\tau}\|h(s_1-\tau)-h(s_2-\tau)\|_Y)d\tau,$$

which goes to zero when $s_1 \to s_2$.
Finally, similar arguments show that Kh is indeed an element of $BC^\eta(\mathbb{R};X)$ and that

$$(2.2) \qquad \|Kh\|^\eta \leq K \left\{\frac{1}{\gamma_+-\varepsilon-\eta} + \frac{1}{\eta-\varepsilon} + \frac{1}{-\gamma_--\varepsilon-\eta}\right\}\|h\|^\eta.$$

Using the semigroup property of $T(s)$ we deduce that $x(s) = (Kh)(s)$ is a solution of

$$(2.3) \qquad x(s) = T(s-\sigma)x(\sigma) + \int_{\sigma}^{s} T(s-\tau)h(\tau)d\tau, \; \sigma \leq s,$$

in $BC^\eta(\mathbb{R};X)$ and that any other solution of (2.3) that belongs to the same space is of the form $x(s) = T(s)\phi + (Kh)(s)$ for some $\phi \in X_0$. From now on we fix η.

To apply a contraction argument we replace h by $N(\mu,x) + L(\mu,x)$. Unfortunately the substition operator $N(\mu,.)$ from $BC^{\eta}(\mathbb{R};X)$ into $BC^{\eta}(\mathbb{R};Y)$, indicated by the same symbol, and defined by $N(\mu,x)(s) = N(\mu,x(s))$, in not differentiable although N itself as a mapping between X and Y is. However, seen as a mapping of $BC^{\eta}(\mathbb{R};X)$ into $BC^{\theta}(\mathbb{R};Y)$, $\theta > k\eta$ the mapping is k times continuously differentiable. This is the content of Lemma 4.1 in appendix 1. Motivated by this remark we choose $\eta_1 = \frac{\eta-\delta}{k}$, where δ is some positive number less than η and solve

$$(2.4) \qquad x = T(.)\phi + K(N(\mu,x)+L(\mu,x)).$$

From (2.2) we infer that norm of K as an element of $L(BC^{\bar{\eta}}(\mathbb{R};X))$ is less than a constant C when $\bar{\eta}$ varies between η_1 and η. We suppose that ν, mentioned in H_4, is so small that $C\nu < 1$. This implies that for small μ the right hand side defines for each $\phi \in X_0$ a contraction in $BC^{\eta}(\mathbb{R};X)$. Because of the smoothness of N and L the mapping

$$(\mu,\phi) \to x^*(\mu,\phi) \qquad |\mu| \le \mu_0, \ \phi \in X_0$$

where $x^*(\mu,\phi)$ is the unique fixed point of the contraction in $BC^{\eta}(\mathbb{R};X)$ is C^k with respect to μ and Lipschitz continuous with respect to ϕ. Moreover $x^*(\mu,\phi)$ is also the unique solution in the smaller space $BC^{\eta_1}(\mathbb{R};X)$. This fact we exploit to prove by induction the k-times differentiability of x^* with respect to ϕ. For $1 \le \ell \le k$ let $L_{(\ell)}(\mu,.)$ and $N_{(\ell)}(\mu,.)$ be the mappings of $[BC^{\eta_1}(\mathbb{R};X)]^{\ell}$ into $BC^{\ell\eta_1}(\mathbb{R};X)$ defined by

$$N_{(\ell)}(\mu,h)g^{\ell}(s) = \frac{\partial^{\ell}}{\partial_x^{\ell}} N(\mu,h(s))\, g(s)^{\ell} \text{ (and similarly for } \mathbb{L}).$$

For fixed but small μ the right hand side of

$$y_1 = T(.) + K((L_{(1)}(\mu,x^*(\mu,\phi))+N_{(1)}(\mu,x^*(\mu,\phi)))y)$$

defines a contraction in $L(X_0;BC^{\eta_1}(\mathbb{R};X))$. The fixed point $y_1^*(\mu,\phi)$ is the derivative of $\phi \to x^*(\mu,\phi)$ as mapping of X_0 into $BC^{\eta}(\mathbb{R};X)$. This is a straightforward consequence of Lemma 4.1. Thus having constructed

$y_1^*,\dots,y_{\ell-1}^*$ we derive the equation for y_ℓ by formal ℓ times differentiation of equation (2.4)

$$y_\ell = K(z(\mu)+(L_{(1)}(\mu,x^*(\mu,\phi)) + N_{(1)}(\mu,x^*(\mu,\phi)))y_\ell),$$

where $z(\mu)$ is an element of $L(X_0^\ell;BC^{\ell\eta_1}(\mathbb{R};X))$ involving y_1 up to $y_{\ell-1}$. A similar reasoning shows that the fixed point $y_\ell^*(\mu,\phi)$ is the ℓ-the derivative of $\phi \to x^*(\mu,\phi)$ as a mapping of X_o into $BC^\eta(\mathbb{R};X)$.

The existence of x^* satisfying (i) a and b has now been proved. If we define $C(\mu,\phi) = x^*(\mu,\phi)(0)$ then (ii)-(iv) follow from Lemma 4.1 (b). Uniqueness implies (v). □

REMARK. The above shows that the k-th derivative is Lipschitz continuous with respect to ϕ.

Standard arguments, see for instance Lemma 2.3 in [1] and section 9 in [8], imply that the center manifold has the property which is known as local attractivity. More precisely

LEMMA 2.1. *There exist positive constans* $\bar{\mu}$, δ, ν, *such that for* $|\mu| \le \bar{\mu}$ *every solution* x *of* (2.1) *that is bounded by* δ *on the interval* $[0,s]$ *satisfies* $\|x(s)- C(\mu,P_ox(s))\| \le K\, e^{-\nu s}$, *where* K *is a constant which depend only on* $x(0) - P_ox(0)$.

REMARK 1. If $X_+ = \{0\}$ then solutions of (2.1) starting in a sufficient small neighbourhood of the origin are bounded by δ for all time. In this case the center manifold is attractive.

REMARK 2. All small periodic solutions lie on the center manifold. For those solutions (2.1) is reduced to an equation in a finite dimensional space, the dimension equals the dimension of X_o.

Since P_o maps X into $\mathcal{D}(A)$ we are allowed to differentiate the P_o projected equation. So we define $y = P_ox$.

(2.5) $$\frac{dy}{ds} = Ay(s) + P_o\{L(\mu,C(\mu,y(s)) + N(\mu,C(\mu,y(s)))\}.$$

At this point we apply the results for Hopf bifurcation in the finite dimensional case. These are obtained most easily using the parametrization as in [14, chapter VIII]. The following theorem is well known. For the sake of completeness we elaborate some formulae which can be found in [15] in appendix 2.

THEOREM 2.2. *Let* $f = f(\mu,u) \in C^k(\Omega x \mathbb{R}^n;\mathbb{R}^n)$, $k \geq 3$, satisfy

(i) $f(\mu,u) = 0$

(ii) $i\omega_o$ *is a simple eigenvalue of* $f_u(0)(=f_u(0,0))$ *with eigenvector* ζ_o *and* $\pm\, i\omega_o \ell \notin \sigma(f_u(0))$ *for* $\ell = 0,2,3,\ldots..$

(iii) $\mathrm{Re} < f_{u\mu}(0)\zeta_o,\zeta_o^* > \neq 0$,

where ζ_o^* *is the eigenvector of* $f_u^*(0)$ *at* $-i\omega_o$, *normalized such that* $< \zeta_o,\zeta_o^* > = 1$. *Then there exist* C^{k-1} *-functions* $\omega(\varepsilon),\mu(\varepsilon)$ *and* $u(\varepsilon)$ *with values in* $\mathbb{R}$, $\mathbb{R}$, $C^1_{2\pi}(\mathbb{R};\mathbb{R}^n)$ *respectively and defined for* ε *sufficiently small, such that* $\omega(0) = \omega_o$, $\mu(0) = 0$, $u(0) = 0$ *and satisfying.*

$$\omega(\varepsilon)\,\frac{du(\varepsilon)}{ds} = f(\mu(\varepsilon),u(\varepsilon)). \tag{2.6}$$

In addition, ω *and* μ *are odd functions of* ε*;*

$$\frac{d^2\mu(0)}{d\varepsilon^2} = \frac{-2\mathrm{Re}c_1}{\mathrm{Re}<f_{u\mu}(0)\zeta_o,\zeta_o^*>},$$

$$\frac{d^2\omega(0)}{d\varepsilon^2} = 2\ \mathrm{Im}c_1 - \frac{2\mathrm{Re}c_1.\mathrm{Im}<f_{u\mu}(0)\zeta_o,\zeta_o^*>}{\mathrm{Re}<f_{u\mu}(0)\zeta_o,\zeta_o^*>},$$

$$\frac{du(0)}{d\varepsilon}(s) = \mathrm{Re}(e^{is}\zeta_o),$$

$$\begin{aligned} c_1 = \tfrac{1}{2} &< f_{uuu}(0)(\zeta_o^2,\bar\zeta_o),\zeta_o^* > + \\ &< f_{uu}(0)(-f_u(0)^{-1}f_{uu}(0)(\zeta_o,\bar\zeta_o),\zeta_o),\zeta_o^* > + \\ \tfrac{1}{2} &< f_{uu}(0)((2i\omega_o-f_u(0))^{-1}\zeta_o^2,\bar\zeta_o),\zeta_o^* >. \end{aligned} \tag{2.7}$$

In a small neighbourhood of the origin the solutions are the unique period-

ic solutions modulo translation.

Combination of the reduction to (2.5) and theorem 2.2 yields the following theorems.

THEOREM 2.3. *Let* $H_1 - H_4$ *be satisfied. Assume furthermore that:*

H_5: $i\omega_o$ *is a simple eigenvalue of* A *with eigenvector* p_o *and*

$$\pm\, i\omega_o \ell \notin \sigma(A) \text{ for } \ell = 0,2,3,\ldots,$$

H_6: $$\operatorname{Re} < P_o L_{\mu x}(0)p,p^* > \neq 0,$$

where p^* *is the eigenvector of* A^* *at* $-i\omega_o$, *normalized such that* $<p,p^*> = 1$. *Then there exist* C^{k-1}*-functions* $\omega(\varepsilon)$, $\mu(\varepsilon)$ *and* $\phi(\varepsilon)$ *with values in* $\mathbb{R},\mathbb{R}$ *and* X_o *respectively, defined for* ε *small enough, such that* $\omega(0) = \omega_o$, $\mu(0) = 0$, $\phi(0) = 0$ *and such that* $x^*(\mu(\varepsilon),\phi(\varepsilon))$ *is a* $2\pi\omega(\varepsilon)^{-1}$*-periodic solution of equation* (2.1). *Moreover if* x *is a small periodic solution of this equation with* μ *close to zero and period close to* $2\pi\omega_o^{-1}$ *then* $\mu = \mu(\bar{\varepsilon})$ *for some* $\bar{\varepsilon}$, *the period is* $2\pi\omega(\bar{\varepsilon})^{-1}$ *and modulo translation* $x^*(\mu(\bar{\varepsilon}),\phi(\bar{\varepsilon}))$. *The functions* ω *and* μ *are odd.*

OUR MAIN RESULT IS STATED IN

THEOREM 2.4. *Under the hypotheses of the previous theorem* μ *satisfies*

$$\mu_2 = \frac{1}{2}\frac{d^2\mu(0)}{d\varepsilon^2} = \frac{-\operatorname{Re} c}{\operatorname{Re}<P_o L_{\mu x}(0)p,p^*>}$$

$$c = \frac{1}{2} < P_o N_{xxx}(0)(p^2,\bar{p}),p^* > +$$

(2.8) $$< P_o N_{xx}(0)(-A^{-1}N_{xx}(0)(p,\bar{p}),p),p^* > +$$

$$\frac{1}{2} < P_o N_{xx}(0)((2i\omega_o - A)^{-1}N_{xx}(0)p^2,\bar{p}),p^* >.$$

As a by-product we also obtain the leading terms in the expansion of

$\omega-\omega_o$ *and* x^* *as functions of* ε.

THEOREM 2.5. *Under the hypothesis of the Theorem* 2.3 *we have*

$$\omega_2 = \frac{1}{2}\frac{d^2\omega(0)}{d\varepsilon^2} = \mathrm{Im}c - \frac{\mathrm{Re}c\ \mathrm{Im}<P_0L_{\mu x}(0)p,p^*>}{\mathrm{Re}<P_oL_{\mu x}(0)p,p^*>},$$

$$x_1 = \frac{dx^*(0)}{d\varepsilon}(s) = \mathrm{Re}(e^{is}p).$$

PROOF OF THEOREM 2.3-2.5. We relate properties of f in (2.5) to properties of A,L and N. Rather than writing (2.5) in its coordinates with respect to a basis in X_o we apply Theorem 2.2 directly to (2.5). Then all the assertions are clear except for (2.8). Application of (2.7) using Theorem 2.1 (ii)-(iv) and the identity $T(\tau)p = e^{i\omega_o\tau}p$ yields

$$\begin{aligned}
c = {} & \frac{1}{2} < P_oN_{xxx}(0)(p^2,\bar{p}),p^* > + \\
& < P_oN_{xx}(0)\Big(\int_0^\infty T(\tau)P_-N_{xx}(0)(p,\bar{p}) + \int_0^{-\infty} T(\tau)P_+N_{xx}(0)(p,\bar{p}),p\Big),p^* > + \\
& \frac{1}{2} < P_oN_{xx}(0)\Big(\int_0^\infty e^{-2i\omega_o\tau}P_-N_{xx}(0)p^2 + \int_0^{-\infty} e^{-2i\omega_o\tau}P_+N_{xx}(0)p^2,\bar{p}\Big),p^* > + \\
& < P_oN_{xx}(0)(-A^{-1}P_oN_{xx}(0)(p,\bar{p}),p),p^* > + \\
& \frac{1}{2} < P_oN_{xx}(0)((2i\omega_o-A)^{-1}P_oN_{xx}(0)p^2,\bar{p}),p^* >.
\end{aligned}$$

The simplification of this identity to the one in (2.8) follows from the observation that for λ on the imaginary axis the action of $(A-\lambda I)^{-1}$ on elements of Y is given by

$$(2.9)\qquad (A-\lambda I)^{-1}y = \int_0^\infty e^{-\lambda\tau}T(\tau)P_-y\,d\tau + (A-\lambda I)^{-1}P_oy + \int_0^{-\infty} e^{-\lambda\tau}T(\tau)P_+y\,d\tau.$$

Compare [9, VIII 1.11, Theorem 11]. □

REMARK 1. The expression for c does not change if we replace P_o by the projection of Y onto $N(A-i\omega_o)$.

REMARK 2. Whether one deals with o.d.e'.s $\mathbb{R}^n$, r.f.d.e'.s, analytical semigroups or parabolic differential equations, the formula that determines the direction of bifurcation is always the same. In all cases one has to know the resolvent at 0 and $2i\omega_o$. In the case of o.d.e'.s in $\mathbb{R}^n$ or r.f.d.e'.s this evolves into the inversion of two matrices. For parabolic equations one may have to invert infinitely many matrices!

REMARK 3. If Re $< P_o L_{\mu x}(0)p,p^* >$ is positive (which means that a branch of eigenvalues of the linearized perturbed equation crosses the imaginary axis at $i\omega_o$ with nonzero speed from the left to the right half plane) and c is negative then the periodic solutions exist for μ positive. These are stable within the center manifold, which itself is local attractive (Lemma 2.1). Hence also in a full neighbourhood of the origin the solutions are attractive provided $X_+ = \{0\}$, see for instance [12, pag 274-276].

3. AUTONOMOUS RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section we mainly follow the notation as used in [10] but $\zeta(\theta)$ instead of $-\eta(-\theta)$. Let C be the space of continuous functions defined on $[-r,o]$ with range in $\mathbb{R}^n$. We assume that $\mu \to \zeta(\mu,\theta)$, $(\mu,\phi) \to r(\mu,\phi)$ $\mu \in \Omega$, $\theta \in [-r,o]$, $\phi \in C$ are C^k-smooth mappings of Ω into $NBV([-r,o];\mathbb{R}^{nxn})$, $\Omega \times C$ into $\mathbb{R}^n$ respectively. The k-th derivative of r is uniformly continuous. Furthermore $r(\mu,0) = 0$, $r_\phi(\mu,0) = 0$. Sometimes we write ζ instead of $\zeta(0,.)$. $\{T(s)\}$ is the strongly continuous semigroup of bounded operators on C defined by $T(s)\phi(\theta) = y_s(\theta) = y(s+\theta)$, $\theta \in [-r,o]$ where

$$(3.1)\qquad \begin{cases} \dot{y}(t) = \int_0^r d\zeta(0,\theta)y(t-\theta) \quad t \geq 0 \\ y(t) = \phi(t) \quad -r \leq t \leq 0. \end{cases}$$

If x is the solution of

$$(3.2)\qquad \begin{cases} \dot{x}(t) = \int_0^r d\zeta(\mu,\theta)x(t-\theta) + r(\mu,x_t) \\ x_\sigma(\theta) = \phi(\theta) \end{cases}$$

then

$$x_s = T(s-\sigma)x_\sigma + \int_\sigma^{s_-} T(s-\tau)X_o \left\{ \int_0^r (d\zeta(\mu,\theta)-d\zeta(0,\theta))x(\tau-\theta) + \right. \tag{3.3}$$

$$\left. + r(\mu,x_\tau) \right\} d\tau, \ -\infty < \sigma \le s < \infty.$$

Here $X_o(\theta) = \begin{cases} \mathrm{Id} & \theta = 0 \\ 0 & -r \le \theta < 0 \end{cases}$, see [10, section 6.2].

Although X_o does not belong to C the action of T(s) is well defined on X_o by $T(s)\ X_o(\theta) = U(s+\theta)$ where

$$\begin{cases} \dfrac{dU}{dt} = \displaystyle\int_0^t d\zeta(0,\theta)U(t-\theta) \quad t \ge 0 \\ U(\theta) = \begin{cases} I, & \theta = 0 \\ 0, & -r \le \theta < 0. \end{cases} \end{cases}$$

$T(s)X_o$ is continuous except at $\theta + s = 0$. If f is a continuous mapping of $[\sigma,s]$ into $\mathbb{R}^n$ then $\int_\sigma^s T(s-\tau)X_o f(\tau)d\tau \in C$. To bring (3.3) in the right form we define $N(\mu,\phi) = X_o r(\mu,\phi)$, $L(\mu,\phi) = X_o \int_0^r (d\zeta(\mu,\theta)-d\zeta(0,\theta))\phi(-\theta)$, with domain $\Omega \times C$ and range in $Y = L_\infty(-r,0) \times \mathbb{R}^n$. T(s) can be extended to a semigroup on Y [5]. Here we deviate a little bit from the general theory in section 2, since C is not a subpase of Y. However, obviously $\phi \to \phi|_{(-r,o)} \times \phi(0)$ maps C into a subspace of Y. As a matter of fact all we need is the extension of $T(s_1)$ to $T(s_2)X_o(s_1,s_2,\ge 0)$, which is given by $T(s_1)T(s_2)X_o(\theta) = U(s_1+s_2+\theta)$, $-r \le \theta \le 0$. The infinitesimal generator of $\{T(s)\}$ is described by

$$A\ \phi = \dot{\phi}$$

$$\mathcal{D}(A) = \{\phi \in C^1 \mid \dot{\phi}\ (0) = \int_0^r d\zeta(0,\theta)\phi(-\theta)\}.$$

A has compact resolvent, $\sigma(A) = \{\lambda \mid \det \Delta(0;\lambda) = 0\}$, where

$$\Delta(\mu,\lambda) = \lambda I - \int_0^r e^{-\lambda\theta} d\zeta(\mu,\theta).$$

For the construction of the projection operators one can make use of the formal adjoint and the bilinear form as in [9]. We advocate an alternative approach using the true adjoint as is briefly pointed out in [6]. It appears that if one chooses as a realization of C^* the space $\widetilde{NBV}$, i.e. the space of bounded variation functions in $\mathbb{R}_+$ such that (i) $f(0) = 0$; (ii) f is constant for $t \geq r$; (iii) f is continuous from the right on $(0,r)$ with the pairing given by

$$< \phi, f > = \int_0^r \phi(-t)\,df(t)$$

then the action of $T^*(s)$ on a forcing function $f \in \widetilde{NBV}$ is given by

$$T^*(s)f(t) = \begin{cases} 0\,, \; t = 0 \\ x_s(t) - \zeta^T * x_s(t), t > 0, \end{cases}$$

where x is the solution of the Volterra convolution equation

$$x = \zeta^T * x + f.$$

$$\mathcal{D}(A^*) = \{\psi \in \widetilde{NBV} \mid \psi(t) = \psi(0+) + \int_0^t \psi'(\tau)d\tau,\; t > 0,\; \psi' \in \widetilde{NBV}\}$$

$$A^*\psi\;(t) = \psi'(t) + \zeta^T(t)\psi(0+),\; t > 0.$$

Using these fact the construction of projection operators corresponding to the decomposition of C is standard.

On our way to a Hopf bifurcation we state two lemmas relating properties of A. The proofs are elementary and are therefore omitted.

LEMMA 3.1. *The following two assertions are equivalent:*

(i) *there exist* $p(0), p^*(0+) \in \mathbb{C}^n$ *such that*

(α) $\Delta(0, i\omega_o)\; v = 0$ *implies* $v = cp(0)$ *for some* $c \in \mathbb{C}$,

(β) $\Delta^T(0, -i\omega_o)\; v = 0$ *implies* $v = cp^*(0+)$ *for some* $c \in \mathbb{C}$,

(γ) $< \frac{\partial}{\partial\mu} \Delta\;(0, i\omega_o)p(0), p^*(0+) > = 1.$

moreover $\Delta(0, \pm i\ell\omega_o)$ *is nonsingular for* $\ell = 0,2,3,\ldots.$

(ii) $i\omega_o$ *is a simple eigenvalue of* A, *with eigenvector* $p(\theta) = p(0)e^{i\omega_o\theta}$, $-r \leq \theta \leq 0$. *The eigenvector* p^* *of* A^* *at* $-i\omega_o$ *satisfy-*

ing $< p,p^* > = 1$ *is such that*
$\frac{d}{dt} p^*(t) = e^{-i\omega_0 t} \int_t^r e^{i\omega_0 \tau} d\zeta^T(\tau)p^*(0+)$. *For* $\ell = 0,2,3,\ldots$: $\pm i\omega_0 \ell \notin \sigma(A)$.

REMARK. The projection operator on $N(A-i\omega_0)$,P, is explicitly given by $P\phi = <\phi,p^*> p$. To satisfy H_3 we are led to define $PX_0 = \overline{p^*(0+)p}$, which is a shorthand for the n×n matrix $(p^*(0+)^1 p,\ldots,p^*(0+)^n p)$. Hence we have

LEMMA 3.2. $< \frac{\partial}{\partial\mu} PL(0)p,p^* > = < \frac{\partial}{\partial\mu} \Delta(0,i\omega_0)p(0),p^*(0+) >$.

The remainder of this section we devote to the verification of the hypotheses $H_1 - H_6$. Once this is done the obvious theorems corresponding to Theorem 2.3-2.5 hold. We do not state them explicitly but we state the eleboration of (2.8).

We already noticed that H_1 is fulfilled. C can be decomposed according to the spectrum of A. T(s) is compact for s ≥ r. Hence H_2 is fulfilled [7, section 7.4]. With respect to H_3 we remark that $T(s)P\_X_0$ is an element of P_C for s ≥ r and bounded on [0,r]. This together with H_2 shows that H_3 (i)-(iii) hold. In order to satisfy H_4 we modify L and N outside some neighbourhood of zero in Y. (Because of the linearity of L in y we are even allowed to omit the modification of L). So let $\xi;\mathbb{R}_+ \to \mathbb{R}$ be a C^∞ function such that (i) $\xi(y) = 1$ for $0 \le y \le 1$; (ii) $0 \le \xi(y) \le 1$ for $1 \le y \le 2$; (iii) $\xi(y) = 0$ for $y \ge 2$. Define for positive δ $\hat{N}_\delta(\mu,\phi) = X_0 r(\mu,\phi)\ \xi\ (\frac{\|r(\mu,\phi)\|}{\delta})$. For δ small enough H_4 is satisfied. The small periodic solutions that we are interested in are not affected by this midification. We suppress the symbols "$\wedge$, $_\delta$". $H_5 - H_6$ are satisfied an account of Lemma 3.1-3.2 under the

Assumptions: (i) $\Delta(0,i\omega_0)$ has zero as a simple eigenvalue and $\Delta(0,\pm i\ell\omega_0)$ is nonsingular for $\ell = 0,2,3,\ldots$,

(3.4)

(ii) $\mathrm{Re} < \frac{\partial}{\partial\mu} \Delta(0,i\omega_0)p(0),p^*(0+) > \neq 0$.

Finally we prove

THEOREM 3.1.

$$c(r.f.d.e) = \frac{1}{2} < r_{xxx}(0)(p^2,\bar{p}),p^*(0+) > +$$

$$< r_{xx}(0)(\Delta(0,0)^{-1}r_{xx}(0)(p,\bar{p}),p),p^*(0+) > +$$

$$\frac{1}{2} < r_{xx}(0)(e^{2i\omega_o\cdot}\Delta(0,2i\omega_o)^{-1}r_{xx}(0)p^2,\bar{p}),p^*(0+) >.$$

PROOF. This is an elaboration of (2.8). The identity

$$\int_0^\infty e^{-\lambda\tau}T(\tau)X_o d\tau = e^{-\lambda\cdot}\Delta(\lambda)^{-1}$$

which hold for Re λ sufficiently large (confert [10, chapter 1]) implies hat for Re λ sufficiently large

$$\int_0^\infty e^{-\lambda\tau}T(\tau)P_-X_o d\tau = e^{-\lambda\cdot}\Delta(\lambda)^{-1} - (A-\lambda I)^{-1}(P_o+P_+)X_o.$$

Because the left hand side is analytic for Re $\lambda \geq 0$ the identity also holds for these values of λ. Using

$$\int_0^{-\infty} e^{-\lambda\tau}T(\tau)P_+X_o d\tau = (A-\lambda I)^{-1}P_+X_o \text{Re } \lambda \leq 0$$

and (2.9) we conclude that

$$(A-\lambda I)^{-1}X_o = e^{-\lambda\cdot}\Delta(\lambda)^{-1}.$$

Finally we use the remark below Lemma 3.1. □

REMARK. While finishing this paper the authors attention was drawn to a preprint of Stech containing formula (3.5). [17, formula (2.10)].

Of course one is able to apply this result to a special kind of nonlinearity. If we consider

$$(3.6) \qquad \dot{x}(t) = \int_0^r d\zeta(\mu,\theta)g(\mu,x(t-\theta))$$

$$g(\mu,x) = x + r(\mu,x), r(\mu,x) = o(|x|),$$

then the following formula is an elaboration of (3.5)(confert Theorem 11.5 in [8]).

THEOREM 3.2. $c(3.6) = \frac{1}{2} < r_{xxx}(0)(p(0)^2,\overline{p(0)}),p^*(0+) > +$

$$< r_{xx}(0)((\Delta(0,0)^{-1}-I)r_{xx}(0)(p(0),\overline{p(0)}),p(0)),p^*(0+) > +$$

$$\frac{1}{2} < r_{xx}(0)((\Delta(0,2i\omega_o)^{-1}-I)r_{xx}(0)p(0)^2,\overline{p(0)}),p^*(0+) >.$$

Acknowledgement

The author likes to thank O. Diekmann for his careful reading of the manuscript which led to many improvements.

4. APPENDIX 1

LEMMA 4.1. Let f be a mapping between the Banach spaces X and Y such that

(i) f is C^k-smooth, $k \geq 0$,

(ii) all derivatives are bounded by a constant M,

(iii) the k-th derivative is uniformly continuous.

Let η_1 and η_2 be positive constants such that $k\eta_1 < \eta_2$. For $h \in BC^{\eta_1}(\mathbb{R};X)$ define $\tilde{f}(h)$ by $\tilde{f}(h)(s) = f(h(s))$. Then

(a) $\tilde{f} : BC^{\eta_1}(\mathbb{R};X) \to BC^{\eta_2}(\mathbb{R}:Y)$ is C^k-smooth,

(b) the derivatives of $\tilde{f}$, $D_\ell\tilde{f}$ satisfy:

$$D_\ell\tilde{f}(h)g^\ell(s) = \frac{d^\ell f}{dx^\ell}(h(s))\, g(s)^\ell,\ 0 \leq \ell \leq k,$$

(c) all derivatives are bounded by M

(d) the j-th derivative is uniformly continuous

PROOF. Let $\delta = \eta_2 - k\eta_1$ and $D_\ell\tilde{f}$ be defined as in (b). We show that this is indeed the ℓ-th derivative of $\tilde{f}$.

$$\sup_{\|g\|^{\eta_1}=1} \frac{1}{\varepsilon^{\ell}} \|\tilde f(h+\varepsilon g) - \sum_{m=0}^{\ell} \frac{1}{m!} D_m \tilde f(h)(\varepsilon g)^m\|^{\eta_2} =$$

$$\sup_{\|g\|^{\eta_1}=1} \sup_{s\in\mathbb{R}} e^{-\eta_2|s|} \frac{1}{\varepsilon^{\ell}} \|f(h(s)+\varepsilon g(s)) - \sum_{m=0}^{\ell} \frac{1}{m!} \frac{d^m f}{dx^m} (h(s)(\varepsilon g(s))^m\| \le$$

$$\sup_{\|g\|^1=1} \sup_{s\in\mathbb{R}} \frac{e^{-\delta|s|}}{\ell!} \| \frac{d^{\ell} f}{dx^{\ell}} (h(s)+\tau\varepsilon g(s)) - \frac{d^{\ell} f}{dx^{\ell}} (h(s))\|.$$

We show that the last term goes to zero when $\varepsilon > 0$. Choose $\xi > 0$. Let $A = A(\eta)$ be a number such that $\frac{2M}{\ell!} e^{-\delta A} < \xi$. On the interval $[-A,A]$ is $\|g(s)\|$ bounded by $e^{\eta A}$. Therefore the uniformly continuity of f implies the existence of $\varepsilon = \varepsilon(\xi)$ such that

$$\sup_{\|g\|^{\eta_1}=1} \sup_{s\in[-A,A]} \sup_{\tau\in[0,1]} \frac{e^{-\delta|s|}}{\ell!} \|\frac{d^{\ell} f}{dx^{\ell}}(h(s)+\tau\varepsilon g(s)) - \frac{d^{\ell} f}{dx^{\ell}}(h(s))\| \le \xi$$

for all $0 \le \varepsilon \le \varepsilon(\xi)$. This together with the choise of A shows that the last inequality is also valid when we replace $s \in [-A,A]$ by $s \in \mathbb{R}$. Hence $\tilde f$ has continuous derivatives up to order k, which are bounded by the same constant M. Along the same lines one can prove that the k-th derivative is uniformly continuous. □

5. APPENDIX 2

PROOF OF THEOREM 2.2 [15]. The existence of functions f,μ,ω satisfying (2.6) has been proved at many places, see for instance [5]. Here we only derive the Taylor series up to and including order three using the parametrization defined below.

$$J_o: C^1_{2\pi}(\mathbb{R};\mathbb{R}^n) \to C_{2\pi}(\mathbb{R};\mathbb{R}^n)$$

$$J_o u = -\omega_o \frac{du}{ds} + f_u(0)u.$$

Then

$$N(J_o) = \{cz+\overline{cz} \mid c \in \mathbb{C}, z = e^{is}\zeta_o\},$$

$$\mathcal{R}(\mathbb{J}_o) = \{x \in C_{2\pi}(\mathbb{R};\mathbb{R}^n) \mid [x,z^*] = \frac{1}{2\pi}\int_0^{2\pi} \langle x(s), e^{is}\zeta_o^*\rangle\, ds = 0\}.$$

In fact this is an equivalent formulation of the Fredholm alternative. The parametrization is choosen such that

$$(5.1) \qquad \varepsilon = [u,z^*]$$

To satisfy equation (2.6) it is necessary that:

$$(5.2) \qquad \mathbb{J}_o u_1 = 0$$

$$(5.3) \qquad \mathbb{J}_0 u_2 - \omega_1 \frac{du_1}{ds} + \mu_1 f_{u\mu}(0)u_1 + \frac{1}{2} f_{uu}(0)u_1^2 = 0.$$

Therefore, using (5.1) and the Fredholm alternative, we infer that

$$(5.4) \qquad u_1 = z + \bar{z},$$

$$-i\omega_1 + \mu_1 \langle f_{u\mu}(0)\zeta_o, \zeta_o^*\rangle = 0.$$

The transversality condition results in

$$(5.5) \qquad \mu_1 = \omega_1 = 0.$$

$$(5.6) \qquad u_2 = \frac{1}{2}(2i\omega_o - f_u(0))^{-1} f_{uu}(0)(\zeta_o)^2 e^{2is} + (-f_u(0))^{-1} f_{uu}(0)(\zeta_o,\bar{\zeta}_o) + \frac{1}{2}(-2i\omega_o - f_u(0))^{-1} f_{uu}(0)(\bar{\zeta}_o)^2 e^{-2is}.$$

In the third order we get:

$$(5.7) \qquad \mathbb{J}_o u_3 = \omega_2 \frac{du_1}{ds} - \mu_2 f_{u\mu}(0)u_1 - f_{uu}(0)(u_1,u_2) - \frac{1}{6} f_{uuu}(0)u_1^3.$$

The inproduct of the right hand side with z^* must vanish. Therefore

(5.8) $$\mu_2 = -\frac{\mathrm{Re}\,c_1}{\mathrm{Re}\langle f_{u\mu}(0)\zeta_o,\zeta_o^*\rangle},$$

$$\omega_2 = \mathrm{Im}\,c_1 - \frac{\mathrm{Re}\,c_1}{\mathrm{Re}\langle f_{u\mu}(0)\zeta_o,\zeta_o^*\rangle}\,\mathrm{Im}\langle f_{u\mu}(0)\zeta_o,\zeta_o^*\rangle.$$

(5.9) $$c_1 = \frac{1}{2}\langle f_{uuu}(0)(\zeta_o,\bar{\zeta}_o),\zeta_o^*\rangle +$$

$$\langle f_{uu}(0)(-f_u(0)^{-1}f_{uu}(0)(\zeta_o,\bar{\zeta}_o),\zeta_o),\zeta_o^*\rangle +$$

$$\frac{1}{2}\langle f_{uu}(0)((2i\omega_o-f_u(0))^{-1}f_{uu}(0)\zeta_o^2,\bar{\zeta}_o),\zeta_o^*\rangle. \qquad \square$$

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