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THE LAGRANGE MULTIPLIER RULE ON MANIFOLDS AND OPTIMAL CONTROL OF NONLINEAR SYSTEMS

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ABSTRACT

In this paper we present a differential geometric approach to the Lagrange problem and the fixed end points, fixed time optimal control problem for nonlinear time-invariant control systems. We restrict attention to first order conditions for optimality. Our treatment of the optimal control problem uses a recently proposed fibre bundle approach for the definition of nonlinear systems.

KEY WORDS & PHRASES: *Nonlinear system theory, Optimal control problems on manifolds, First order conditions, Lagrange multiplier rule*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we present a differential geometric approach to the Lagrange problem and the fixed end points, fixed time optimal control problem for nonlinear time-invariant control systems. Herein we restrict attention to first order conditions; i.e. to the problem of finding stationary curves rather than optimal curves. The approach is based on a generalization of the Lagrange multiplier rule. This generalization is, in a crude form, given in unpublished course notes by TAKENS [1978]. In fact, the results about the variational and Lagrange problem given in Sections 2 up to 4 are not new. Basic references to these results are CARTAN [1922], CARATHEODORY [1935], HERMANN [1962,1977]. However, the presentation differs and we worked out some questions about the relation between formal stationarity (roughly speaking stationarity w.r.t. variations satisfying the restrictions only to first order) and stationarity (propositions 4.2 and 5.3). We like to emphasize that our treatment of the Lagrange and the optimal control problem is essentially global and no assumptions are made about regularity of the cost function to obtain the characterization of optimal trajectories. Locally, the results are shown to imply the well-known Lagrange equation and Pontryagin's maximum principle.

In our setup we use the definition of control systems as first proposed by BROCKETT [1977] and WILLEMS [1979] and worked out by NIJMEIJER and VAN DER SCHAFT [1982].

The differential geometric notations follows closely that of SPIVAK [1979, I & II]. For instance, if M is a smooth manifold, TM is its tangent bundle (T_xM is the tangent space at $x \in M$) and T^*M is the cotangent bundle. If $f\colon M \to N$ is a smooth mapping between smooth manifolds M and N then $f_*\colon TM \to TN$ is its lift to the tangent bundles and for any k-form ω on N, $f^*\omega$ is a k-form on M which is defined by $(f^*\omega)(v) = \omega(f_*v)$ for all $v \in TM$. Some minor deviations from Spivak's notation occur. The set of smooth vector fields on a smooth manifold is denoted by $\mathcal{X}(M)$. Furthermore, given a k-form ω and a vectorfield X on M, we define the *contraction* $\iota_X\omega$ of ω with respect to X, to be the (k-1)-form on M defined by

$$\iota_X\omega(X_1,\ldots,X_{k-1}) = \omega(X,X_1,\ldots,X_{k-1})$$

for

$$X_i \in \mathcal{X}(M) \qquad (i = 1,\dots,k-1).$$

Unless states otherwise all manifolds, mappings, forms and vector fields are assumed to be smooth, i.e. C^∞.

2. THE UNRESTRICTED VARIATIONAL PROBLEM

Let M be a manifold with dim M = m and α a 1-form on M. Let I denote some closed interval, [a,b] say, in $\mathbb{R}$. Then, for C^∞ curves $\phi: I \to M$ we can define the *action* of α along ϕ by

$$(2.1) \qquad \mathcal{J}(\phi) = \int_\phi \alpha = \int_I \phi^*\alpha.$$

(In the first integral the integration path is Im ϕ.) The *variational problem* on M with respect to α is to find curves which are locally optimal, i.e. which produce an optimal value for the action relative to small variations of the curves. We shall restrict ourselves to *first order necessary conditions*, hence to *stationarity* rather than optimality of the action. The following definition is standard in the calculus of variations.

DEFINITION 2.1. A mapping $\tilde{\phi}: (-\delta,\delta) \times I \to M$ (for some $\delta > 0$) is called a *variation keeping end point fixed* (k.e.p.f.) of $\phi: I \to M$ if:

(i) $\tilde{\phi}$ is C^∞ in each variable;

(ii) $\tilde{\phi}(0,t) = \phi(t)$ for all $t \in I$;

(iii) $\tilde{\phi}(\varepsilon,a) = \phi(a)$, $\tilde{\phi}(\varepsilon,b) = \phi(b)$ for all $\varepsilon \in (-\delta,\delta)$.

The set of variations k.e.p.f. of ϕ is denoted by $\mathcal{V}_\phi$ and for short we write $\phi_\varepsilon(t) = \tilde{\phi}(\varepsilon,t)$.

Stationary curves for the action are curves which make the first variation of the action vanish. The following definition makes this precise.

DEFINITION 2.2. A curve $\phi: I \to M$ is *stationary* with respect to α, if for all $\phi_\varepsilon \in \mathcal{V}_\phi$ we have

$$(2.2) \qquad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0.$$

From now on we shall assume that the curves we consider are injective immersions. This is a rather natural assumption as curves with double points are usually not optimal, because of occurrence of a loop. In such cases we can formulate the variational problem for piecewise injective curves as a sum of variational problems for each piece (see also SPIVAK [1979, II ch. 6.14]).

We can give another, equivalent, definition of stationarity in terms of vector fields along ϕ. By a *vector field along* a curve ϕ: I $\to$ M we mean a smooth function V: I $\to$ TM which satisfies $V(t) \in T_{\phi(t)}M$. Clearly, each variation $\tilde{\phi} \in V_\phi$ defines a vector field V along ϕ by the formula

$$(2.3) \qquad V(t) = \tilde{\phi}_*(0,t)(\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}), \qquad t \in I,$$

with $V(a) = V(b) = 0$.

Conversely, given any vector field V along ϕ, with $V(a) = V(b) = 0$, we can extend it (as ϕ is an injective immersion) to a vector field $X \in \mathcal{X}(M)$ and construct a variation k.e.p.f. of ϕ by

$$(2.4) \qquad \phi_\varepsilon(t) = \gamma_X(\varepsilon)(\phi(t)),$$

where $\gamma_X(\varepsilon)$ denotes the flow of X over ε. Let now ω be an arbitrary 1-form on M and let $L_X\omega$ denote its *Lie-derivative* w.r.t. X. Then

$$(2.5) \qquad \begin{aligned} \phi^* L_X\omega &= \phi^*(\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}[(\gamma_X(\varepsilon))^*\omega - (\gamma_X(0))^*\omega]) \\ &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[(\gamma_X(\varepsilon)\circ\phi)^*\omega] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\phi_\varepsilon^*\omega). \end{aligned}$$

We also have the well-known relation:

$$(2.6) \qquad L_X\omega = \iota_X d\omega + d\iota_X\omega.$$

Given V along ϕ, we have for an arbitrary smooth extension X of V:

$$\phi^* L_X\omega(\frac{\partial}{\partial t}) = d\omega(V(t),\phi_*(\frac{\partial}{\partial t})) + d(\omega(V(t)))(\frac{\partial}{\partial t}).$$

(With $\frac{\partial}{\partial t}$ we mean the tangent vector evaluated at t.) This shows that $\phi^* L_X\omega$ depends on V only, so that we shall write $\phi^* L_V\omega$ to be the 1-form on I defined by

(2.7) $$\phi^* L_V\omega(\tfrac{\partial}{\partial t}) = d\omega(V(t),\phi_* \tfrac{\partial}{\partial t}) + d(\omega(V(t)))(\tfrac{\partial}{\partial t}).$$

Then, for all extensions X of V and induced variations cf. (2.4) we have the equality

(2.8) $$\phi^* L_V\omega = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_\varepsilon^*\omega.$$

So any vector field V along ϕ with $V(a) = V(b) = 0$ defines a class of variations k.e.p.f. ϕ_ε of ϕ satisfying (2.8).

These relations between vector fields along ϕ which vanish at the end points and variations k.e.p.f. of ϕ show that we can equivalently define stationarity by:

DEFINITION 2.2' ϕ is *stationary* with respect to α on M, if for all vector fields V along ϕ with $V(a) = V(b) = 0$ we have

(2.9) $$\int_I \phi^* L_V\alpha = 0.$$

This definition easily leads to a useful and well-known characterization of stationary curves.

PROPOSITION 2.3. *ϕ is stationary with respect to α, if and only if for all* $t \in I$:

(2.10) $$\phi_*(\tfrac{\partial}{\partial t}) \in \ker d\alpha,$$

where $\ker d\alpha = \{v \in TM \mid d\alpha(v,w) = 0,\ \forall w \in T_{\pi(v)}M\}$ *and* $\pi\colon TM \to M$ *the natural projection.*

PROOF. For any vector field V along ϕ with $V(a) = V(b) = 0$ we have, using Stokes theorem and (2.7):

$$(2.11)\qquad \int_I \phi^* L_V\alpha = \int_I \phi^* \iota_V d\alpha,$$

where $\phi^*\iota_V d\alpha$ has to be interpreted in the obvious way: $\phi^*\iota_V d\alpha(\frac{\partial}{\partial t}) = d\alpha(V(t), \phi_*(\frac{\partial}{\partial t}))$. The proof of the proposition follows immediately from Definition 2.2 and equality (2.10). □

From this proposition we conclude that $d\alpha$ is an *integral invariant* for stationary curves w.r.t. α (cf. CARTAN [1922]). Curves ϕ: I → M satisfying (2.10) are called *characteristic curves* of $d\alpha$.

3. THE RESTRICTED VARIATIONAL PROBLEM

A natural way to impose restrictions on curves in M is by use of 1-forms. Let β be a 1-form on M which is nowhere zero, then the equality $\phi^*\beta = 0$ defines a restriction on curves ϕ in M. In fact it expresses that ϕ lies in an integral manifold of the distribution which is defined pointwise by $S(x) = \{v \in T_xM \mid \beta(v) = 0\}$. In this case S is integrable. More generally, we can define restrictions by smooth distributions defining only locally a set of basic 1-forms. In such a setting we define an admissible curve as follows.

<u>DEFINITION 3.1.</u> A curve ϕ: I → M is called *admissible* under restriction distribution S on M if

$$\phi_*(\frac{\partial}{\partial t}\Big|_t) \in S(\phi(t)), \qquad \forall t \in I.$$

Given S we can define its *annihilator* $E = S^\perp$, $E \subset T^*M$, pointwise by

$$(3.1)\qquad E(x) = \{\beta_x \in T_x^*M \mid \beta_x(s) = 0,\ \forall s \in S(x)\}.$$

We shall assume throughout this paper that S has constant dimension. Then the smoothness implies that E is a smooth codistribution, consisting of all smooth 1-forms which lie pointwise in E(x), $x \in M$. That means that E is locally spanned by smooth 1-forms $\beta_1,\dots,\beta_p$ (p = codim S) and in this neighbourhood admissible curves ϕ satisfy: $\phi^*\beta_i = 0$ $(i = 1,\dots,p)$. Now denote the set of admissible variations k.e.p.f. by $\mathcal{V}_\phi^E$:

(3.2) $$V_\phi^E = \{\xi_\varepsilon \in V_\phi \mid \xi_\varepsilon^*\beta = 0,\ \forall\beta \in E\}.$$

Then the following definition is natural.

DEFINITION 3.2. An admissible curve $\phi\colon I \to M$ is *stationary* w.r.t. a 1-form α *under smooth restriction distribution* S, if for all $\phi_\varepsilon \in V_\phi^E$ ($E = S^\perp$):

(3.3) $$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0.$$

Note that this definition implies that isolated admissible curves are stationary as only the trivial variation $\phi_\varepsilon(t) = \phi(t)\ \forall\varepsilon$, is admissible.

We see that condition (3.3) is of first order in ε. This suggests that the higher order part of the variation is of no interest as long as the variation is admissible. Moreover, the question arises whether we might restrict attention to variations which are admissible only to first order in ε. If this is true we might expect a considerable simplification of the theory and the practical computations. We shall not answer this question for general smooth restriction distributions. We prove that the answer is positive for (at least a large class of) the problems that we consider in this paper. (Propositions 4.2 and 5.3). So, in the cases of interest to us, the following stronger concept of "formal stationarity" reduces to stationarity. This notion, together with Theorem 3.8 is suggested in TAKENS [1978].

DEFINITION 3.3. An admissible curve $\phi\colon I \to M$ is *formally stationary* w.r.t. a 1-form α *under restriction distribution* S (smooth and of constant dimension) if, with $E = S^\perp$:

(3.4) $$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0 \qquad \forall\phi_\varepsilon \in W_\phi^E,$$

where the set of *formally admissible variations* W_ϕ^E is defined by

(3.5) $$W_\phi^E = \{\xi_\varepsilon \in V_\phi \mid \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \xi_\varepsilon^*\beta = 0,\ \forall\beta \in E\}.$$

Clearly, formal stationarity implies stationarity as $V_\phi^E \subset W_\phi^E$. The converse is not true for arbitrary distribution S, but has been proven for the cases of interest to us.

Similar to Definition 2.2' we can give an equivalent definition for formal stationarity:

DEFINITION 3.3'. ϕ is *formally stationary* w.r.t. α *under restriction distribution* S if ϕ is admissible and for all vector fields V along ϕ with $V(a) = V(b) = 0$ we have

$$\phi^* L_V \beta = 0, \ \forall \beta \in E \Rightarrow \int_I \phi^* L_V \alpha = 0.$$

Before expressing the main results about these notions we shall dwell for some time upon the global character of these results. In fact the global problem can easily be broken up in finitely many equivalent local problems. This follows from the following proposition.

PROPOSITION 3.4. *Let* ϕ: I $\to$ M *be a given injective immersion. Let* $\{I^\mu\}$ *be a finite collection of closed subintervals of* I *such that* $\{\mathrm{int}\ I^\mu\}$ *is an open covering of* I. *Define* $\phi^\mu = \phi|I^\mu$, *the restriction of* ϕ *to* I^μ. *Then* ϕ *is (formally) stationary w.r.t. a* 1-*form* α *(under restriction* S*) if and only if* ϕ^μ *is (formally) stationary w.r.t.* α *(under restriction* S*) for all* μ.

PROOF. First let ϕ be stationary and let ϕ^μ_ε be a variation k.e.p.f. of ϕ^μ. Then $\hat{\phi}_\varepsilon$ defined by

$$\hat{\phi}_\varepsilon(t) = \phi^\mu_\varepsilon(t), \quad t \in I^\mu,$$

$$= 0 \quad , \quad t \notin I^\mu,$$

is not necessarily smooth at the end points of the subinterval I^μ, so that it is no variation k.e.p.f. of ϕ on I. However, we can find a smooth variation ϕ_ε of ϕ on I which is arbitrarily close to $\hat{\phi}_\varepsilon$ on I. Hence stationarity of ϕ implies stationarity of ϕ^μ, for all μ.

Conversely, let ϕ^μ be stationary w.r.t. α for all μ. Let V be a vector field along ϕ with $V(a) = V(b) = 0$. We can choose f^μ: I $\to \mathbb{R}$ such that $\mathrm{supp}\ f^\mu \subset I^\mu$, $\Sigma_\mu f^\mu(t) = 1$, $\forall t \in I$. Then $f^\mu V = V_\mu$ is a vector field along ϕ^μ, which vanishes at the end points. Then stationarity of ϕ^μ implies:

$$\int_I \phi^* L_V \alpha = \sum_\mu \int_{I_\mu} (\phi^\mu)^* L_{V_\mu} \alpha = 0.$$

For formal stationarity we require the additional equality (use (2.7)):

$$\phi^* L_{V_\mu} \beta = f^\mu \phi^* L_V \beta + \beta(V(t)) df(\tfrac{\partial}{\partial t}) = f^\mu \phi^* L_V \beta$$

as $\beta(V(t)) = \phi^*\beta(X) = 0$ for any extension X of V as ϕ is admissible. This implies

$$\phi^* L_V \beta = 0 \iff \phi^* L_{V_\mu} \beta = 0, \quad \forall \mu.$$

Using this we can give a proof for formal stationarity that is similar to the one above for stationarity. □

This proposition allows us to assume that $\phi(I)$ lies entirely in a coordinate neighbourhood of M, as far as unrestricted stationarity and restricted formal stationarity is concerned.

After this intermezzo we return to the development of the main theorem of this section. We need the following definitions.

DEFINITION 3.5. Let M be a manifold with cotangent bundle T^*M and natural projection $\pi: T^*M \to M$. Then, the *canonical* 1-*form* θ on T^*M is defined by

(3.6) $$\theta(\xi) = \pi^*\xi$$

for all $\xi \in T^*M$.

REMARK 3.6. By definition of π^*, (3.6) implies:

(3.7) $$\theta(\xi)(v) = \xi(\pi_* v),$$

for all $\xi \in T^*M$, $v \in T_\xi T^*M$ $(\pi : T^*M \to M)$. If we choose coordinates $x_1,\ldots$ $\ldots,x_m$ in some open neighbourhood in M, then we can define *canonical coordinates* $\bar{x}_i, p_i$, $i = 1,\ldots,m$ by

(3.8) $$\bar{x}_i(\xi) = x_i(\pi\xi); \qquad p_i(\xi) = \xi\left(\frac{\partial}{\partial x_i}\bigg|_{\bar{x}(\xi)}\right).$$

We shall identify x_i and $\bar{x}_i$. In canonical coordinates, the canonical 1-form θ on T^*M is given by

$$(3.9) \qquad \theta = \sum_{i=1}^{m} p_i dx_i .$$

DEFINITION 3.7. Let M be a manifold with 1-form α and distribution S on M. Let $E = S^\perp$ (cf. (3.1)). Then the *Cartan form* θ_α on E, associated with α is defined by:

$$(3.10) \qquad \theta_\alpha = \pi_E^* \alpha + \theta_E,$$

where π_E is the restriction to E of the natural projection π: $T^*M \to M$ and θ_E is the restriction to E of the canonical 1-form on T^*M.

Now we are ready to formulate the basic theorem.

THEOREM 3.8. *Let* M *be a manifold with* 1*-form* α *and distribution* S *of constant dimension. Then, an injective immersion* ϕ: I $\to$ M *is formally stationary with respect to* α *under restriction* S *if and only if there exists an injective immersion* η: I $\to$ E *with* $\eta(t) \in \pi_E^{-1}(\phi(t))$ *and* η *stationary in* E *with respect to the Cartan form* θ_α.

PROOF. First let an injective immersion η: I $\to$ E be given with $\eta(t) \in \pi_E^{-1}(\phi(t))$ and η stationary with respect to α. By Proposition 3.4 we can restrict ourselves to curves in a coordinate neighbourhood such that E is spanned by forms $\beta_1,\ldots,\beta_p$ on this neighbourhood. Furthermore, note that an arbitrary vector field along η yields a projected vector field along ϕ as ϕ and η are injective immersions.

To prove that ϕ is formally stationary we first have to prove that ϕ is admissible. Therefore choose local coordinates x for M and let $\beta_1,\ldots,\beta_p$ be a local basis for E. Then we can give coordinates (x,y) for E. I.e. an element $(x,\Sigma_{i=1}^p y_i\beta_i(x)) \in E$ has coordinates (x,y) $(y = (y_1,\ldots,y_p))$. By definition of the canonical form on $E \subset T^*M$ we obtain

$$\theta_E(x,y)(v) = (\sum_{i=1}^{p} y_i\beta_i(x))(\pi_{E*}v) = \sum_{i=1}^{p} y_i(\pi_E^*\beta_i)(v),$$

for $v \in T_{(x,y)}E$. So

$$\theta_E(x,y) = \sum_{i=1}^{p} y_i(\pi_E^*\beta_i)(x,y). \tag{3.11}$$

Therefore, given an arbitrary vector field X on E,

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{p} Y_j \frac{\partial}{\partial y_j},$$

we obtain:

$$\begin{aligned}(L_X\theta_E)(x,y) &= \sum_{i=1}^{p} L_X(y_i\pi_E^*\beta_i)(x,y) \\ &= \sum_{i=1}^{p} Y_i(\pi_E^*\beta_i)(x,y) + \sum_{i=1}^{p} y_i L_X(\pi_E^*\beta_i)(x,y).\end{aligned} \tag{3.12}$$

Now let in these coordinates η be given by

$$\eta(t) = (\phi(t),\lambda(t)) \tag{3.13}$$

(ϕ and λ are x and y coordinates, respectively) and define

$$W_i(t) = w_i(t)\left.\frac{\partial}{\partial y_i}\right|_{\eta(t)}, \quad i = 1,\dots,p,$$

where w_i arbitrary on I with $w_i(a) = w_i(b) = 0$. Clearly W_i $(i = 1,\dots,p)$ are vector fields along η with projection $\pi_{E*}W_i = 0$. We have

$$\begin{aligned}\int_I \eta^* L_{W_i}\theta_\alpha &= \int_I \eta^* L_{W_i}(\pi_E^*\alpha) + \int_I \eta^* L_{W_i}\theta_E = \int_I \eta^* L_{W_i}\theta_E \\ &\overset{(3.12)}{=} \int_I w_i(t)\eta^*\pi_E^*\beta_i + \sum_{i=1}^{p}\int_I \lambda_i(t)\eta^* L_{W_i}(\pi_E^*\beta_i).\end{aligned} \tag{3.14}$$

As $\pi_{E*}W_i = 0$ the last term equals zero because

$$\eta^* L_{W_i}(\pi_E^*\beta_i) \overset{(2.7)}{=} \pi_E^* d\beta_i(W_i(t),\eta_*(\tfrac{\partial}{\partial t})) + d(\pi_E^*\beta_i(W_i(t)))(\tfrac{\partial}{\partial t}) = 0.$$

Then the stationarity of η makes the left-hand side of (3.14) equal to zero, so that

$$0 = \int_I w_i(t)\eta^*\pi_E^*\beta_i = \int_I w_i(t)\phi^*\beta_i,$$

for arbitrary w_i. This proves that $\phi^*\beta_i = 0$ $(i = 1,\dots,p)$, hence ϕ is admissible.

To prove the formal stationarity of ϕ let a vector field V along ϕ with $V(a) = V(b) = 0$ be given in coordinates:

$$V(t) = \sum_{i=1}^{n} v_i(t) \left.\frac{\partial}{\partial x_i}\right|_{\phi(t)} .$$

Define a vector field W along η by

$$W(t) = \sum_{i=1}^{n} v_i(t) \left.\frac{\partial}{\partial x_i}\right|_{\eta(t)} .$$

Then $\pi_{E*}W = V$ and the $\partial/\partial y_i$ - components of W are zero. So use of (3.12) yields:

$$\begin{aligned}\int_I \eta^* L_W \theta_\alpha &= \int_I \eta^* L_W(\pi_E^*\alpha) + \int_I \eta^* L_W \theta_E \\ &= \int_I \eta^* L_W(\pi_E^*\alpha) + \sum_{i=1}^{p} \int_I \lambda_i(t)\eta^* L_W(\pi_E^*\beta_i).\end{aligned} \tag{3.15}$$

Moreover, using (2.7):

$$\begin{aligned}\eta^* L_W(\pi_E^*\beta_i)\left(\left.\frac{\partial}{\partial t}\right|_{\eta(t)}\right) &= \pi_E^* d\beta_i(W(t),\eta_*(\tfrac{\partial}{\partial t})) + d(\pi^*\beta_i(W(t)))(\tfrac{\partial}{\partial t}) \\ &= dw(V(t),\phi_*(\tfrac{\partial}{\partial t})) + d(\beta_i(V(t)))(\tfrac{\partial}{\partial t}) \\ &= \phi^* L_V\beta_i\left(\left.\frac{\partial}{\partial t}\right|_{\phi(t)}\right).\end{aligned}$$

Substituting this in (3.15) yields

$$\int_I \eta^* L_W \theta_\alpha = \int_I \phi^* L_V \alpha + \sum_{i=1}^{p} \int_I \lambda_i(t)\phi^* L_V \beta_i .$$

Stationarity of η makes the left-hand vanish. Hence $\phi^*L_V\beta_i = 0$ $(i = 1,\ldots,p)$ yields $\int_I \phi^*L_V\alpha = 0$ which implies formal stationarity.

To prove the converse, let ϕ be formally stationary. Given any vector field W along η with $W(a) = W(b) = 0$ we obtain, using (3.12):

$$\int_I \eta^*L_W\theta_\alpha = \int_I \eta^*L_W(\pi_E^*\alpha) + \sum_{i=1}^{p} \int_I W_{y_i}\phi^*\beta_i + \sum_{i=1}^{p} \int_I \lambda_i\eta^*L_W(\pi_E^*\beta_i),$$

with W_{y_i} the $\partial/\partial y_i$ - component of W. As ϕ is admissible $(\phi^*\beta_i = 0)$ we obtain, with $V = \pi_{E*}W$:

$$\int_I \eta^*L_W\theta_\alpha = \int_I \phi^*L_V\alpha + \sum_{i=1}^{p} \int_I \lambda_i\phi^*L_V\beta_i. \tag{3.16}$$

Hence, we have to prove that we can find $\lambda_i\colon I \to \mathbb{R}$ $(i = 1,\ldots,p)$ such that for all V, V a vector field along ϕ with $V(a) = V(b) = 0$, the following equality is satisfied

$$\int_I \phi^*L_V\alpha = -\sum_{i=1}^{p} \int_I \lambda_i\phi^*L_V\beta_i \ . \tag{3.17}$$

Note that we then have $\eta(t) = \Sigma_{i=1}^{p} \lambda_i(t)\beta_i(\phi(t))$ satisfying the conditions of the theorem. For simplicity we assume that $p = 1$, i.e. E is spanned by one 1-form. We omit the subscripts for λ and β. To find an appropriate λ in this case define a vector field Z along ϕ such that $\beta(Z) = 1$ along ϕ. Let

$$F_1 = \{V \mid V \text{ vector field along } \phi,\ \phi^*L_V\beta = 0,\ V(a) = 0\},$$

$$F_2 = \{V \mid V \text{ vector field along } \phi,\ V = \psi Z,\ \psi(a) = 0\}.$$

Then, any vector field V along ϕ with $V(a) = V(b) = 0$ can be written uniquely as the sum

$$V = V_1 + V_2, \qquad V_1 \in F_1,\ V_2 \in F_2.$$

This is shown by the following argument. Given V the differential equation:

$$\begin{aligned} &\phi^*L_V\beta(\tfrac{\partial}{\partial t}) = \psi(t)d\beta(Z(t),\phi_*(\tfrac{\partial}{\partial t})) + d\psi(\tfrac{\partial}{\partial t}) \\ &\psi(a) = 0 \end{aligned} \tag{3.18}$$

defines ψ: $I \to \mathbb{R}$ uniquely. Now define

$$V_2 = \psi Z; \quad V_1 = V - \psi Z$$

then we have the appropriate splitting as $V_2 \in F_2$ by choice and $V_1 \in F_1$ because

$$\phi^* L_{V_1} \beta(\frac{\partial}{\partial t}) = \phi^* L_V \beta(\frac{\partial}{\partial t}) - \phi^* L_{\psi Z} \beta(\frac{\partial}{\partial t})$$

$$\overset{(2.7)}{=} \phi^* L_V \beta(\frac{\partial}{\partial t}) - \psi(t) d\beta(Z(t), \phi_*(\frac{\partial}{\partial t})) + d\psi(\frac{\partial}{\partial t}) \overset{(3.18)}{=} 0.$$

Note that $V_1(b) = -V_2(b) = -\psi(b)Z(b)$ is not necessarily equal to zero. Now let V be arbitrary with $V(a) = V(b) = 0$ and $V = V_1 + V_2 = V_1 + \psi Z$ its unique splitting. Then (2.7) and Stokes theorem yield:

$$\int_I \phi^* L_V \alpha = \int_I \phi^* \iota_V d\alpha + \int_I d(\alpha(V)) = \int_I \phi^* \iota_{V_1} d\alpha + \int_I \phi^* \iota_{V_2} d\alpha,$$

where $\phi^* \iota_V d\alpha(\partial/\partial t) = d\alpha(V(t), \phi_*(\partial/\partial t))$, by definition.
If $\psi(b) \neq 0$ $(V_1(b) \neq 0)$ we define a constant C_0 such that

$$\int_I \phi^* L_V \alpha = \int_I \phi^* \iota_{V_2} d\alpha - C_0 \psi(b). \tag{3.19}$$

If $\psi(b) = 0$ then $\int_I \phi^* L_V \alpha = \int_I \phi^* \iota_{V_2} d\alpha$ by formal stationarity of ϕ, so that we can choose C_0 arbitrarily and (3.19) still holds. Then define Ψ_1, Ψ_2: $I \to \mathbb{R}$ by

$$\Psi_1 dt = \phi^* \iota_Z d\beta; \quad \Psi_2 dt = \phi^* \iota_Z d\alpha \tag{3.20}$$

and λ: $I \to \mathbb{R}$ by

$$\dot{\lambda} = \Psi_2 + \Psi_1 \lambda, \quad \lambda(b) = C_0. \tag{3.21}$$

Then we have:

$$-\int_I \lambda\phi^* L_V\beta \overset{(2.7)}{=} -\int_I \lambda(\psi\Psi_1 dt + d\psi)$$

$$= \int_I (\psi\dot{\lambda} - \lambda\Psi_1\psi)dt - \lambda(b)\psi(b)$$

$$\overset{(3.21)}{=} \int_I \psi\Psi_2 dt - C_0\psi(b) \overset{(3.20)}{\underset{(3.19)}{=}} \int_I \phi^* L_V\alpha.$$

So the chosen λ satisfies (3.17) for $p = 1$. Hence η, given by $\eta(t) = \lambda(t)\beta(\phi(t))$, is stationary w.r.t. θ_α and $\pi_E\eta = \phi$. For $p > 1$ the proof is similar. □

In fact, Theorem 3.8 is a generalization of the Lagrange multiplier rule: it proves existence of Lagrange multiplier $\lambda_i(t)$ such that solutions of a restricted problem in variables x can be found as solutions to an unrestricted problem in variables (x,y) with $y = \lambda$ at the solution.

Theorem 3.8 forms the heart of this paper. It enables us to formulate the Lagrange problem and the optimal control problem as a problem of finding characteristic curves of the differential of a certain Cartan form (recall Proposition 2.3).

REMARK 3.9. We can set up the theory of Section 2 and 3, including Theorem 3.8, for the free end point variational problem with action function

$$\mathcal{J}(\phi) = h(\phi(b)) + \int_I \phi^*\alpha$$

with α a 1-form on M, $h: M \to \mathbb{R}$ denoting the terminal cost and for $\phi: I \to M$ with $\phi(a)$ fixed and $\phi(b) \in F$ (F a submanifold of M of dimension unequal zero). This can be done by slight modifications in definitions and theorems. For clearness of exposition, as we do not work it out for the Lagrange and optimal control problem, the details are given in Appendix A.

4. THE LAGRANGE PROBLEM

Consider a smooth manifold Q (the *configuration space*) with dim Q = n and a function $L: TQ \times I \to \mathbb{R}$ which is called the *Lagrangian* (I is a closed interval in $\mathbb{R}$ as before). Then we can seek for curves $\psi\colon I \to Q$ which minimize the action integral

$$(4.1) \qquad \mathcal{J}(\psi) = \int_I L(\psi(t),\psi'(t),t)dt.$$

This is called the *Lagrange problem*. We can formulate this problem according to Section 3. To do so, choose a coordinate t on I, let

$$(4.2) \qquad M = TQ \times I, \qquad \alpha = Ldt,$$

and define a mapping ℓ from the set of curves $\psi\colon I \to Q$ to the set of curves $\phi\colon I \to M$ by

$$(4.3) \qquad \phi(t) = \ell(\psi)(t) = (\psi_*(\frac{\partial}{\partial t}\Big|_t),t) \quad \text{for all } t \in I.$$

(In canonical coordinates we have $\phi(t) = (\psi(t),\dot{\psi}(t),t)$.) Subsequently, we can define a codistribution $E \subset T^*M$ by

$$(4.4) \qquad E = \{\beta \in T^*M \mid \forall\psi\colon I \to Q \text{ we have } (\ell(\psi))^*\beta = 0\}.$$

E is a codistribution as the following coordinate representation shows.

REMARK 4.1. Choosing canonical coordinates q, $\dot{q}$, t on M, it can easily be shown that the fibres of E are spanned by 1-forms β_i $(i = 1,\dots,n)$ locally, given by

$$(4.5) \qquad \beta_i = dq_i - \dot{q}_i dt.$$

We have dim $E = n$ and the restriction distribution S for the Lagrange problem is defined as the annihilator $S = E^{\perp}$ $(S = \bigcap_{i=1}^{n} \ker \beta_i)$.

For curves $\phi\colon I \to M$ satisfying the restrictions (i.e. $\phi(t) = (\psi(t), \dot{\psi}(t),t)$ in canonical coordinates for some $\psi\colon I \to Q$) we have

$$(4.6) \qquad \phi^*\alpha = \phi^*(Ldt) = L(\psi(t),\dot{\psi}(t),t)dt.$$

So, the Lagrange problem can be formulated as the problem of minimizing $J(\phi) = \int_I \phi^*\alpha$ over curves in M under restriction distribution S.

Note that there is some inconsistency in this approach. On one hand we define everything coordinate free in TQ, and on the other hand we choose a global coordinate t on I. In fact this gives us an easy way to express that we consider curves modulo a time transformation on I. A more general approach might also easily hide the relation with the original Lagrangian problem.

We restrict attention to first order conditions (stationarity), so in view of Theorem 3.8 the following proposition is relevant.

PROPOSITION 4.2. *For the above restricted variational problem we have equivalence between formal stationarity and stationarity of curves.*

PROOF. We already know that formal stationarity implies stationarity. So the reverse remains to be proven. We consider curves modulo a time transformation, so every formally admissible variation k.e.p.f. ϕ_ε of $\phi = (\psi,\dot{\psi},t)$ can be given as

$$(4.7) \qquad \phi_\varepsilon(t) = (\phi_\varepsilon^q(t),\phi_\varepsilon^{\dot{q}}(t),t),$$

and we can restrict ourselves to vector fields V along ϕ which can be given in canonical coordinates by

$$(4.8) \qquad V(t) = V^q(t)\,\frac{\partial}{\partial q}\Big|_{\phi(t)} + V^{\dot{q}}(t)\,\frac{\partial}{\partial \dot{q}}\Big|_{\phi(t)},$$

with V(a) = V(b) = 0. We use Definition 3.3'. Suppose such a vector field satisfies

$$(4.9) \qquad \phi^* L_V\beta = 0, \qquad \forall\beta \in \mathrm{span}\{dq_i - \dot{q}_i dt\}.$$

(Note that we may work locally, by Proposition 3.4.) We first assume that Q is 1-dimensional, so E is spanned by the form $\beta = dq - \dot{q}dt$. Thus (4.9) implies, using (2.7):

$$0 = \phi^* L_V \beta(\frac{\partial}{\partial t}) = d\beta(V(t), \phi_*(\frac{\partial}{\partial t})) + d(\beta(V(t)))(\frac{\partial}{\partial t})$$

$$= -d\dot{q} \wedge dt(V(t), \phi_*(\frac{\partial}{\partial t})) + d(V^q(t))(\frac{\partial}{\partial t}).$$

So

$$\frac{dV^q(t)}{dt} = V^{\dot{q}}(t). \tag{4.10}$$

Now choose ϕ_ε by

$$\phi_\varepsilon(t) = (\psi(t) + \varepsilon V^q(t), \dot{\psi}(t) + \varepsilon V^{\dot{q}}(t), t). \tag{4.11}$$

Then ϕ_ε is a variation k.e.p.f. of ϕ according to Definition 2.1 with

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \phi_\varepsilon(t) = (V^q(t), V^{\dot{q}}(t), 0). \tag{4.12}$$

Moreover

$$\phi_\varepsilon^* \beta(\frac{\partial}{\partial t}) = \beta(\dot{\psi}(t) + \varepsilon \dot{V}^q(t), \ddot{\psi}(t) + \varepsilon \dot{V}^{\dot{q}}(t), 1)$$

$$= \dot{\psi}(t) + \varepsilon \dot{V}^q(t) - (\dot{\psi}(t) + \varepsilon V^{\dot{q}}(t)) = 0,$$

using (4.10). So ϕ_ε is an admissible variation k.e.p.f. of ϕ, so that by stationarity and (4.12)

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi_\varepsilon^* \alpha = \int_I \phi^* L_V \alpha.$$

This proves the theorem for dim Q = 1. For dim Q > 1 the proof is similar. □

A direct consequence of Proposition 4.2 and Theorem 3.8 is the following corollary.

COROLLARY 4.3. *An injective curve* ψ: I → Q *is a stationary curve for the Lagrange problem if and only if there exists an injective curve* η: I → E *(Def. (4.4)) with* $\pi_E \circ \eta = \ell(\psi)$ *and* η *stationary with respect to the Cartan form*

$$\theta_L = \pi_E^*(L dt) + \theta_E$$

(see (3.12) *with* $\alpha = L dt$*).*

This corollary reduces the first order Lagrange problem on M to an unrestricted stationarity problem in the higher dimensional bundle E over M; the solutions of the last problem are the characteristic curves of $d\theta_L$. Let us look at how this works out locally. Let q, $\dot{q}$, t denote coordinates for $M = TQ \times I$ (q, $\dot{q}$ n-dimensional) and let λ (n-dimensional) denote coordinates for the fibres of E.($\beta \in E \Rightarrow \beta = \Sigma_{i=1}^n \lambda_i \beta_i$, β_i given by (4.5)). Then

$$\theta_L = \sum_{i=1}^{n} \lambda_i \beta_i + L dt. \tag{4.13}$$

Characteristic curves of $d\theta_L$ are integral curves of a vector field X on E satisfying

$$\iota_X d\theta_L = 0. \tag{4.14}$$

As solution curves project diffeomorphically on I we may, modulo a time parametrization, assume that X has $\partial/\partial t$-component equal to 1.

Denote

$$X = X_{q_i} \frac{\partial}{\partial q_i} + X_{\dot{q}_i} \frac{\partial}{\partial \dot{q}_i} + X_{\lambda_i} \frac{\partial}{\partial \lambda_i} + \frac{\partial}{\partial t},$$

where summation over $i = 1,\dots,n$ is assumed. Substitution of X and θ_L ((4.13)) in (4.14) yields by collecting terms in dq_i, $d\dot{q}_i$, $d\lambda_i$ and dt, respectively:

$$X_{\lambda_i} - \frac{\partial L}{\partial q_i} = 0 \tag{4.15}$$

$$\lambda_i - \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{4.16}$$

$$\dot{q}_i - X_{q_i} = 0 \tag{4.17}$$

$$-\dot{q}_i X_{\lambda_i} - \lambda_i X_{\dot{q}_i} + \frac{\partial L}{\partial q_i} X_{q_i} + \frac{\partial L}{\partial \dot{q}_i} X_{\dot{q}_i} = 0. \tag{4.18}$$

It is easily seen that the first three equations imply the fourth so that we have 3n equations. Equation (4.16) defines a (2n+1)-dimensional submanifold $N \subset E$. So, stationary curves lie in N. In order that $X|_N$ is a vector field on N we must have:

$$X(\lambda_i - \frac{\partial L}{\partial \dot{q}_i}) = 0 \qquad i = 1,\dots,n,$$

which implies:

$$-X_{q_j} \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} - X_{\dot{q}_j} \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} + X_{\lambda_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i} = 0 \qquad i = 1,\dots,n.$$

Using (4.15) and (4.17) we obtain:

$$(4.19) \qquad \frac{\partial}{\partial q} \frac{\partial L}{\partial \dot{q}} \frac{dq}{dt} + \frac{\partial}{\partial \dot{q}} \frac{\partial L}{\partial \dot{q}} X_{\dot{q}} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

We see that the vector field on N is (uniquely) defined if and only if

$$(4.20) \qquad \mathrm{rk}(\frac{\partial^2 L}{\partial \dot{q}^2}) = n.$$

Then, the solution curves can be found as integral curves in N of $X|_N$, which yields the Hamiltonian formalism.

If, in general, η: $t \mapsto (\psi(t),\zeta(t),\nu(t),t)$ is an integral curve of the vector field X satisfying (4.15) up to (4.17), then

$$(4.21) \qquad X_q(\eta(t)) = \frac{d\psi(t)}{dt} \overset{(4.17)}{=\!=} \zeta(t),$$

$$(4.22) \qquad X_{\dot{q}}(\eta(t)) = \frac{d\zeta(t)}{dt} \overset{(4.21)}{=\!=} \frac{d^2\psi(t)}{dt^2},$$

$$(4.23) \qquad X_\lambda(\eta(t)) = \frac{d\nu(t)}{dt} \overset{(4.15)}{=\!=} \frac{\partial L(\psi(t),\dot{\psi}(t),t)}{\partial q}.$$

Moreover, $\eta(t)$ has to lie in N by (4.16), which implies similar to (4.19):

$$(4.24) \qquad \frac{d}{dt}(\frac{\partial L(\psi,\dot{\psi},t)}{\partial \dot{q}}) - \frac{\partial L(\psi,\dot{\psi},t)}{\partial q} = (\frac{\partial^2 L}{\partial \dot{q}^2})(\frac{d^2\psi}{dt^2} - X_{\dot{q}}(\eta(t))) = 0,$$

using (4.22). So ψ has to satisfy the well-known Lagrange equation and ζ and ν are defined by (4.21) and application of (4.16):

$$\text{(4.25)} \qquad \nu(t) = \frac{\partial L(\psi,\dot{\psi},t)}{\partial \dot{q}_i} \; .$$

This shows that solutions may exist independent of condition (4.20). $X|_N$ does not have to exist nor to be unique as a vector field on N. It is enough that there exists an integral curve of an X (satisfying (4.14)) on M, which satisfies the end point conditions and which lies in N.

REMARK 4.4. If we choose $\alpha = Ldt + \beta$ in (4.2) for any $\beta \in E$ then $\phi^*\alpha = \phi^*(Ldt)$ for all admissible ϕ. Therefore, such a choice does not change the solution of the Lagrange problem. E.g. $\beta = \Sigma_{i=1}^n \bar{\lambda}_i\beta_i$ yields for θ_L

$$\theta_L = \sum_{i=1}^{n} (\lambda_i+\bar{\lambda}_i)\beta_i + Ldt.$$

This just results in a translation of λ_i over $\bar{\lambda}_i$ ($i = 1,\ldots,n$), i.e. a translation of the Lagrange multipliers.

5. THE NONLINEAR OPTIMAL CONTROL PROBLEM

We shall first recall the notion of a general nonlinear control system as given by BROCKETT [1977] and WILLEMS [1979] and worked out by NIJMEIJER & VAN DER SCHAFT [1982].

DEFINITION 5.1. A *nonlinear* (time-invariant) *control system* $\Sigma = \Sigma(Q,B,f)$ is defined by a smooth manifold Q, a fibre bundle $\tau\colon B \to Q$ and a smooth map $f\colon B \to TQ$ such that the following diagram commutes

$$\text{(5.1)} \qquad \begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\tau"'] & & TQ \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd}$$

We call Σ *affine* if B is a vector bundle and f restricted to the fibres of B is an affine map into the fibres of TQ.

Σ is called *analytic* if B and Q are analytic manifolds and f is an analytic map.

We say that ψ: I $\to$ Q is a *trajectory* of Σ if ψ is absolutely continuous and

$$\psi_*(\frac{\partial}{\partial t}\Big|_t) \in f(\tau^{-1}(\psi(t))),$$

almost everywhere on I. With each trajectory ψ we can associate a *trajectory-input* ζ: I $\to$ B such that

$$(5.2) \qquad \tau(\zeta(t)) = \psi(t), \quad \psi_*(\frac{\partial}{\partial t}\Big|_t) = f(\zeta(t)), \qquad t \in I.$$

Q is called the *configuration space* in this context cf. the Lagrange context. The fibres of B represent the (state dependent) input spaces. In local coordinates q for Q and u for the fibres $\tau^{-1}(q)$ we obtain the familiar equation $\dot{q} = f(q,u)$ (with abuse of notation f: $(q,u) \mapsto (q,f(q,u))$). A trajectory-input ζ will in such coordinates often be denoted by: $\zeta(t) = (\psi(t), \nu(t))$, ψ and ν denoting the q and u coordinates resp. In the sequel we will use f in both ways, how it is used will be clear from the context. If Σ is affine then, in coordinates, f has the form

$$(5.3) \qquad f(q,u) = A(q) + \sum_{i=1}^{m} u_i B_i(q),$$

with $u_i \in \mathbb{R}$, A and B_i vector fields on Q ($i = 1,\dots,m$).

We shall assume in the rest of this paper that f is an injective immersion.

Now, an optimal control problem can be interpreted as a certain variational problem on the space of states and inputs, i.e. B, under certain restrictions, one of these being the restriction to curves in B which are trajectory-inputs of the given system. In fact, the approach to the Lagrangian problem for curves in Q can be followed here with respect to curves in B. Therefore, let us first assume to be given a function G: TB $\times$ I $\to \mathbb{R}$, in analogy with the Lagrangian L in Section 4. What G appears to be in the specific case of optimal control will be discussed later. The problem becomes to find stationary curves ϕ: $I \to TB \times I$ of the action integral

$$J(\phi) = \int_I G dt,$$

with ϕ restricted to curves of the form $\phi(t) = (\zeta_*(\frac{\partial}{\partial t}\big|_t), t)$ where ζ: $I \to B$ is a trajectory-input of the system, with $\zeta(a)$ and $\zeta(b)$ given. Let $\tilde{\tau}_*$: $TB \times I \to TQ$ be defined by $\tilde{\tau}_*(v,t) = \tau_*(v)$, for all $v \in TB$, then we easily deduce from (5.2) for such curves ϕ:

$$\tilde{\tau}_*\phi(t) = \tau_*\zeta_*(\frac{\partial}{\partial t}\big|_t) = f(\zeta(t)), \qquad t \in I. \tag{5.4}$$

Now define with natural projection $\tilde{\pi}_B$: $TB \times I \to B$:

$$M = \{w \in TB \times I \mid f \circ \tilde{\pi}_B(w) = \tilde{\tau}_*(w)\}. \tag{5.5}$$

Then, the curves satisfying the restrictions lie in M (trivial from (5.4)). Moreover, M is a submanifold of $TB \times I$. Namely, if (q,u) are coordinates in B and $(q,u,\dot{q},\dot{u},t)$ denote canonical coordinates in $TB \times I$, then elements of M can locally be given by $(q,u,f(q,u),\dot{u},t)$. As f is an injective immersion M can be coordinatized locally by $(q,u,\dot{u},t)$. Finally, the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\tilde{\tau}_*"] \arrow[d, "\tilde{\pi}_B"'] & TQ \arrow[d, "\pi_Q"] \\ B \arrow[ur, "f"] \arrow[r, "\tau"'] & Q \end{tikzcd} \tag{5.6}$$

So our variational problem may be restricted to M and the restriction codistribution $E = S^{\perp}$ on M with

$$E = \{\beta \mid \beta \text{ 1-form on } M, \ \phi^*\beta = 0 \text{ for all admissible } \phi: I \to M\}$$

is locally represented as follows.

PROPOSITION 5.2. *Let* $(q,u,\dot{u},t)$ *denote coordinates on* M *as above. Then* E *is spanned locally by* (n+m) 1-*forms:*

$$\text{(5.7)} \qquad \begin{aligned} \beta_i &= dq_i - f_i(q,u)dt & i &= 1,\dots,n, \\ \beta_{n+j} &= du_j - \dot{u}_j dt & j &= 1,\dots,m. \end{aligned}$$

Here $f_i(q,u)$ *denotes the* i-*th coordinate of* $f(q,u)$.

PROOF. For an admissible curve $\phi: I \to M$ in coordinates given by $\phi(t) = (\phi_q(t),\phi_u(t),\phi_{\dot{u}}(t),t)$ we have imbedded in $TB \times I$ (imbedding i):

$$i\circ\phi(t) = (\phi_q(t),\phi_u(t),f(\phi_q(t),\phi_u(t)),\phi_{\dot{u}}(t),t) = (\zeta_*(\frac{\partial}{\partial t}\Big|_t),t)$$

for some trajectory input ζ. Clearly $\zeta = (\phi_q,\phi_u)$ and

$$\dot{\phi}_q = f(\phi_q,\phi_u); \qquad \dot{\phi}_u = \phi_{\dot{u}} .$$

It follows immediately that $\phi^*\beta_i = 0$, $i = 1,\dots,n+m$. As β_i $(i = 1,\dots,n+m)$ are independent and the dimension of E_x equals n+m $(x \in M)$ due to determination of curves in M by trajectory-inputs, the proof is completed. □

In order to use Theorem 3.8 profitably for the restricted variational problem defined above, we have to investigate, analogous to Proposition 4.2, the equivalence between stationarity and formal stationarity under this specific restriction distribution. It appears that equivalence holds for the special but important class of affine analytic systems. We shall state the proposition here; however, the rather technical proof is postponed to Appendix B.

PROPOSITION 5.3. *Let* $\Sigma(Q,B,f)$ *be an analytic affine control system and let* $G: M \to \mathbb{R}$ *be given. Then a curve* $\phi: I \to M$ *is formally stationary w.r.t.* Gdt *under restriction codistribution* E *(cf. Proposition* 5.2*) if and only if it is stationary w.r.t. to* Gdt *under restriction* E.

We shall now first discuss the choice of G in the case of *fixed end points, fixed time optimal control problem* (OCP). Such a problem is defined by:

OCP: 1. a nonlinear time-invariant control system $\Sigma(Q,B,f)$;

2. a *cost-function* $g\colon \mathbb{B} \to \mathbb{R}$;

3. two points $\zeta_a, \zeta_b \in B$.

We search for trajectory-inputs $\zeta\colon I \to B$ such that $\zeta(a) = \zeta_a$, $\zeta(b) = \zeta_b$ and

$$\int_I g(\zeta(t))dt$$

is minimal.

Clearly, a natural choice for $G\colon M \to \mathbb{R}$ is:

$$G(m) = g(\pi_M(m)), \quad m \in M \tag{5.8}$$

with $\pi_M\colon M \to B$ the natural projection. Each trajectory-input $\zeta = (\psi,\nu)$ is uniquely associated with an admissible curve $\phi = (\psi,\nu,\dot\nu,t)$ in M, and each admissible curve in M projects on a trajectory-input in B. So finding an optimal trajectory-input is equivalent to finding an optimal admissible curve in M.

Note that, similar to Remark 4.4, we might have added a 1-form $\beta \in E$ to $G dt$, resulting in a translation of the coordinates on the fibres of E (the Lagrange multipliers).

The final result is a compilation of the above translation of an OCP together with Theorem 3.8 and Proposition 5.3.

COROLLARY 5.4. *Let an analytic affine OCP be given as above. Then a trajectory-input ζ is a stationary solution if and only if there exists an injective immersion $\eta\colon I \to E$ (E as in Proposition 5.2) which is stationary w.r.t. the Cartan form*

$$\theta_G = \pi_E^*(G dt) + \theta_E \tag{5.9}$$

such that $\pi_M \circ \pi_E \circ \eta = \zeta$, where $\pi_M\colon M \to B$ and $\pi_E\colon E \to M$ natural projections and θ_E the canonical 1-form on T^*M restricted to E.

PROOF. Directly from the above translation of the OCP, together with Proposition 5.3 and Theorem 3.8. □

So, with use of proposition 2.3, we may conclude that $\zeta\colon I \to B$ is a stationary trajectory-input for the OCP if it is the projection of a characteristic curve of $d\theta_G$ on E. Let us work out this characterization as we did for the Lagrange problem at the end of section 4. Denote coordinates for E by $(q,u,\dot{u},\lambda,\mu,t)$ with $(q,u,\dot{u},t)$ coordinates for M and $\beta \in E$ is given by $\beta = \sum_{i=1}^{n} \lambda_i\beta_i + \sum_{j=1}^{m} \mu_j\beta_{n+j}$ (β_k cf.(5.7)). Then (5.8) and (5.9) yields

$$\theta_G = g(q,u)dt + \sum_{i=1}^{n} \lambda_i(dq_i - f_i(q,u)dt) + \sum_{j=1}^{m} \mu_j(du_j - \dot{u}_j dt).$$

Substituting $X = X_q \frac{\partial}{\partial q} + X_u \frac{\partial}{\partial u} + X_{\dot{\mu}} \frac{\partial}{\partial \dot{u}} + X_\lambda \frac{\partial}{\partial \lambda} + X_\mu \frac{\partial}{\partial \mu} + \frac{\partial}{\partial t}$

in $\iota_X d\phi_G = 0$ and collecting terms for $d\lambda_i$, dq_i du_j, $d\dot{u}_j$ and $d\mu_j$, respectively $(i=1,\dots,n, j=1,\dots,m)$ yield

(5.10) $$X_q = f(q,u),$$

(5.11) $$X_\lambda = \frac{\partial g(q,u)}{\partial q} - \left(\frac{\partial f(q,u)}{\partial q}\right)^T \lambda,$$

(5.12) $$X_\mu = \frac{\partial g(q,u)}{\partial u} - \left(\frac{\partial f(q,u)}{\partial u}\right)^T \lambda,$$

(5.13) $$\mu = 0,$$

(5.14) $$X_u = \dot{u}.$$

The equation resulting from the dt term is satisfied by substituting (5.10) up to (5.14). If $\phi(t) = (\phi^q(t), \phi^u(t), \phi^{\dot{u}}(t), \phi^\lambda(t), \phi^\mu(t), t)$ is an integral curve of X as given above (note $dt(X)=1$) then we must have

(5.15) $$\frac{d\phi^q}{dt} = f(\phi^q, \phi^u),$$

(5.16) $$\frac{d\phi^\lambda}{dt} = \frac{\partial g(\phi^q,\phi^u)}{\partial q} - \left(\frac{\partial f(\phi^q,\phi^u)}{\partial q}\right)^T \phi^\lambda,$$

(5.17) $$\phi^\mu = 0,$$

$$(5.18)\qquad \frac{\partial g(\phi^q,\phi^u)}{\partial u} - \left(\frac{\partial f(\phi^q,\phi^u)}{\partial u}\right)^T \phi^\lambda = X_\mu = \frac{d\phi^\mu}{dt} = 0,$$

$$(5.19)\qquad X_u = \frac{d\phi^u}{dt} = \phi^{\dot{u}}.$$

So a stationary curve satisfies $\phi = (\phi^q,\phi^u,\frac{d\phi^u}{dt},\phi^\lambda,0,t)$ with ϕ^q, ϕ^u, ϕ^λ satisfying (5.15), (5.16) and (5.18). Note that, with the definition

$$H(q,\lambda,u) = g(q,u) - \lambda^T f(q,u)$$

we obtain for these three equations

$$(5.20)\qquad \begin{cases} \dot{\phi}^q = -\dfrac{\partial}{\partial\lambda} H(\phi^q,\phi^\lambda,\phi^u), \\ \dot{\phi}^\lambda = \dfrac{\partial}{\partial q} H(\phi^q,\phi^\lambda,\phi^u), \\ \dfrac{\partial}{\partial u} H(\phi^q,\phi^\lambda,\phi^u) = 0, \end{cases}$$

which are the well-known equations resulting from Pontryagin's maximum principle. So we see that a trajectory-input $\zeta = (\phi^q,\phi^u)$ of a nonlinear analytic affine optimal control system is stationary, keeping end points fixed, if and only if there exists a $\phi^\lambda: I \to \mathbb{R}^n$ such that equations (5.20) are satisfied. Note that this only gives a necessary condition for optimality.

The above coordinate-dependent characterization is for illustrative purposes only. The value of corollary 5.4 is its coordinate-free description of stationary curves for OCP as characteristic curves of $d\theta_G$.

Note that we did not assume any regularity of the cost function. As in the elaboration on the Lagrange problem, such conditions come in at the moment we want to define a submanifold of M in which the solution curves lie. For instance the "(q,λ)-manifold" with solutions defined by the Hamiltonian system given by the first two equations of (5.20) for certain ϕ^u. This is obtained by choosing $\mu = 0$, so that $X_\mu = 0$. Then (5.12) can be used to solve for u if its Jacobian w.r.t. u is nonsingular.

6. CONCLUSIONS

We presented a general formulation of first order conditions for the restricted variational problem with fixed end points and applied it to the Lagrange problem and the fixed end points, fixed time optimal control problem. Moreover, it is pointed out how the free end point variational problem can be handled (see Appendix A). This opens the way to formulation of free end point or infinite-time control problems on manifolds. Future research efforts will be in that direction and in the application of these results.

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APPENDIX A: THE FREE END POINT PROBLEM

Consider the free end point variational problem of finding stationary curves $\phi\colon I \to M$, with $\phi(a) = \phi_a$ fixed, $\phi(b) \in F \subset M$ (we call F the *target set*), of the action function

$$J(\phi) = h(\phi(b)) + \int_I \phi^*\alpha,$$

for some 1-form α on M and $h\colon F \to \mathbb{R}$. We assume that F is a smooth submanifold of M of dimension $r \neq 0$ and h is smooth. To handle this problem we have to adapt the definition of variation and (formal) stationarity in order to obtain similar results.

DEFINITION A.1. A mapping $\tilde{\phi}\colon (-\delta,\delta) \times I \to M$ is called a *free end point variation* of $\phi\colon I \to M$ if (i) and (ii) of definition 2.1 are satisfied and moreover: $\tilde{\phi}(\varepsilon,a) = \phi(a)$, $\tilde{\phi}(\varepsilon,b) \in F$, $\forall\varepsilon \in (-\delta,\delta)$.

It is easily seen that we can again associate such variations with vector fields V along ϕ which now satisfy: $V(a) = 0$, $V(b) \in T_{\phi(b)}F$, the tangent space to F at $\phi(b)$. We denote the set of free end point variations of ϕ by $\mathcal{V}_\phi$.

DEFINITION A.2. A curve ϕ: $I \to M$ is *stationary* for the free end point problem with α and h if

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} [h(\phi_\varepsilon(b)) + \int_I \phi_\varepsilon^*\alpha] = 0 \qquad \forall\, \phi_\varepsilon \in \bar{V}_\phi .$$

This definition appears to be equivalent to

DEFINITION A.2'. ϕ is *stationary* for the free end point problem, if for all vector fields V along ϕ with $V(a) = 0$, $V(b) \in T_{\phi(b)}F$ we have

$$\text{(A.1)} \qquad dh(V(b)) + \int_I \phi^* L_V\alpha = 0.$$

Then we can easily prove the following characterization of stationary curves, analogous to proposition 2.3.

PROPOSITION A.3. *ϕ is stationary for the free end point problem with α and h if and only if*

(i) $\phi_*(\frac{\partial}{\partial t}) \in \ker d\alpha \ \forall t \in I$,

(ii) $(dh+\alpha)|_F(\phi(b)) = 0$,

where $|_F$ *denotes restriction of this form to* F.

PROOF. Sufficiency is trivial. If ϕ is stationary, then (A.1) with Stokes theorem yields

$$\text{(A.2)} \qquad 0 = (dh+\alpha)(V(b)) + \int_I \phi^* \iota_V d\alpha$$

for arbitrary V along ϕ ($V(a) = 0, V(b) \in T_{\phi(b)}F$). Suppose $\phi_*(\frac{\partial}{\partial t}) \notin \ker d\alpha$ for some $t \in I$. Then by the smoothness we can construct a V along ϕ with $V(a) = 0$, $V(b) = 0 \in T_{\phi(b)}F$ with $\int_I \phi^*\iota_V d\alpha \neq 0$. This contradicts (A.2). So $\phi_*(\frac{\partial}{\partial t}) \in \ker d\alpha$ $(t \in I)$. Then (A.2) implies (ii). □

Condition (i) is the same condition as in proposition 2.3. Condition (ii) is the so-called *transversality condition* in the endpoint.
Definition of (formal) stationarity under restriction distribution is clear. We use definition A.2 but restrict ϕ_ε to admissable variations, or we use definition A.2' with V such that $\phi^* L_V\beta = 0 \ \forall \beta \in E$. Finally we get the

following version of theorem 3.8.

PROPOSITION A.4. *An injective immersion* ϕ: I $\to$ M *is formally stationary for the free end point problem with 1-form* α, *end cost* h *and target manifold* F, *under restriction* S, *if and only if there exists an injective immersion* η: I $\to$ $E(=S^{\perp})$ *with* $\eta(t) \in \pi_E^{-1}(\phi(t))$ *and* η *stationary for the free end point problem with Cartan form* θ_α, *end cost* $h_E = h\circ\pi_E$ *and target manifold* $F_E = \{e \in E | \pi_E(e) \in F\}$.

PROOF. The prove goes along the same lines as the proof of theorem 3.8. There is a slight difference where we use Stokes theorem in the definition of C_0. Here we choose C_0 such that

$$\text{(A.3)} \qquad dh(V(b)) + \int_I \phi^* L_V \alpha = \int_I \phi^* \iota_{V_2} d\alpha + C_0 \psi(b),$$

which is fine for $\psi(b) \neq 0$. If $\psi(b) = 0$, then $V_2(b) = \psi(b)Z(b) = 0 \in T_{\phi(b)}F$ and as $V = V_2 + V_1 \in T_{\phi(b)}F$ we also have $V_1(b) \in T_{\phi(b)}F$. Then formal stationarity with $V_2(b) = 0$ yields

$$dh(V(b)) + \int_I \phi^* L_V \alpha = (dh+\alpha)(V_1(b)) + (dh+\alpha)(V_2(b)) + \int_I \phi^* \iota_{V_1} d\alpha + \int_I \phi^* \iota_{V_2} d\alpha = \int_I \phi^* \iota_{V_2} d\alpha,$$

so that (A.3) is also satisfied if $\psi(b) = 0$ for arbitrary choice of C_0. For the rest there is no essential difference with the proof of theorem 3.8. □

APPENDIX B: PROOF OF PROPOSITION 5.3

For ease of notation we write $\alpha = Gdt$ and $I = [0,1]$. Let ϕ: I $\to$ M be given locally in coordinates for M as in proposition 5.2:

$$\phi(t) = (\phi^q(t), \phi^u(t), \phi^{\dot{u}}(t), t),$$

with $\phi^*\beta = 0 \ \forall \beta \in E$. Recall notations (3.2) and (3.5). We shall show that, symbolically, (B.1) $$\mathcal{W}_\phi^E = \mathcal{V}_\phi^E + O(\varepsilon^2),$$

i.e. every formal variation can be written as an order ε^2 perturbation of a variation. The proof then follows immediately from:

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_I \phi_\varepsilon^* \alpha = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_I (\overline{\phi}_\varepsilon + O(\varepsilon^2))^* \alpha = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_I \overline{\phi}_\varepsilon^* \alpha = 0,$$

for $\phi_\varepsilon \in \mathcal{W}_\phi^E$ with $\phi_\varepsilon = \overline{\phi}_\varepsilon + O(\varepsilon^2)$ and $\overline{\phi}_\varepsilon \in \mathcal{V}_\phi^E$.
To prove (B.1) write $\xi \in \mathcal{W}_\phi^E$:

$$\xi(\varepsilon,t) = (\xi^q(\varepsilon,t), \xi^u(\varepsilon,t), \xi^{\dot{u}}(\varepsilon,t), t).$$

Then we have, as E is spanned by forms (5.7):

(B.2) $$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left(\frac{\partial}{\partial t}\xi^q(\varepsilon,t) - f(\xi^q(\varepsilon,t), \xi^u(\varepsilon,t))\right) = 0$$

(B.3) $$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left(\frac{\partial}{\partial t}\xi^u(\varepsilon,t) - \xi^{\dot{u}}(\varepsilon,t)\right) = 0.$$

From (B.3), together with $\frac{\partial}{\partial t}\,\xi^u(0,t) - \xi^{\dot{u}}(0,t) = \frac{\partial}{\partial t}\,\phi^u(t) - \phi^{\dot{u}}(t) = 0$ by admissibility of ϕ, we obtain:

(B.4) $$\xi^{\dot{u}}(\varepsilon,t) = \frac{\partial}{\partial t}\,\xi^u(\varepsilon,t) + C^{\dot{u}}(\varepsilon,t),$$

with $C^{\dot{u}}(\varepsilon,t) = O(\varepsilon^2)$ for $\varepsilon \to 0$. Denote

(B.5) $$\eta(t) = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \xi^q(\varepsilon,t); \quad \mu(t) = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \xi^u(\varepsilon,t).$$

Then (B.2) yields

$$\dot{\eta}(t) = \frac{\partial f}{\partial q}(\phi^q(t), \phi^u(t)).\ \eta(t) + \frac{\partial f}{\partial u}(\phi^q(t), \phi^u(t)).\ \mu(t)$$

(B.6)

$$\eta(0) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \xi^q(\varepsilon,0) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi^q(0) = 0.$$

The homogeneous part of (B.6) is the linear *equation of variations* associated with the solution ϕ^q of the nonlinear equation $\dot{q} = f(q,\phi^u)$ (see ARNOLD [1978]). In fact it represents a vector field on TQ. If γ denotes the flow of the nonlinear vector field and $\gamma(t)q_0$ is a solution with initial point q_0, then the solution of the homogeneous part of (B.5) is given by

$$\eta(t) = \gamma(t)_*(q_0)(\eta(0)).$$

Note that $\eta(t) \in T_{\gamma(t)q_0}Q$. Substituting (5.3) we obtain, by using the variation of constants formula, for the solution of the linear inhomogeneous equation:

$$\eta(t) = \sum_{i=1}^{m} \int_0^t \gamma(t-\sigma)_* B_i(\phi^q(\sigma))\mu_i(\sigma)d\sigma. \tag{B.7}$$

Note that $\eta(1) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \xi^q(\varepsilon,1) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi^q(1) = 0.$

Taylor's theorem with (B.5) and $\xi(0,t) = \phi(t)$ yields

$$\begin{aligned} \xi^q(\varepsilon,t) &= \phi^q(t) + \varepsilon\eta(t) + \overline{C}^q(\varepsilon,t), \\ \xi^u(\varepsilon,t) &= \phi^u(t) + \varepsilon\mu(t) + \overline{C}^u(\varepsilon,t), \end{aligned} \tag{B.8}$$

where $\overline{C}^q(\varepsilon,t)$ and $\overline{C}^u(\varepsilon,t)$ are $0(\varepsilon^2)$ for $\varepsilon \to 0$.
Now consider, for arbitrary $C_i^u(\varepsilon,t) = 0(\varepsilon^2)$, the equation on I:

$$\begin{aligned} \dot{q}(t) &= A(q(t)) + \sum_{i=1}^{m} B_i(q(t))\phi_i^u(t) \\ &\quad + \sum_{i=1}^{m} B_i(q(t))(\varepsilon\mu_i(t)+C_i^u(\varepsilon,t)), \\ q(0) &= \phi^q(0)(\overset{\nabla}{=}q_0). \end{aligned} \tag{B.9}$$

Then for $\varepsilon = 0$ $\phi^q(t)$ is a solution, as ϕ^u, ϕ^q is a trajectory-input of the system. Then we know (see BROCKETT [1976, thm 6] and CROUCH [1981]) that for $\varepsilon\mu(t) + C^u(\varepsilon,t)$ small enough any solution of (B.9) can be written as a unique uniform convergent Volterra series:

$$\zeta(\varepsilon,t) = W^0(t)(q_0) + \sum_{i_1=1}^{m} \int_0^t W^1_{i_1}(t,\sigma_1)(q_0)(\varepsilon\mu_{i_1}(\sigma_1) + C^u_{i_1}(\varepsilon,\sigma_1))d\sigma_1$$

(B.10)
$$+ \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \int_0^t \int_0^{\sigma_1} W^2_{i_1 i_2}(t,\sigma_1,\sigma_2)(q_0)(\varepsilon\mu_{i_1}(\sigma_1) +$$

$$+ C^u_{i_1}(\varepsilon,\sigma_1))(\varepsilon\mu_{i_2}(\sigma_2) + C^u_{i_2}(\varepsilon,\sigma_2))d\sigma_1 d\sigma_2 + \dots .$$

The kernels are defined as follows. Let $\tilde{\gamma}$ denote the flow for $\varepsilon = 0$ (note that $\tilde{\gamma}(t)q_0 = \gamma(t)q_0$ and γ as in (B.7)). Define vector fields $B_i(\sigma)$ on Q by

(B.11) $$B_i(\sigma)(q) = \tilde{\gamma}(-\sigma)_* B_i(\tilde{\gamma}(\sigma)q).$$

Then the definition is by recursion

$$W^0(t)(q) = h \circ \tilde{\gamma}(t)(q),$$

$$W^k_{i_1 \dots i_k}(t,\sigma_1,\dots,\sigma_k)(q) = B_{i_k}(\sigma_k) W^{k-1}_{i_1 \dots i_{k-1}}(t,\sigma_1,\dots,\sigma_{k-1})(q),$$

where h denotes the coordinate function $h\colon Q \to \mathbb{R}^n$. We see that $W^k_{i_1 \dots i_k}$ $(k=0,\dots)$ are $\mathbb{R}^n$ valued functions, and (B.10) gives the solution in coordinates. As $\phi^q(t)$ is a solution for $\varepsilon = 0$ we must have

$$W^0(t)(q_0) = \phi^q(t).$$

Furthermore,

$$\sum_{i=1}^{m} \int_0^t W^1_i(t,\sigma)(q_0)(\varepsilon\mu_i(\sigma))d\sigma =$$

$$\varepsilon \sum_{i=1}^{m} \int_0^t \tilde{\gamma}(-\sigma)_* B_i(\tilde{\gamma}(\sigma)q_0) W^0(t)(q)\big|_{q=q_0} \mu_i(\sigma)d\sigma =$$

$$\varepsilon \sum_{i=1}^{m} \int_0^t \gamma(t-\sigma)_* B_i(\phi^q(\sigma))\mu_i(\sigma)d\sigma = \varepsilon\eta(t).$$

So (B.10) gives

(B.12)
$$\zeta(\varepsilon,t) = \overset{q}{\phi}(t)+\varepsilon\eta(t) + \sum_{i_1=1}^{m} \int_0^1 W^1_{i_1}(t,\sigma_1)(q_0)\overset{u}{C}_{i_1}(\varepsilon,\sigma_1)d\sigma_1$$

$$+ \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \int_0^t \int_0^{\sigma_1} W^2_{i_1 i_2}(t,\sigma_1,\sigma_2)(q_0)(\varepsilon\mu_{i_1}(\sigma_1)+C^u_{i_1}(\varepsilon,\sigma_1))$$

$$(\varepsilon\mu_{i_2}(\sigma_2)+C^u_{i_2}(\varepsilon,\sigma_2))d\sigma_1 d\sigma_2 + \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \sum_{i_3=1}^{m} \cdots .$$

Now assume that we choose $C^u_i(\varepsilon,\sigma)$ analytic:

$$C^u_i(\varepsilon,\sigma) = \sum_{k=2}^{\infty} c_{ik}(\sigma)\varepsilon^k .$$

Then we can collect terms in ε^2:

$$\varepsilon^2 \sum_{i_1=1}^{m} \int_0^t \Big(W^1_{i_1}(t,\sigma_1)(q_0)c_{i_1 2}(\sigma_1\; d\sigma_1$$

$$+ \mu_{i_1}(\sigma_1) \sum_{i_2=1}^{m} \int_0^{\sigma_1} W^2_{i_1 i_2}(t,\sigma_1,\sigma_2)(q_0)\mu_{i_2}(\sigma_2)d\sigma_2\Big)d\sigma_1 .$$

We can choose $c_{i_1 2}(\sigma)(\sigma\epsilon I)(i_1=1,\dots,m)$ such that this expression equals zero for $t \in I$. Moreover, as $\mu_{i_1}(\sigma_1)$ appears in the second term and $\mu_{i_1}(a) = \mu_{i_1}(b) = 0$ we have $c_{i_1 2}(a) = c_{i_1 2}(b) = 0$, $i = 1,\dots,m$. Similarly, we can choose c_{i3}, c_{i4}, ... such that all nonlinear terms in ε in equation (B.12) vanish, with $c_{ik}(a) = c_{ik}(b) = 0$, $k = 3,4,\dots$.
So the choice

(B.13)
$$\overline{\xi}^q(\varepsilon,t) = \phi^q(t) + \varepsilon\eta(t),$$
$$\overline{\xi}^u(\varepsilon,t) = \phi^u(t) + \varepsilon\mu(t) + C^u(\varepsilon,t),$$

satisfies the system equation for all ε small enough. Moreover, as $C^u(\varepsilon,t) = O(\varepsilon^2)$ and $C^u(\varepsilon,a) = C^u(\varepsilon,b) = 0$ it is easily seen that for $\overline{\xi}$ defined by

$$\overline{\xi}(\varepsilon,t) = (\overline{\xi}^q(\varepsilon,t),\overline{\xi}^u(\varepsilon,t),\frac{\partial}{\partial t}\,\overline{\xi}^u(\varepsilon,t),t),$$

we have $\overline{\xi} \in \mathcal{W}_\phi^E$. Therefore, using (B.8) yields

$$\xi(\varepsilon,t) = \overline{\xi}(\varepsilon,t) + O(\varepsilon^2),$$

which proves (B.1).

<u>REMARK</u>. The restriction to affine systems does not seem to be essential. All arguments, including the Volterra series solution can be given for general nonlinear systems too. The restriction to analytic systems is essential for the method of proof as otherwise the Volterra series does not have to converge. We conjecture however that even in that case the theorem is valid.