stichting
mathematisch
centrum

AFDELING MATHEMATISCHE STATISTIEK SW 88/82 SEPTEMBER
(DEPARTMENT OF MATHEMATICAL STATISTICS)

W.R. VAN ZWET

RANKS AND ORDER STATISTICS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: Primary: 62E20
Secondary: 62F05, 62F12

Ranks and order statistics *)

by

W.R. van Zwet

ABSTRACT

In this paper it is shown that under very general conditions, asymptotic normality of a two-sample linear rank statistic under a fixed alternative follows from asymptotic normality of an appropriate linear function of order statistics.

KEY WORDS & PHRASES: *asymptotic normality, two-sample ranktests, fixed alternatives, linear functions of order statistics*

---

*) This report will be submitted for publication elsewhere

1. INTRODUCTION.

In their famous 1958 paper, Chernoff and Savage [4] proved the asymptotic normality of linear rank statistics for the two-sample problem under fixed alternatives. Such asymptotic normality proofs had been given before, but the degree of generality in this paper far surpassed these earlier efforts. The result was obtained for scores generated by a very smooth function J on $(0,1)$ of controlled growth near 0 and 1 and for almost any fixed alternative. Classes of alternatives for which the convergence to normality is uniform were also investigated, thus extending the result to sequences of alternatives within such a class. The paper validated the normal approximation and the computation of asymptotic efficiencies for most two-sample rank statistics that one is likely to come across. It also struck terror into the hearts of graduate students at the time because of - what was then considered - its extreme technicality; in order to approximate the rank statistic by a sum of independent random variables no fewer than six remainder terms were shown to tend to zero, each for its own particular reason. Unfortunately, the number of such remainder terms has increased monotonically over the years and nowadays authors in this area appear to need at least fifteen.

It is hard to overestimate the influence of the Chernoff-Savage paper. It started a steady stream of research resulting in a voluminous literature on the asymptotics of rank statistics. Many extensions to more general and more complicated rank tests were obtained and at the same time technical refinements have led to improved conditions. Even though contiguity arguments later took over part of the field, work along Chernoff-Savage lines is continuing to the present day.

Another one of Herman Chernoff's contributions to asymptotic statistics - and one that was almost as influential - is the 1967 paper by Chernoff, Gastwirth and Johns [3] on the asymptotic normality of linear functions of order statistics (LFO's). For uniform order statistics $U_{1:N} < U_{2:N} < \ldots < U_{N:N}$ and weights $a_{j,N}$ generated by a function J on $(0,1)$, they prove asymptotic normality of $\sum a_{j,N}\, \psi(U_{j:N})$ for smooth ψ and under growth conditions on both J and ψ. The result provided normal approximations and an asymptotic theory for linear estimators. The proof is based on transforming to exponential order statistics and exploiting their very special structure. However, the authors point out that an alternative approach based on the methods of Chernoff and Savage would also have been possible.

The research on the asymptotics of LFO's that was initiated by the Chernoff-Gastwirth-Johns paper, is again quite substantial. Various techniques have been applied and have led to different and gradually improved sets of conditions and the end of this process doesn't yet seem to be in sight. It is interesting that there is a trade-off between assumptions on J and on ψ ; one can either assume very little about J but a lot about ψ , or the other way round.

When one looks at the literature on the asymptotic normality of rank statistics and of LFO's, one can't help noticing a striking similarity of the techniques employed in the two areas. It was noted above that the Chernoff-Savage method is applicable to the study of LFO's too, but the similarity doesn't end there. Almost any technical device that has worked in one area, has worked for the other problem also. When viewing the research in the two areas, the image of two armies marching on parallel roads readily comes to mind. But this raises the further question whether perhaps the two seemingly very different problems of proving asymptotic normality for rank statistics and for LFO's, are essentially the same or at least more intimately connected than one would think at first sight. Or, in terms of our admittedly fanciful image: are the two armies perhaps going to the same place?

Basically, I believe they are. In this paper it will be shown that under very general conditions, asymptotic normality of a two-sample linear rank statistic under a fixed alternative follows from asymptotic normality of an appropriate LFO. Since the possibility of a result in the other direction won't even be considered in this paper, I can't possibly claim to have shown that the two problems are the same, but only that they are intimately related. Like everything else in this area, the proof of the result is highly technical; a part of it that can't possibly be of general interest, will be left to the interested reader with appropriate hints being given in the appendix. Problems concerning the uniformity of the convergence to normality of the rank statistic will be avoided by restricting attention to a single fixed alternative and a rapidly converging sample ratio. Finally, I should perhaps make it clear that I'm not advocating that one should use the result of this paper to prove asymptotic normality for a rank statistic by first proving it for the corresponding LFO. What motivates the result is not its possible application, but only the light it may throw on the connection between the two problems.

2. THE RESULT.

Let F and G be two continuous distribution functions (d.f.'s) on the real line with densities f and g. For $N = 1,2,\ldots$, consider independent random variables $X_{1,N},X_{2,N},\ldots,X_{N,N}$ and assume that $X_{1,N},\ldots,X_{m_N,N}$ have common d.f. F and $X_{m_N+1,N},\ldots,X_{N,N}$ have common d.f. G. This is the two-sample situation with sample sizes m_N and $n_N = N - m_N$; $\lambda_N = n_N/N$ indicates the relative size of the second sample. Let $X_{1:N} < X_{2:N} <\ldots< X_{N:N}$ denote the combined sample $X_{1,N},\ldots,X_{N,N}$ arranged in increasing order and define the antiranks $D_{1,N},\ldots,D_{N,N}$ by $X_{D_{j,N},N} = X_{j:N}$.

The random variables

$$V_{j,N} = \begin{cases} 1 & \text{if } m_N + 1 \le D_{j,N} \le N \\ 0 & \text{otherwise} \end{cases}$$

indicate from which sample each of the ordered sample elements originates. For real numbers $a_{1,N},\ldots,a_{N,N}$ called scores, the two-sample linear rank statistic is defined by

$$(2.1) \qquad T_N = \sum_{j=1}^{N} a_{j,N} V_{j,N} .$$

Suppose that $\lambda_N \to \lambda \in (0,1)$ as $N \to \infty$ and define

$$(2.2) \qquad H = \lambda G + (1-\lambda)F , \quad h = \lambda g + (1-\lambda)f ,$$

$$(2.3) \qquad \psi = \frac{\lambda g \circ H^{-1}}{h \circ H^{-1}} ,$$

$$(2.4) \qquad \pi_{j,N} = N \cdot \int_{(j-1)/N}^{j/N} \psi(x)dx ,$$

$$(2.5) \qquad \bar{a}_N = \frac{\sum_{j=1}^{N} \pi_{j,N}(1-\pi_{j,N})a_{j,N}}{\sum_{j=1}^{N} \pi_{j,N}(1-\pi_{j,N})} ,$$

$$(2.6) \qquad \tau_N^2 = \frac{1}{N} \sum_{j=1}^{N} \pi_{j,N}(1-\pi_{j,N})(a_{j,N}-\bar{a}_N)^2 .$$

Let $U_1,U_2,\ldots$ be independent and identically distributed random variables with a common uniform distribution on $(0,1)$ and let $U_{1:N} < U_{2:N} <\ldots< U_{N:N}$ denote the order statistics corresponding to $U_1,\ldots,U_N$. Define

$$(2.7) \qquad \begin{aligned} L_N &= \sum_{j=1}^{N} a_{j,N}\, \psi(U_{j:N}) , \\ \hat{L}_N &= \sum_{j=1}^{N} E(L_N|U_j) - (N-1)\, E\, L_N , \end{aligned}$$

and note that $\hat{L}_N$ is the L_2 - projection of L_N . Finally, let 1_A denote the indicator of a set A , let $\sigma^2(Y)$ denote the variance of a random variable Y and let $\overset{\mathcal{D}}{\to} N(0,1)$ denote convergence in distribution to the standard normal.

THEOREM 2.1.

Assume that $\lambda \in (0,1)$ *and* $\delta > 0$ *exist such that*

$$(2.8) \qquad \lim_{N\to\infty} N^{\frac{1}{2}}(\lambda_N - \lambda) = 0 ,$$

$$(2.9) \qquad \liminf_N \tau_N^2 > 0 ,$$

$$(2.10) \qquad \limsup_N \frac{1}{N} \sum_{j=1}^{N} |a_{j,N}|^{2+\delta} < \infty .$$

If $\sigma^2(N^{-\frac{1}{2}}L_N)$ *is bounded and*

$$(2.11) \qquad \frac{L_N - EL_N}{\sigma(\hat{L}_N)} \overset{\mathcal{D}}{\to} N(0,1) ,$$

then there exists a bounded sequence of positive numbers $\sigma_N \geq \tau_N$ *such that*

$$(2.12) \qquad \frac{T_N - EL_N}{N^{\frac{1}{2}}\sigma_N} \overset{\mathcal{D}}{\to} N(0,1) .$$

Some brief comments on assumptions (2.8) - (2.10) may be in order. First of all, (2.8) ensures an almost constant sample-ratio $\lambda_N = \lambda + o(N^{-\frac{1}{2}})$ and together with the fact that F and G are fixed, this prevents uniformity problems. Another important aspect is that λ_N remains bounded away from 0 and 1 as $N \to \infty$. Without this, both $N^{-\frac{1}{2}}T_N$ and $N^{-\frac{1}{2}}L_N$ will degenerate as $N \to \infty$ and technical complications arise. Assumptions (2.9) and (2.10) together ensure that the scores $a_{j,N}$ are roughly of the order of 1 as $N \to \infty$, which is merely a norming convention. Apart from this, (2.9) prevents a more general kind of degeneration of $N^{-\frac{1}{2}}T_N$ which would occur if the $V_{j,N}$ would degenerate for certain indices j , whereas the scores $a_{j,N}$ would be almost constant for the remaining indices; of course (2.9) implies that $\lambda \in (0,1)$ in (2.8). Assumption (2.10) controls the growth of the scores.

3. PROOF.

Recall (2.1) - (2.7) and define in addition

$$(3.1) \qquad P_{j,N} = \psi(U_{j:N}) \ , \quad P = P_N = (P_{1,N},\ldots,P_{N,N}) \ ,$$

$$(3.2) \qquad \omega(P) = N^{-\frac{1}{2}} \sum_{j=1}^{N} (P_{j,N}-\lambda_N) \ ,$$

$$(3.3) \qquad \sigma^2(P) = \frac{1}{N} \sum_{j=1}^{N} P_{j,N}(1-P_{j,N}) \ ,$$

$$(3.4) \qquad \bar{a}(P) = \frac{\sum_{j=1}^{N} P_{j,N}(1-P_{j,N})a_{j,N}}{\sum_{j=1}^{N} P_{j,N}(1-P_{j,N})} \ ,$$

$$(3.5) \qquad \tau^2(P) = \frac{1}{N} \sum_{j=1}^{N} P_{j,N}(1-P_{j,N})(a_{j,N}-\bar{a}(P))^2 \ .$$

The following lemma will be the starting point of the proof.

LEMMA 3.1.

If (2.8) *and* (2.10) *are satisfied, then for every positive* δ *and* ε ,

$$E \exp\{it\, N^{-\frac{1}{2}}T_N\} = E \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{\sigma(P)} \exp\left\{ -\frac{\omega^2(P)}{2\sigma^2(P)} + \right.$$

$$(3.6) \qquad \left. - \tfrac{1}{2}t^2\tau^2(P) - it\,\omega(P)\,\bar{a}(P) + it\, N^{-\frac{1}{2}}\sum a_{j,N}P_{j,N} \right\} \cdot 1_{\{\tau^2(P)\geq\varepsilon\}} +$$

$$+ O(N^{\frac{1}{2}}P(\sigma^2(P)<\delta) + P(\tau^2(P)<\varepsilon)) + o(1)$$

as $N \to \infty$.

This lemma may be proved by modifying an argument in Bickel and Van Zwet [2]. Since this is a highly technical matter, the interested reader is referred to the appendix for details.

Define

(3.7) $$S_N = \sum_{j=1}^{N} (\psi(U_j) - \lambda) ,$$

(3.8) $$\sigma_0^2 = \lambda(1-\lambda) \int_{-\infty}^{\infty} \frac{f(x)g(x)}{h(x)} dx ,$$

and let $\pi_{j,N}$, $\bar{a}_N$ and τ_N^2 be given by (2.4) – (2.6). The next lemma is needed to simplify (3.6).

LEMMA 3.2.

If (2.8) - (2.10) *are satisfied, then the following statements hold with probability* 1:

(3.9) $$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} |P_{j,N} - \pi_{j,N}| = 0 ,$$

(3.10) $$\lim_{N\to\infty} |\omega(P) - N^{-\frac{1}{2}} S_N| = 0 ,$$

(3.11) $$\lim_{N\to\infty} \sigma^2(P) = \sigma_0^2 > 0 ,$$

(3.12) $$\lim_{N\to\infty} |\bar{a}(P) - \bar{a}_N| = 0$$

(3.13) $$\lim_{N\to\infty} |\tau^2(P) - \tau_N^2| = 0 .$$

Proof.

Define the function ψ_N on $[0,1)$ by

(3.14) $\psi_N(t) = \psi(U_{j:N})$ for $\frac{j-1}{N} \le t < \frac{j}{N}$,

for $j = 1,\ldots,N$. Lemma 2.1 in Van Zwet [5] ensures that with probability 1 (w.p.1) ψ_N converges to ψ in Lebesgue measure. As $|\psi_N|$ and $|\psi|$ are bounded by 1 and

$$\frac{1}{N}\sum_{j=1}^{N} |P_{j,N}-\pi_{j,N}| \le \int_0^1 |\psi_N(x) - \psi(x)|dx ,$$

(3.9) follows. This implies that

$$\lim_{N\to\infty} |\sigma^2(P) - \frac{1}{N}\sum_{j=1}^{N} \pi_{j,N}(1-\pi_{j,N})| = 0$$

w.p.1. On the other hand, the strong law yields

$$\lim_{N\to\infty} \sigma^2(P) = \int_0^1 \psi(x)(1-\psi(x))dx = \sigma_0^2$$

w.p.1. Hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \pi_{j,N}(1-\pi_{j,N}) = \sigma_0^2$$

and since (2.9) and (2.10) imply that the left-hand side is positive, (3.11) is proved. Because $\sigma_0^2 > 0$, (2.10) yields (3.12) and another application of (2.10) proves (3.13). As (3.10) is an immediate consequence of (2.8), the proof of the lemma is complete. □

Assumption (2.10) implies that on the set where $\tau^2(P) \ge \varepsilon$, $\sigma^2(P)$ is also bounded away from zero. Hence the expression following the expectation sign on the right in (3.6) is bounded and we may replace $\omega(P)$, $\sigma(P)$, $\bar{a}(P)$ and $\tau(P)$ in this expression by $N^{-\frac{1}{2}}S_N$, σ_0 , $\bar{a}_N$ and τ_N , provided only that (2.8) - (2.10) hold. Take $\delta = \frac{1}{2}\sigma_0^2 > 0$ and $\varepsilon = \frac{1}{2} \liminf \tau_N^2 > 0$. Because of (3.11) and (3.13), combined with the fact that $\sigma^2(P)$ is a mean of independent, identically distributed and bounded random variables, one finds that

$$N^{\frac{1}{2}} P(\sigma^2(P) < \delta) + P(\tau^2(P) < \varepsilon) = o(1)$$

as $N \to \infty$. Hence lemmas 3.1 and 3.2 together yield

LEMMA 3.3.

If (2.8) - (2.10) *are satisfied, then, as* $N \to \infty$,

$$\phi_N(t) = E \exp\{itN^{-\frac{1}{2}}(T_N - EL_N)\} = \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{\sigma_0} \exp\{-\tfrac{1}{2}\tau_N^2 t^2\} \cdot$$

(3.15)

$$\cdot E \exp\left\{ - \frac{S_N^2}{2\sigma_0^2 N} - it\bar{a}_N N^{-\frac{1}{2}} S_N + itN^{-\frac{1}{2}}(L_N - EL_N)\right\} + o(1) .$$

The next step is to establish the asymptotic normality of the projection $\hat{L}_N$ of L_N .

LEMMA 3.4.

If (2.10) *is satisfied and* $\liminf_N \sigma^2(N^{-\frac{1}{2}}\hat{L}_N) > 0$, *then*

$$(3.16) \quad \frac{\hat{L}_N - EL_N}{\sigma(\hat{L}_N)} \overset{\mathcal{D}}{\to} N(0,1) .$$

Proof.

A straightforward calculation shows that

$$(3.17) \quad \hat{L}_N - EL_N = \sum_{i=1}^{N} \{\alpha_N(U_i) - \beta_N(U_i) + \gamma_N(U_i)\} \quad \text{where}$$

$$\alpha_N(u) = \sum_{j=1}^{N-1} \frac{N}{N-j} a_{j,N} E\, \psi(U_{j:N-1})(U_{j:N-1} - \tfrac{j}{N})\, 1_{\{U_{j:N-1} \le u\}} ,$$

$$\beta_N(u) = \sum_{j=1}^{N-1} \frac{N}{j} a_{j+1,N} E\, \psi(U_{j:N-1})(U_{j:N-1} - \tfrac{j}{N})\, 1_{\{U_{j:N-1} > u\}} ,$$

$$\gamma_N(u) = \psi(u) \sum_{j=1}^{N} a_{j,N} \binom{N-1}{j-1} u^{j-1}(1-u)^{N-j} .$$

Because $|\psi| \le 1$ and $E\, U_{j:N-1} = j/N$,

$$|\alpha_N(u)| \le \left|\sum_{j=1}^{N-1} \frac{N}{N-j} a_{j,N}\, E\,\psi(U_{j:N-1})\, E(U_{j:N-1} - \frac{j}{N})\, 1_{\{U_{j:N-1}\le u\}}\right| +$$

$$+ \left|\sum_{j=1}^{N-1} \frac{N}{N-j} a_{j,N}\, E\{\psi(U_{j:N-1}) - E\,\psi(U_{j:N-1})\}(U_{j:N-1} - \frac{j}{N})\, 1_{\{U_{j:N-1}\le u\}}\right| \le$$

$$\le \sum_{j=1}^{N-1} \frac{N}{N-j}|a_{j,N}| \left|E(U_{j:N-1} - \frac{j}{N})\, 1_{\{U_{j:N-1}\le u\}}\right| +$$

$$+ \sum_{j=1}^{N-1} \frac{N}{N-j}|a_{j:N}|\,\sigma(\psi(U_{j:N-1}))\,\sigma(U_{j:N-1}) \,.$$

Now $\sigma^2(U_{j:N-1}) = j(N-j)/\{N^2(N+1)\}$ and by lemma A2.1 in Albers, Bickel and Van Zwet [1]

$$P(\sigma^{-1}(U_{j:N})|U_{j:N} - \frac{j}{N+1}| \ge t) \le 2\exp\left\{-\frac{3t^2}{6t+8}\right\}$$

for all $j = 1,\ldots,N$, $N = 1,2,\ldots$ and $t \ge 0$. It follows that

$$\left|E(U_{j:N-1} - \frac{j}{N})\, 1_{\{U_{j:N-1}\le u\}}\right| \le C\left(\frac{j(N-j)}{N^3}\right)^{\frac12} \exp\left\{-\alpha\frac{|u-j/N|}{\{j(N-j)\}^{\frac12}} N^{3/2}\right\}$$

for positive constants C and α . Hence

$$|\alpha_N(u)| \le C\sum_{j=1}^{N-1}|a_{j,N}|\left(\frac{j}{N(N-j)}\right)^{\frac12} \exp\left\{-\alpha\frac{|u-j/N|}{\{j(N-j)\}^{\frac12}} N^{3/2}\right\} +$$

$$+ \sum_{j=1}^{N-1}|a_{j,N}|\left(\frac{j}{N(N-j)}\right)^{\frac12}\sigma(\psi(U_{j:N-1})) \,.$$

Take $\delta > 0$ as in assumption (2.10). There exist positive numbers $p \le 2+\delta$, $q < 2$ and $r > 2$ such that $p^{-1} + q^{-1} + r^{-1} = 1$ and repeated use of Hölder's inequality yields

$$|\alpha_N(u)| \le C\, N^{-\frac12}\left(\sum_{j=1}^{N-1}|a_{j,N}|^p\right)^{1/p}\left(\sum_{j=1}^{N-1}\left(\frac{j}{N-j}\right)^{\frac12 q}\right)^{1/q} \cdot$$

$$\cdot\left[\left(\sum_{j=1}^{N-1}\exp\{-2\alpha r N^{\frac12}|u-j/N|\}\right)^{1/r} + \left(\sum_{j=1}^{N-1}\sigma^r(\psi(U_{j:N-1}))\right)^{1/r}\right] .$$

For ψ_N as in (3.14), we can argue as in the proof of lemma 3.2 to find that, as $N \to \infty$,

$$\frac{1}{N}\sum_{j=1}^{N-1} \sigma^r(\psi(U_{j:N-1})) \le E\int_0^1 |\psi_{N-1}(x) - \psi(x)|^r dx = o(1)$$

because ψ_N and ψ are bounded and ψ_N converges to ψ in Lebesgue measure with probability 1. Bounding the other sums by the corresponding integrals or by using (2.10) we arrive at

$$\sup_{0<u<1} |\alpha_N(u)| = \mathcal{O}(N^{-\frac{1}{2}+p^{-1}+q^{-1}})\{\mathcal{O}(N^{\frac{1}{2}r^{-1}}) + o(N^{r^{-1}})\} = o(N^{\frac{1}{2}})$$

as $N \to \infty$. Similarly, $\sup|\beta_N(u)| = o(N^{\frac{1}{2}})$ and $\sup|\gamma_N(u)| \le \max_j |a_{j,N}| = o(N^{\frac{1}{2}})$ in view of (2.10). Since $\sigma(N^{-\frac{1}{2}}\hat{L}_N)$ is bounded away from zero, (3.17) and the central limit theorem yield (3.16). □

The next lemma deals with the asymptotic equivalence of L_N and $\hat{L}_N$.

LEMMA 3.5.

Suppose that $\liminf_N \sigma^2(N^{-\frac{1}{2}}\hat{L}_N) > 0$ *and* $\sigma^2(N^{-\frac{1}{2}}L_N)$ *is bounded and that* (2.10) *and* (2.11) *hold. Then* $\sigma^{-1}(\hat{L}_N)(L_N-\hat{L}_N)$ *tends to zero in probability as* $N \to \infty$.

Proof.

Write $Z_N = \sigma^{-1}(\hat{L}_N)(L_N-EL_N)$ and $\hat{Z}_N = \sigma^{-1}(\hat{L}_N)(\hat{L}_N-EL_N)$. The conditions of lemma 3.4 are satisfied so that we may assume that

$$E\, Z_N = 0\ ,\quad E\, Z_N^2 \le C\ ,\quad Z_N \xrightarrow{\mathcal{D}} N(0,1)\ ,$$

$$E\, \hat{Z}_N = 0\ ,\quad E\, \hat{Z}_N^2 = 1\ ,\quad \hat{Z}_N \xrightarrow{\mathcal{D}} N(0,1)$$

for a positive constant C . We have to show that $\Delta_N = Z_N - \hat{Z}_N$ converges to zero in probability.

Because $\hat{Z}_N$ is the projection of Z_N we have $E\,\hat{Z}_N\Delta_N = 0$ for every N . Moreover, $E\,\Delta_N^2 \le E\,Z_N^2 \le C$ and since $E\,\hat{Z}_N^2 = 1$ and $\hat{Z}_N \xrightarrow{\mathcal{D}} N(0,1)$, the sequence $\{\hat{Z}_N^2\}$ is uniformly integrable. This implies that the sequence

$\{\hat{Z}_N\Delta_N\}$ is uniformly integrable.

The sequence of joint distributions of $(Z_N, \hat{Z}_N, \Delta_N)$ is clearly tight. Take any weakly converging subsequence

$$(Z_{N_k}, \hat{Z}_{N_k}, \Delta_{N_k}) \xrightarrow{\mathcal{D}} (Z, \hat{Z}, \Delta) \text{ , say.}$$

Obviously $Z = \hat{Z} + \Delta$ with probability 1, $E\, Z^2 = E\, \hat{Z}^2 = 1$ and as $\{\hat{Z}_N\Delta_N\}$ is uniformly integrable with $E\, \hat{Z}_N\Delta_N = 0$, we have $E\, \hat{Z}\Delta = 0$. It follows that $E\, \Delta^2 = E\, Z^2 - E\, \hat{Z}^2 = 0$ so that $\Delta_{N_k} \xrightarrow{\mathcal{D}} 0$. Hence $\Delta_N \xrightarrow{\mathcal{D}} 0$ and the lemma is proved. □

We are now in a position to prove the theorem.

<u>Proof of Theorem 2.1.</u>

Assume first that $\liminf \sigma^2(N^{-\frac{1}{2}}\hat{L}_N) > 0$ so that the conclusion of lemma 3.5 holds. In view of (3.7), (3.17), the boundedness of ψ and the proof of lemma 3.4, we see that

(3.18) $$\lim_{N\to\infty}[P(N^{-\frac{1}{2}}S_N \le x, N^{-\frac{1}{2}}(L_N - EL_N) \le y) - P(A_N \le x, B_N \le y)] = 0$$

for all x and y, where A_N and B_N are jointly normally distributed with $E\, A_N = E\, B_N = 0$,

(3.19) $$E\, A_N^2 = \int_0^1 \psi^2(x)dx - \lambda^2 = \lambda(1-\lambda) - \sigma_0^2 \text{ ,}$$

(3.20) $$E\, B_N^2 = \int_0^1 \{\alpha_N(x) - \beta_N(x) + \gamma_N(x)\}^2 dx \text{ ,}$$

(3.21) $$E\, A_N B_N = \int_0^1 \psi(x)\{\alpha_N(x) - \beta_N(x) + \gamma_N(x)\}dx \text{ .}$$

This is still true without the assumption that $\liminf \sigma^2(N^{-\frac{1}{2}}L_N) > 0$, the only difference being that (3.20) is now not necessarily bounded away from 0 for large N. To see this, note that if $\sigma^2(N^{-\frac{1}{2}}\hat{L}_N)$ (or any sub-sequence) tends to zero, then $N^{-\frac{1}{2}}(L_N - EL_N)$ (or its sub-sequence) will tend to zero in probability because of (2.11).

Since $\sigma_0^2 > 0$, assumption (2.10) ensures that $\bar{a}_N$ is bounded. It follows from lemma 3.3 that, for $N \to \infty$ and for every fixed t ,

$$(3.22)\quad \phi_N(t) = \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{\sigma_0} \exp\{-\tfrac{1}{2}\tau_N^2 t^2\} \cdot$$

$$\cdot\ E \exp\left\{-\frac{A_N^2}{2\sigma_0^2} - it\ \bar{a}_N A_N + it\ B_N\right\} + o(1)$$

$$= \exp\{-\tfrac{1}{2}\sigma_N^2 t^2\} + o(1) ,$$

where $\sigma_N \geq \tau_N$ and hence $\liminf \sigma_N > 0$ by (2.9). As $E\,B_N^2 = \sigma^2(N^{-\frac{1}{2}}\hat{L}_N) \leq \sigma^2(N^{-\frac{1}{2}}L_N)$ and $\sigma^2(N^{-\frac{1}{2}}L_N)$ is bounded, it is easy to see that $\{\sigma_N\}$ is bounded. This completes the proof of the theorem. □

APPENDIX

In this appendix we indicate how lemma 3.1 may be obtained by modifying an argument in Bickel and Van Zwet [2]. When referring to numbered formulas, lemmas etc. in that paper, we shall add an asterisk to avoid confusion; thus (2.10*) and lemma 2.3* refer to (2.10) and lemma 2.3 in Bickel and Van Zwet [2].

Since λ as defined in (2.5*) is the same as $\lambda_N = n_N/N$ in the present paper, we have to be careful to replace λ by λ_N in formulas such as (2.5*), (2.10*) and (2.14*). However, in (2.4*) we don't make this substitution, thus in effect replacing λ_N by its limiting value λ ; note, however, that (2.3*) and (2.6*) remain valid. Similarly, at the beginning of section 3* we don't replace λ by λ_N in the definitions of H and h , thus making them coincide with definition (2.2) in the present paper. It is easy to check that lemma 3.1* remains valid with these modifications, i.e.

$$(A.1)\quad E \exp\{it\ N^{-\frac{1}{2}}T_N\} = \frac{E_H \nu(t,P)\exp\{it\ N^{-\frac{1}{2}}\sum a_j P_j\}}{2\pi\ N^{\frac{1}{2}}\ B_{N,n}(\lambda)} .$$

Next we need to establish conditions under which for every fixed t , (A.2) $$\nu(t,p) = \frac{(2\pi)^{\frac{1}{2}}}{\sigma(p)} \exp\left\{ -\frac{\omega^2(p)}{2\sigma^2(p)} - \frac{\tau^2(p)t^2}{2} - i\,\omega(p)\bar{a}(p)t\right\} + \mathcal{O}(1) .$$

This is a weaker version of the conclusion of lemma 2.3* for which assumptions (2.21*) and (2.22*) would be needed. In the first place it is weaker because we are not concerned with values of $|t|$ tending to infinity with N, and therefore we can dispense with assumption (2.22*). Secondly, we don't need the asymptotic expansion for $\nu(t,p)$ established in lemma 2.3*, but only its leading term (A.2) and inspection of the proof of the lemma reveals that (2.21*) may be replaced by assumption (2.10) in the present paper and

(A.3) $$\tau^2(p) \geq \varepsilon \qquad \text{for some } \varepsilon > 0 .$$

Together, (2.10) and (A.3) guarantee the validity of (A.2).

If $\tau^2(p) < \varepsilon$, we can bound $|\nu(t,p)|$ as follows

$$|\nu(t,p)| \leq |\nu(0,p)| \leq 2\pi\cdot\min(N^{\frac{1}{2}},\sigma^{-1}(p)) .$$

To see this, note that $|\rho(t,p)| \leq 1$ and $0 \leq c(p) \leq 1$ in (2.13*), that the first inequality in (2.24*) yields $|\psi(s,0,p)| \leq \exp\{-[\frac{1}{2}-(\pi^2/24)]\sigma^2(p)s^2\}$ for $|s| \leq \pi N^{\frac{1}{2}}$, and apply (2.11*). Hence, for positive ε and δ,

(A.4) $$\begin{aligned} E_H|\nu(t,P)|\, 1_{\{\tau^2(P)<\varepsilon\}} &\leq E_H|\nu(t,P)|\, 1_{\{\sigma^2(P)<\delta\}} + \\ &+ E_H|\nu(t,P)|\, 1_{\{\tau^2(P)<\varepsilon,\sigma^2(P)\geq\delta\}} \leq \\ &\leq 2\pi N^{\frac{1}{2}} P_H(\sigma^2(P)<\delta) + 2\pi\delta^{-\frac{1}{2}} P_H(\tau^2(P)<\varepsilon) . \end{aligned}$$

Assumption (2.8) of the present paper ensures that

(A.5) $$\lim_{N\to\infty} 2\pi N^{\frac{1}{2}} B_{N,n}(\lambda) = \left[\frac{2\pi}{\lambda(1-\lambda)}\right]^{\frac{1}{2}} \in (0,\infty)$$

and, combining (A.1), (A.2), (A.4) and (A.5) we arrive at

$$\begin{aligned}
&E \exp\{it\, N^{-\frac{1}{2}}T_N\} = E_H \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{\sigma(P)} \exp\Bigg\{ - \frac{\omega^2(P)}{2\sigma^2(P)} + \\
\text{(A.6)}\quad & - \tfrac{1}{2}t^2\tau^2(P) - it\, \omega(P)\bar{a}(P) + it\, N^{-\frac{1}{2}}\sum a_j P_j \Bigg\} \cdot 1_{\{\tau^2(P)\geq\varepsilon\}} + \\
&+ O(N^{\frac{1}{2}}P_H(\sigma^2(P)<\delta) + P_H(\tau^2(P)<\varepsilon)) + o(1) ,
\end{aligned}$$

as $N \to \infty$, for every positive ε and δ . The assumptions needed to prove this are (2.8) and (2.10).

Finally we note that (2.4^*) as modified above, implies that under H the vector $P = (P_1,\ldots,P_N)$ is distributed as $(\psi(U_{1:N}),\ldots,\psi(U_{N:N}))$ where ψ is given by (2.3) and $(U_{1:N},\ldots,U_{N:N})$ are order statistics of a sample of size N from the uniform distribution on (0,1). Substituting this in (A.6) we obtain (3.6).

REFERENCES

[1] Albers, W., Bickel, P.J. and Van Zwet, W.R. (1976). Asymptotic expansions for the power of distributionfree tests in the one-sample problem. *Ann. Statist.* 4, 108-156.

[2] Bickel, P.J. and Van Zwet, W.R. (1978). Asymptotic expansions for the power of distributionfree tests in the two-sample problem. *Ann. Statist.* 6, 937-1004.

[3] Chernoff, H., Gastwirth, J.L. and Johns, M.V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation. *Ann. Math. Statist.* 38, 52-72.

[4] Chernoff, H. and Savage, I.R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* 29, 972-994.

[5] Van Zwet, W.R. (1980). A strong law for linear functions of order statistics. *Ann. Probability* 8, 986-990.

36266