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H.J.J. TE RIELE

ON GENERATING NEW AMICABLE PAIRS FROM GIVEN AMICABLE PAIRS

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On generating new amicable pairs from given amicable pairs*)

by

Herman J.J. te Riele

ABSTRACT

Rules are given for constructing new amicable pairs from given amicable pairs. By applying these rules to 1575 "mother" pairs known to the author, 1782 new amicable pairs were generated, so that the average "offspring" of these mother pairs is greater than 1. By a second, much more restricted application of the rules to the 400 smallest "daughter" pairs, 88 new "granddaughter" amicable pairs were generated.

KEY WORDS & PHRASES: *amicable pair*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

An amicable pair is a pair of positive integers (m_1,m_2), $m_1 \neq m_2$, such that $\sigma(m_1) = \sigma(m_2) = m_1 + m_2$, where $\sigma(\cdot)$ denotes the sum of divisors function. These number pairs have a long and interesting history (cf. [5]). Euler ([4]) was the first who systematically studied amicable pairs, and a great part of the known pairs were found with his methods and the use of electronic computers.

It is believed that there are infinitely many amicable pairs, although this has never been proved. In this note we present rules for finding new amicable pairs from given pairs. By applying these rules to a large list of known pairs, we have generated an average "offspring" of *more than one* "daughter" pair per given "mother" pair. We consider this result as strong numerical evidence for the existence of infinitely many amicable pairs.

In Section 2 we present one very simple rule, and two rules derived from it, by which we have generated 1782 new amicable pairs from a list of 1575 known mother pairs. These mother pairs were collected from lists of Lee and Madachy ([5], 1108 pairs), Costello ([3], 7 pairs), Borho ([1], 3 pairs), Woods ([10], 456 pairs) and te Riele ([7], 1 pair). By a second (more restricted, cf. Section 3) application of the rules to the 400 smallest pairs in the list of 1782 daughter pairs mentioned above, still 88 more new "granddaughter" amicable pairs were generated. All these 1870 new pairs are explicitly given in [8], including their prime factorizations. It should be noted that other rules for finding new amicable pairs from known pairs are already known (so-called Thabit-rules, see [1]), but up till now relatively few new pairs have actually been found with these rules (cf. [1,2,6]).

In Section 3 we give details of our computations, including some information about the mother pairs from which the largest numbers of daughter pairs were generated.

This work was inspired by the following observations. By inspecting the Lee & Madachy-list we found that there are certain pairs $(m_1^{(i)}, m_2^{(i)})$, $i = 1,2,\dots,t$, $t \geq 2$, which share some common part, i.e., for which a pair of numbers (a_1,a_2) exists with $a_1 > 1$ and $a_2 > 1$ such that a_1 is a divisor of $m_1^{(i)}$ with $\gcd(a_1,m_1^{(i)}/a_1) = 1$ and a_2 is a divisor of $m_2^{(i)}$ with $\gcd(a_2,m_2^{(i)}/a_2) = 1$, for $i = 1,2,\dots,t$. For example, the five pairs (wich are numbered 1, 3, 4, 9 and 24, respectively, in [5]):

$$\begin{cases}220 = 2^2 5.11\\ 284 = 2^2 71\end{cases} \begin{cases}2620 = 2^2 5.131\\ 2924 = 2^2 17.43\end{cases} \begin{cases}5020 = 2^2 5.251\\ 5564 = 2^2 13.107\end{cases}$$

$$\begin{cases}63020 = 2^2 5.23.137\\ 76084 = 2^2 23.827\end{cases} \begin{cases}308620 = 2^2 5.13.1187\\ 389924 = 2^2 43.2267\end{cases}$$

share the common part $(a_1,a_2) = (2^2 5,2^2)$. Other examples are the five pairs numbered 6, 42, 90, 126 and 442 in [5] which share $(2^3 17,2^3 23)$ and the eight pairs numbered 8, 19, 20, 29, 121, 213, 260 and 282 in [5] which share $(2^4 23,2^4)$.

Notation. We write $\bar{n}$ for $\sigma(n)$ and $\tilde{n}$ for $\sigma(n) - n$.

2. THE RULES

RULE 1 (to find an amicable pair $(a_1p_1,a_2q_1q_2)$)
Find a solution (q_1,q_2) *of the bilinear Diophantine equation*

$$(2.1) \qquad Dq_1q_2 - \tilde{a}_1\bar{a}_2(q_1+q_2) = a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2, \; D = a_1a_2 - \tilde{a}_1\tilde{a}_2,$$

for which both q_1 *and* q_2 *are primes,* $q_1 \neq q_2$, *and* $\gcd(a_2,q_1q_2) = 1$. *For such a solution, compute* p_1 *from*

$$(2.2) \qquad \tilde{a}_1p_1 = a_2q_1q_2 - \bar{a}_1.$$

If p_1 *is prime and* $\gcd(a_1,p_1) = 1$ *then* $(a_1p_1,a_2q_1q_2)$ *is an amicable pair.* □

In fact, this rule is well-known (cf. [5, p. 81, formulas (12)-(15)]), but the difficulty is to find numbers a_1 and a_2 for which q_1, q_2 and p_1 are *integral. Now we choose* a_1 *and* a_2 *as follows.* Let (m_1,m_2) be a *known* amicable pair and write it as (a_1p,a_2q) where p and q are primes and $\gcd(a_1,p) = \gcd(a_2,q) = 1$. By using the definition of an amicable pair, one easily verifies that p and q can be expressed in terms of a_1 and a_2 as follows:

$$(2.3) \qquad p = D^{-1}(\tilde{a}_2\bar{a}_1+a_2\bar{a}_2), \; q = D^{-1}(a_1\bar{a}_1+\tilde{a}_1\bar{a}_2), \text{ where D is defined in (2.1).}$$

So we have that D is a divisor of $a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2$. Moreover, experiments have shown, that often D is also a divisor of $\tilde{a}_1\bar{a}_2$, or at least that D and $\tilde{a}_1\bar{a}_2$ have a large common divisor. This means that for this choice of a_1 and a_2 the coefficient of q_1q_2 in (2.1) can be made 1 (or at least a small integer > 1) by dividing (2.1) by the greatest common divisor of D, $\tilde{a}_1\bar{a}_2$ and $a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2$. This is a favorable situation for the existence of *integral* solutions q_1 and q_2. Moreover, it turns out that often p_1 defined by (2.2) is integral when q_1 and q_2 are integral.

EXAMPLE If we choose $(a_1,a_2) = (2^25,2^2)$ from the smallest known amicable pair (see Section 1), then we have $D = 14$, $\tilde{a}_1\bar{a}_2 = 154 = 11\cdot 14$ and $a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2 = 994 = 71\cdot 14$. Eq. (2.1) reads $q_1q_2 - 11(q_1+q_2) = 71$, or, equivalently, $(q_1-11)(q_2-11) = 192 = 2^63$. The solutions $(q_1,q_2) = (17,43)$ and $(13,107)$ give $p_1 = 131$ and $p_1 = 251$, respectively, so that with this rule we have generated the (known) pairs numbered 3 and 4 in [5] (cf. Section 1). □

The following two rules are generalizations of Rule 1, simply obtained by replacing a_1 by a_1p_2 (to give Rule 2) and a_2 by a_2q_3 (to give Rule 3). Here, p_2 resp. q_3 are "suitably" chosen primes (cf. (2.4) resp. (2.7) below).

RULE 2 (to find an amicable pair $(a_1p_1p_2, a_2q_1q_2)$)
Choose a prime p_2 *such that* $\gcd(a_1,p_2) = 1$ *and*

$$(2.4) \qquad Dp_2 - \tilde{a}_2\bar{a}_1 > 0.$$

Find a solution (q_1,q_2) *of the bilinear Diophantine equation*

$$(2.5) \qquad (Dp_2-\tilde{a}_2\bar{a}_1)q_1q_2 - (\tilde{a}_1p_2+\bar{a}_1)\bar{a}_2(q_1+q_2) = a_1\bar{a}_1p_2^2 + (a_1\bar{a}_1+\tilde{a}_1\bar{a}_2)p_2 + \bar{a}_1\bar{a}_2$$

for which both q_1 *and* q_2 *are primes,* $q_1 \neq q_2$, *and* $\gcd(a_2,q_1q_2) = 1$. *For such a solution compute* p_1 *from*

$$(2.6) \qquad (\tilde{a}_1p_2+\bar{a}_1)p_1 = a_2q_1q_2 - \bar{a}_1(p_2+1).$$

If p_1 *is prime,* $p_1 \neq p_2$, *and* $\gcd(a_1,p_1) = 1$ *then* $(a_1p_1p_2, a_2q_1q_2)$ *is an amicable pair.* □

<u>RULE 3</u> (to find an amicable pair $(a_1p_1,a_2q_1q_2q_3)$)
Choose a prime q_3 *such that* $\gcd(a_2,q_3) = 1$ *and*

$$(2.7) \qquad Dq_3 - \tilde{a}_1\bar{a}_2 > 0.$$

Find a solution (q_1,q_2) *of the bilinear Diophantine equation*

$$(2.8) \qquad (Dq_3-\tilde{a}_1a_2)q_1q_2 - \tilde{a}_1\bar{a}_2(q_3+1)(q_1+q_2) = a_1\bar{a}_1 + \tilde{a}_1\bar{a}_2(q_3+1)$$

for which both q_1 *and* q_2 *are primes,* $q_1 \neq q_2$, $q_1 \neq q_3$, $q_2 \neq q_3$, *and* $\gcd(a_2,q_1q_2) = 1$. *For such a solution compute* p_1 *from*

$$(2.9) \qquad \tilde{a}_1p_1 = a_2q_1q_2q_3 - \bar{a}_1.$$

If p_1 *is prime and* $\gcd(a_1,p_1) = 1$ *then* $(a_1p_1,a_2q_1q_1q_3)$ *is an amicable pair.* □

Of crucial importance in Rules 2 and 3 is the choice of the primes p_2 resp. q_3. As with (2.1), p_2 and q_3 should be chosen such that the coefficient of q_1q_2 in (2.5) resp. (2.8) is as small as possible (preferably 1) after dividing by the greatest common divisor of the three coefficients.

3. COMPUTATIONAL DETAILS

We have applied Rules 1-3 to the 1575 mother amicable pairs mentioned in the introduction. For each of these pairs (m_1,m_2) the following computations were carried out. We computed $g := \gcd(m_1,m_2)$ and the quotients m_1/g and m_2/g. From these quotients all possible pairs of primes (p,q) were selected such that $(m_1,m_2) = (a_1p,a_2q)$ with $\gcd(a_1,p) = \gcd(a_2,q) = 1$. To all the pairs (a_1,a_2) (and (a_2,a_1)) obtained in this way, we applied Rules 1, 2 and 3 with the following restrictions. Let the bilinear equations (2.1), (2.5) and (2.8) be written as

$$(2.10) \qquad c_1q_1q_2 - c_2(q_1+q_2) = c_3,$$

with $\gcd(c_1,c_2,c_3) = 1$. This is equivalent to

$$(c_1q_1-c_2)(c_1q_2-c_2) = c_1c_3 + c_2^2 =: C. \tag{2.10'}$$

In those cases where C was smaller than 10^{25}, C was completely factorized and for all possible products $C = B_1B_2$ the corresponding numbers $q_i = c_1^{-1}(c_2+B_i)$, $i = 1,2$, were computed, checked on being integral, checked on primality, etc. In Rules 2 and 3 we chose for p_2 resp. q_3 successively the smallest ten primes satisfying (2.4) resp. (2.7), in order to retain c_1 in (2.10) as small as possible.

In this way we have generated 1782 new amicable pairs from 1575 mother pairs, namely 170 with Rule 1, 1523 with Rule 2 and 89 with Rule 3. Of course, also (relatively few) *known* pairs were generated. The smallest new pairs found with Rules 1, 2 and 3 are, respectively:

$$\begin{cases} 114944072 = 2^3 17.19.44483 \\ 125269528 = 2^3 53.439.673 \end{cases} \text{with Rule 1 from} \begin{cases} 726104 = 2^3 17.19.281 \\ 796696 = 2^3 53.1879 \end{cases} ([5,\#37]),$$

$$\begin{cases} 323401712 = 2^4 23.431.2039 \\ 332270608 = 2^4 101.127.1619 \end{cases} \text{with Rule 2 from} \begin{cases} 176272 = 2^4 23.479 \\ 180848 = 2^4 89.127 \end{cases} ([5,\#20]),$$

$$\begin{cases} 1699746184 = 2^3 23.41.233.967 \\ 1725115256 = 2^3 17.12684671 \end{cases} \text{with Rule 3 from} \begin{cases} 10744 = 2^3 17.79 \\ 10856 = 2^3 23.59 \end{cases} ([5,\#6]).$$

The largest pair found is the pair of 38-digit numbers:

$$\begin{cases} 84939420717490497547044056177577599145 = E.359.40939.44296620189660299 \\ 85084703583907612558249457600803200855 = E.911.119883499.5971269576509, \end{cases}$$

where $E = 3^6 5.19.23.137.547.1093$, generated with Rule 2 from the pair (E.359.144779,E.911.57149), which is pair # 1083 of [5].

From 533 of the 1575 mother pairs we actually generated one or more new daughter pairs. The three "champion" mother pairs are:

$$\begin{cases} 77306245632044 = 2^2 11.13.47.6829.421079 \\ 85036870195156 = 2^2 11.1932656140799 \end{cases}$$ which is pair # 952 of [5],

from which 85 new pairs were generated with Rule 2,

$$\begin{cases} 3693013664 = 2^5 41.131.21487 \\ 3812143072 = 2^5 119129471 \end{cases}$$ which is pair # 441 of [5],

from which 37 new pairs were generated with Rule 2, and

$$\begin{cases} 76809600128 = 2^7 149.1151.3499 \\ 77414399872 = 2^7 604799999 \end{cases}$$ which is pair # 609 of [5],

from which 31 new pairs were generated with Rule 2. We leave it to the reader to find out why in particular from *these* mother pairs so many daughter pairs could be generated. In [8] a frequency table is given consisting of all the mother pairs and the corresponding *numbers* of new daughter pairs generated from them with Rules 1, 2 and 3.

In a second step, we applied Rules 1, 2 and 3 to the 400 smallest of the 1782 daughter pairs, with the restriction that per daughter pair (m_1,m_2) we only considered one choice of (a_1,a_2), viz., the pair obtained by dropping the largest prime divisor of m_1 from m_1 and the largest prime divisor of m_2 from m_2. As in our first step, we also considered only those cases for which $C < 10^{25}$ in (2.10'). This reduced the number of cases drastically, also because the average daughter pair considered was much larger in size than the average mother pair considered, which led to much larger values of C in (2.10'). Nevertheless, we found 88 new "granddaughter" pairs from the smallest 400 daughter pairs, by using Rules 1, 2 and 3. These are also given in [8] together with a frequency table, similar to the one described above.

Finally, we have listed in [8] a compressed numbered list of all the 1575 mother pairs, for referencing and checking purposes.

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