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HOPF BIFURCATION FOR VOLTERRA CONVOLUTION EQUATIONS

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Hopf bifurcation for Volterra convolution equations *)

by

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ABSTRACT

In this paper we discuss Hopf bifurcation for nonlinear Volterra integral equations of convolution type. Starting point is a semiflow associated with the equation and acting on a space of compactly supported forcing functions.

KEY WORDS & PHRASES: *Volterra integral equation, convolution type, dynamical system, nonlinear, variation-of-constants formula, center manifold, Hopf bifurcation*

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1. INTRODUCTION

Mathematical models from population dynamics and epidemiology often lead to (systems of) integral equations of the type

$$x(t) = \int_0^1 B(\tau)g(x(t-\tau))d\tau.$$

See, for instance, [1,2,4,5,8,10].

In general, one first looks for constant solutions. In the hyperbolic case the stability character of such a constant solution can be deduced from the linearized equation and therefore from the position of the roots of a characteristic equation in the complex plane. Frequently, g and/or B depend on parameters (describing the biological population or external conditions) and roots of the characteristic equation cross the imaginary axis when such parameters are varied [2,4,10]. When a pair of conjugated roots crosses the imaginary axis, one expects to find the bifurcation of periodic solutions [2,3,5,6,7].

In the following we shall give a standard *dynamical* description of this so called Hopf bifurcation starting from a somewhat unusual semigroup construction. It will appear that, by choosing a space of forcing functions as the state space, certain technical difficulties are avoided (which could arise because we deal with integral equations and not with integro-differential equations). We use a variation-of-constants formula to construct a center manifold. On this finite dimensional manifold the flow is governed by an o.d.e. and so we are at familiar grounds. The intimate relation between the direction of bifurcation and stability has motivated us to derive a formula for the direction in terms of computable quantities. Complete proofs will be published elsewhere [3].

2. PRELIMINARIES

In the following let Ω be an open subset of $\mathbb{R}$ and let $B = B(\mu,\tau)$ be a mapping of $\Omega \times \mathbb{R}_+$ into $\mathbb{R}^{n\times n}$ such that:

H_B(i) supp(B) is contained in $\Omega \times [0,1]$,

H_B(ii) for each $\mu \in \Omega$, $B(\mu,\cdot) \in L^2(\mathbb{R}_+;\mathbb{R}^{n\times n})$,

H_B(iii) the mapping $\mu \mapsto \int_0^\sigma B(\mu,\tau)d\tau$, $\sigma \in [0,1]$ is a C^k-smooth ($k\geq 1$) mapping of Ω into $NBV([0,1]; \mathbb{R}^{n\times n})$.

Furthermore, let $g \in C^k(\Omega\times\mathbb{R}^n;\mathbb{R}^n)$ be such that

H_g(i) $g(\mu,x) = x + r(\mu,x)$,

H_g(ii) $r(\mu,0) = 0$, $r_x(\mu,0) = 0$ and $|r_x(\mu,x)| \leq M < \infty$.

The condition on g guarantees that $x \equiv 0$ is a solution of

$$(2.1) \qquad x(t) = \int_0^1 B(\mu,\tau)g(\mu,x(t-\tau))d\tau.$$

In the linear case, i.e. $g(\mu,x) = x$, the mapping $t \mapsto e^{\lambda t}$ is a solution if and only if λ is a zero of det $\Delta(\mu,\lambda)$, where Δ is the characteristic function

$$(2.2) \qquad \Delta(\mu,\lambda) = I - \int_0^1 e^{-\lambda\tau} B(\mu,\tau)d\tau.$$

We assume that det $\Delta(\mu_0,\pm i\omega) = 0$ for some $\mu_o \in \Omega$ and $\omega > 0$. In terms of the characteristic function we express in H_B(iv) a criterion for simple eigenvalues, in H_B(v) a non-resonance condition and in H_B(vi) a transversality condition (see section 5).

H_B(iv) There exist a column n-vector $p \neq 0$ and a row n-vector $q \neq 0$ such that

(α) $\Delta(\mu_o,i\omega)v = 0$ implies $v = cp$ for some $c \in \mathbb{C}$,

(β) $w\Delta(\mu_o,i\omega) = 0$ implies $w = cq$ for some $c \in \mathbb{C}$,

(γ) $q \frac{\partial}{\partial\lambda} \Delta(\mu_o,i\omega)p = 1$.

H_B(v) det $\Delta(\mu_o,0) \neq 0$ and det $\Delta(\mu_o,\pm\ell\omega) \neq 0$ for $\ell = 2,3,\ldots$.

H_B(vi) $q \frac{\partial}{\partial\mu} \Delta(\mu_o,i\omega)p \neq 0$.

3. DEFINITION OF THE SEMIGROUP

Following Miller [9] we define below the action of a semigroup on forcing functions by means of translations of the solution. Since the kernel B has bounded support we can choose as our state space

$$X = \{f \in C(\mathbb{R}_+;\mathbb{R}^n) \mid f(t) = 0,\ t \geq 1\},$$
$$\|f\|_X = \sup_{[0,1]} |f(t)|$$

which is, in some sense, as small as possible. This is an important point which reflects that we want to study an autonomous problem. We define, for $s \geq 0$, $S(s)f$ by the relation

$$x_s(t) = \int_0^t B(\mu,\tau)g(\mu,x_s(t-\tau))d\tau + S(s)f(t).$$

Here $x_s(t) = x(s+t)$, $t \geq 0$. Note that $S(s)f$ is indeed an element of X and that there is an obvious way to return from the forcing data into $\mathbb{R}^n$, the solution space:

$x(s) = \alpha(S(s)f)$, where α is the bounded linear operator of X into $\mathbb{R}^n$ defined by $\alpha(f) = f(0)$.

THEOREM 3.1. *The mapping* $s \mapsto S(s)f$ *defines a strongly continuous semigroup of continuous (nonlinear) operators on* X.

In the linear case ($g(\mu,x) = x$) the above semigroup consists of bounded linear operators and will be denoted by $\{T(s)\}$.

THEOREM 3.2. *The infinitesimal generator* A *of* $\{T(s)\}$ *is characterized by*

$$\mathcal{D}(A) = \{f \in X \mid f' \in L^2 \ \&\ f'(\cdot) + B(\mu,\cdot)\alpha(f) \in X\},$$

$$(Af)(t) = f'(t) + B(\mu,t)\alpha(f).$$

The closed operator A has compact resolvent and

$$\sigma(A) = P_\sigma(A) = \{\lambda \mid \det \Delta(\mu,\lambda) = 0\}.$$

The last identity is a consequence of the fact that elements of X have bounded support. On account of H_B(iv) a (finite!) number of eigenvalues of A are on the imaginary axis at $\mu = \mu_0$. We decompose the state space X into $X_- \oplus X_o \oplus X_+$; the corresponding projection operators are P_-, P_o and P_+. Both X_o and X_+ are finite dimensional spaces. On these subspaces T(s) can be extended to a (differentiable) group. X_+ (X_-) consists of those elements that decay under T(s) at μ_0 exponentially with exponent $\gamma_+ - \varepsilon$ ($\gamma_-+\varepsilon$) as $s \to -\infty$ ($s\to\infty$). Here $\gamma_+ = \inf\{\lambda \in \sigma(A) \mid \mathrm{Re}(\lambda) > 0\}$, $\gamma_- = \sup\{\lambda \in \sigma(A) \mid \mathrm{Re}(\lambda) < 0\}$ and ε is some small positive number; $\gamma_+ \neq 0$, $\gamma_- \neq 0$. X_o consists of those elements of X that are exponentially bounded under T(s) at μ_o for $s \in \mathbb{R}$ with exponent ε.

4. THE VARIATION-OF-CONSTANTS FORMULA

For fixed μ, T(s)f is the Fréchet derivative of S(s)f at f = 0. This follows from the variation-of-constants formula

$$(4.1) \qquad S(s)f = T(s)f + \int_0^s T(s-\tau)B(\mu,\cdot)r(\mu,\alpha(S(\tau)f))d\tau.$$

Here we use that T(s) extends to L^2-functions and that the integration with respect to τ produces a continuous function again. Let us denote S(s)f by F(s) and T at μ_o by T_o. Formal differentiation of (3.1) yields

$$\frac{dF(s)}{ds} = A_\mu F(s) + B(\mu,\cdot)r(\mu,\alpha(F(s))).$$

Using Theorem 3.2 we infer that

$$\frac{dF(s)}{ds} = A_{\mu_o}F(s)+(B(\mu,\cdot)-B(\mu_o,\cdot))g(\mu,\alpha(F(s)))+B(\mu_o,\cdot)r(\mu,\alpha(F(s))).$$

Integrating again we find

$$(4.2)\qquad F(s) = T_o(s-\sigma)F(\sigma) + \int_\sigma^s T_o(s-\tau)\{(B(\mu,\cdot)-B(\mu_o,\cdot))g(\mu,\alpha(F(\tau))) + B(\mu_o,\cdot)r(\mu,\alpha(F(\tau)))\}d\tau.$$

This motivates

THEOREM 4.1. *Let* $\sigma \in \mathbb{R}$ *and* $F(\sigma) \in X$ *be given. Let* $x: [\sigma,\infty) \to \mathbb{R}^n$ *denote the unique solution of*

$$(4.3)\qquad x_\sigma = B(\mu,\cdot) * g(\mu,x_\sigma) + F(\sigma).$$

Define $F: [\sigma,\infty) \to X$ *by*

$$x_s = B(\mu,\cdot) * g(\mu,x_s) + F(s).$$

Then F *satisfies* (4.2). *Conversely if* $F: [\sigma,\infty) \to X$ *is a continuous function which satisfies* (4.2) *then* $x(s) = \alpha(F(s))$ *satisfies* (4.3).

5. THE CENTER MANIFOLD

We are interested in small solutions of equation (2.1) and therefore it is appropriate to study solutions of the modified equation

$$(5.1)\qquad x(t) = \int_0^1 B(\mu_o,\tau)x(t-\tau)d\tau + \int_0^1 \{(B(\mu_o,\tau)-B(\mu,\tau))\tilde{g}(\mu,x(t-\tau))+B(\mu_o,\tau)\tilde{r}(\mu,x(t-\tau))\}d\tau,$$

where $\tilde{g}$ and $\tilde{r}$ are obtained from g and r by truncation outside a suitably small ball. Equally well we may look for solutions of $(\widetilde{4.2})$ which is (4.2) with g and r replaced by the truncated functions.

THEOREM 5.1. *Fix* η *in the interval* $(0,\min\{-\gamma_-,\gamma_+\})$. *There exist a neighbourhood* Ω_o *of* μ_o *in* Ω *and a continuous function* C *of* $\Omega_o \times X_o$ *into* X *such that*

(i) $C(\mu,\phi) = F^*(\mu,\phi)(0)$ *where* $F^*(\mu,\phi)(s)$ *is the unique solution of* $(\widetilde{4.2})$ *such that*

  (α) $P_oF^*(\mu,\phi)(0) = \phi$,

  (β) $\sup_{s\in\mathbb{R}} e^{-\eta|s|}\|F^*(\mu,\phi)(s)\|_X < \infty$,

(ii) $\alpha(F^*(\mu,\phi))$, *which is the corresponding solution of* (5.1) *is* C^k*-smooth with respect to* (μ,ϕ). *We will denote this solution by* $x^*(\mu,\phi)$.

(iii) $F^*(\mu,\phi)(s) = C(P_o(F^*(\mu,\phi)(s)))$, *the invariance property,*

(iv) ImC *is tangent to* X_o *at zero for* $\mu = \mu_o$, *i.e.* $\frac{\partial C}{\partial \phi}(\mu_o,0)\psi = \psi$.

REMARK. H_B(iii) includes situations where the delay is the parameter. This hypothesis is too weak to guarantee the differentiability of the solution F^* in the state space, but fortunately the solution in $\mathbb{R}^n$, x^*, is differentiable.

6. HOPF BIFURCATION

All small periodic solutions (that exist for all time!) lie on the center manifold. The flow on this finite dimensional manifold is governed by an o.d.e. Define $y(s) = P_o F^*(\mu,\phi)(s)$, then locally near zero we find

$$\frac{dy}{ds} = Ay(s) + P_o\{(B(\mu,\cdot)-B(\mu_o,\cdot))g(\mu,\alpha C(\mu,y(s))) + B(\mu_o,\cdot)r(\mu,\alpha C(\mu,y(s)))\}. \tag{6.1}$$

It is known [3] that H_B(iv)-(vi) represent a criterion for simple eigenvalues, satisfying a non-resonance and a transversality condition respectively.

THEOREM 6.1. *Assume* $k \geq 2$. *Then there exist* C^1 *functions* $\mu^*(\varepsilon),\phi^*(\varepsilon),\rho^*(\varepsilon)$ *(with values in* Ω, X_o *and* $\mathbb{R}$ *respectively and defined for* ε *sufficiently small) such that* $\mu^*(0) = \mu_o$ *and* $\rho^*(0) = 2\pi\omega^{-1}$ *and such that* $x^*(\mu^*(\varepsilon),\phi^*(\varepsilon)) = \alpha C(\mu^*(\varepsilon),P_oF^*(\mu^*(\varepsilon),\phi^*(\varepsilon))(\cdot))$ *is a* $\rho^*(\varepsilon)$*-periodic solution of the equation* (2.1) *with* $\mu = \mu^*(\varepsilon)$. *Moreover, if* x *is any small periodic solution of this equation with* μ *close to* μ_o *and period close to* $2\pi\omega^{-1}$ *then* $\mu = \mu^*(\varepsilon)$, *the period is* $\rho^*(\varepsilon)$ *and modulo translation* $x = x^*(\mu^*(\varepsilon),\phi^*(\varepsilon))$; μ^* *and* ρ^* *are even functions of* ε *and* $x^*(\mu^*(\varepsilon),\phi^*(-\varepsilon))(t) = x^*(\mu^*(\varepsilon),\phi^*(\varepsilon))(t+\tfrac{1}{2}\rho^*(\varepsilon))$.

If we assume some more smoothness, i.e. $k \geq 3$, we are able to derive a Taylor expansion up to and including order two. Generically the expression below will not vanish and so determines the direction of bifurcation.

THEOREM 6.2. *Assume* $k \geq 3$, *then*

$$\mu^*(\varepsilon) = \frac{\mathrm{Re}\ c_1}{\mathrm{Re}\ q\,\frac{\partial}{\partial\mu}\Delta(\mu_o,i\omega)p}\,\varepsilon^2 + o(\varepsilon^2),$$

where

$$c_1 = \tfrac{1}{2}q \frac{\partial^3 r}{\partial x^3}(\mu_o,0)(p^2,\bar{p}) +$$
$$q \frac{\partial^2 r}{\partial x^2}(\mu_o,0)(p,(\Delta(\mu_o,0)^{-1}-I)\frac{\partial^2 r}{\partial x^2}(\mu_o,0)(p,\bar{p})) +$$
$$\tfrac{1}{2}q \frac{\partial^2 r}{\partial x^2}(\mu_o,0)(\bar{p},(\Delta(\mu_o,2i\omega)^{-1}-I)\frac{\partial^2 r}{\partial x^2}(\mu_o,0)p^2).$$

Finally we state that the Principle of Exchange of Stability holds also in this case.

THEOREM 6.3. *Assume* $k \geq 3$ *and assume in addition that at* $\mu = \mu_o$

(i) $X_+ = \{0\}$,

(ii) $-q \frac{\partial}{\partial \mu} \Delta(\mu_o, i\omega)p > 0$,

(iii) $\pm i\omega$ *are the only eigenvalues on the imaginary axis,*

(iv) $\mathrm{Re}\ c_1 \neq 0$.

Then the bifurcating periodic solution is asymptotically orbitally stable with asymptotic phase if and only if $\mu^*(\varepsilon) > 0$ *for small* $\varepsilon > 0$.

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