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THE TRIANGULAR DECOUPLING PROBLEM FOR NONLINEAR CONTROL SYSTEMS

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The triangular decoupling problem for nonlinear control systems*)

by

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ABSTRACT

In this note a solution of the Triangular Decoupling Problem (T.D.P.) for nonlinear control systems is presented. This problem has been solved completely for linear systems by using the geometric approach. Here we show that the differential geometric approach to nonlinear systems enables us to solve the nonlinear T.D.P. locally. As a final result a simple but illustrative example is given.

KEY WORDS & PHRASES: *nonlinear control systems; differential geometric methods; controllability distributions; triangular decoupling*

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*) This paper will be submitted for publication elsewhere.

1. INTRODUCTION

Consider a control system of the form

$$(1.1a) \qquad \dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)u_i$$

$$(1.1b) \qquad z_i = H_i(x), \quad i = 1,\dots,p$$

where x are local coordinates of a smooth n-dimensional manifold M, $A, B_1, \dots, B_m$ are smooth vector fields on M and $H_i : M \to N_i$ is a smooth output map from M to a smooth p_i-dimensional manifold N_i for $i = 1,\dots,p$. We assume that each H_i is a surjective submersion.

In this note we will study the (*static state feedback*) *Triangular Decoupling Problem* (T.D.P.). That is, we seek a control law of the form

$$(1.2) \qquad u = \alpha(x) + \beta(x)v$$

where $\alpha: M \to \mathbb{R}^m$, $\beta: M \to \mathbb{R}^{m\times m}$ are smooth maps, $\beta(x) = (\beta_{ij}(x))$ is nonsingular for all x in M and $v = (v_1,\dots,v_m)^t \in \mathbb{R}^m$. Let $\tilde{A}(x) = A(x) + \Sigma_{i=1}^m B_i(x)\alpha_i(x)$ and $\tilde{B}_i(x) = \Sigma_{j=1}^m B_j(x)\beta_{ji}(x)$. Then the modified dynamics $\dot{x} = \tilde{A}(x) + \Sigma_{i=1}^m \tilde{B}_i(x)v_i$ should control the output z_i, $i = 1,\dots,p$ sequentially, i.e. $\bar{v}_1$ controls z_1, possibly changing the values $z_2,\dots,z_p$, then $\bar{v}_2$ controls z_2, possibly changing the values of $z_3,\dots,z_p$, with the requirement that z_1 be left unaffected and so forth, with $\bar{v}_p$ controlling z_p without influencing $z_1,\dots,z_{p-1}$ (here the $\bar{v}_i$ are vectors such that $(v_1,\dots,v_m) = (\bar{v}_1,\dots,\bar{v}_p)$). For linear systems the Triangular Decoupling Problem has been solved completely, see [3,11,12,21]. In the solution we present here we use as key tools the so called regular controllability distributions, introduced in [14]. In this way our approach completely fits in the systematic work on the generalization of the geometric approach to linear systems, see e.g. [6-10,13-18]. Note that in the T.D.P. the partial decoupling of the outputs is weaker then achieving complete dynamic interacting, which for a special case - the Restricted Decoupling Problem - has been solved in [16].

2. PROBLEM FORMULATION

Recall the following definitions, see [6-10,14].

DEFINITION 2.1. An involutive distribution D of fixed dimension, on M, is *controlled invariant* for the system (1.1a) if there exists a feedback of the form (1.2) such that the modified dynamics $\dot{x} = \tilde{A}(x) + \Sigma_{i=1}^{m} \tilde{B}_i(x)v_i$ leaves D invariant, i.e.

$$[\tilde{A},D] \subset D$$

$$[\tilde{B}_i,D] \subset D, \; i = 1,\ldots,m.$$

DEFINITION 2.2. An involutive distribution of fixed dimension, on M, is a *regular controllability distribution* of the system (1.1a) if it is controlled invariant for the system and moreover

D = involutive closure of $\{ad_{\tilde{A}}^{k}\tilde{B}_i \mid k \in \mathbb{N},\ i \in I\}$ for a certain subset $I \subset \{1,\ldots,m\}$.

Instead of the above notion of controlled invariance we will use a slightly weaker concept, which is much easier to handle.

DEFINITION 2.3. An involutive distribution D of fixed dimension, on M, is *locally controlled invariant* for the system (1.1a) if locally around each point $x_0 \in M$ there exists a feedback of the form (1.2) such that the modified dynamics $\dot{x} = \tilde{A}(x) + \Sigma_{i=1}^{m} \tilde{B}_i(x)v_i$ leaves D invariant.

A locally controlled invariant distribution can easily be characterized, see [8,13].

THEOREM 2.4. *Let* D *be an involutive distribution of fixed dimension on* M *and suppose that the distribution* $D \cap \text{Span}\{B_1,\ldots,B_m\}$ *has fixed dimension. Then* D *is locally controlled invariant if and only if*

$$[A,D] \subset D + \text{Span}\{B_1,\ldots,B_m\}$$

$$[B_i,D] \subset D + \text{Span}\{B_1,\ldots,B_m\},\; i = 1,\ldots,m.$$

Similarly one defines a local version of definition 2.2: the regular local controllability distributions.

Finally we need a definition of *output controllability*, see also [16]. Consider the system (1.1a) together with an output function $H:M \to N$. Assume that H is a surjective submersion. Let D be the controllability distribution of (1.1a), see [14,20], i.e. D = involutive closure of $\{ad_A^k B_i \mid k \in \mathbb{N},\ i = 1,\dots,m\}$. Then we have

DEFINITION 2.5. The system (1.1a) with output function $H:M \to N$ is *output controllable* if $H_*(D) = TN$, where D is the controllability distribution of (1.1a).

REMARK: This notion of output controllability is similar to the notion of strong accessibility for a system, [20]. Namely if we denote by $R_t(x_0)$ the reachable set of (1.1a) at time t from x_0, then the system is output controllable if $H(R_t(x_0))$ has non-empty interior in N.

It is now easy to see that the local version of the Triangular Decoupling Problem can be formalized, as for linear systems, in the following way: *Given the system* (1.1a,b) *find* (*if possible*) *a local feedback law of the form* (1.2) *and regular local controllability distributions* $R_1,\dots,R_p$ *such that we have*

(2.1) $$R_i \subset \bigcap_{j=1}^{i-1} \operatorname{Ker} H_{j*} \qquad i = 1,\dots,p$$

and

(2.2) $$R_i + \operatorname{Ker} H_{i*} = TM.$$

In (3.2) the vacuous condition at i = 1 just says $R_1 \subset TM$.

Define R_i^* = supremal regular local controllability distribution in $\bigcap_{j=1}^{j-1} \operatorname{Ker} H_{j*}$.

REMARK. R_i^* is well defined, see [10,14], but the dimension of $R_i^*(x)$ may change if x varies in M.

3. MAIN THEOREM

THEOREM 3.1. *Under the assumption that each* R_i^* *has fixed dimension for*

$i = 1,\dots,p$, T.D.P. *is solvable in a local fashion if and only if*

$$(3.1) \qquad R_i^* + \operatorname{Ker} H_{i*} = TM, \quad i = 1,\dots,p.$$

PROOF. The necessity of (2.2) follows from the maximality of the R_i^*. For sufficiency we have to show that the R_i^* are *compatible*; although each R_i^* is locally controlled invariant, by no means it follows that there exists a local feedback law (1.2) which leaves each of them invariant. From (2.1) it is clear that

$$(3.2) \qquad R_1^* \supset R_2^* \supset \dots \supset R_p^*$$

According to [19] we can choose local coordinates $(x_1,\dots,x_{p+1})$ on M such that

$$R_p^* = \operatorname{Span}\{\frac{\partial}{\partial x_1}\}, \quad R_{p-1}^* = \operatorname{Span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\},\dots,$$

$$R_1^* = \operatorname{Span}\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_p}\}, \text{ each } x_i \text{ possibly being a vector.}$$

R_p^* is locally controlled invariant, so

$$(3.3) \qquad \begin{cases} [A,R_p^*] \subset R_p^* + \operatorname{Span}\{B_1,\dots,B_m\} \\ [B_i,R_p^*] \subset R_p^* + \operatorname{Span}\{B_1,\dots,B_m\}, \quad i = 1,\dots,m. \end{cases}$$

By theorem 2.4 this is equivalent to the fact that there exists a local feedback $u = \alpha(x) + \beta(x)v$, such that

$$(3.4) \qquad \begin{cases} [\tilde{A},R_p^*] \subset R_p^* \\ [\tilde{B}_i,R_p^*] \subset R_p^*, \quad i = 1,\dots,m \end{cases}$$

(here $\tilde{A}$ and $\tilde{B}_i$ are defined as in section 1). In our local coordinates this means that

$$(3.5) \qquad \tilde{A}(x) = \begin{bmatrix} \tilde{A}^1(x_1,\dots,x_{p+1}) \\ \tilde{A}^2(x_2,\dots,x_{p+1}) \end{bmatrix}, \quad \tilde{B}_i(x) = \begin{bmatrix} \tilde{B}^1(x_1,\dots,x_{p+1}) \\ \tilde{B}^2(x_2,\dots,x_{p+1}) \end{bmatrix},$$

$i = 1,\dots,m$, where $\tilde{A}^1$, respectively $\tilde{B}_i^1$, represents the first x_1-dimensional (=dim R_p^*) component of the vector field $\tilde{A}$, respectively $\tilde{B}_i$ and $\tilde{A}^2$, respectively $\tilde{B}_i^2$, the remaining components of $\tilde{A}$ respectively $\tilde{B}_i$. Also R_{p-1}^* is locally controlled invariant, so

$$(3.6) \qquad \begin{cases} [\tilde{A},R_{p-1}^*] \subset R_{p-1}^* + \text{Span}\{\tilde{B}_1,\dots,\tilde{B}_m\} \\ [\tilde{B}_i,R_{p-1}^*] \subset R_{p-1}^* + \text{Span}\{\tilde{B}_1,\dots,\tilde{B}_m\}, \ i = 1,\dots,m. \end{cases}$$

By using the second component of the vector fields $\tilde{A}$ and $\tilde{B}_i$ as in (3.5), we deduce, according to [6,8,13], that we can find a local feedback $v = \bar{\alpha}(x) + \bar{\beta}(x)\bar{v}$ such that the new vector fields $\bar{A}$ and $\bar{B}_i$ satisfy (3.4) as well as

$$(3.7) \qquad \begin{cases} [\bar{A},R_{p-1}^*] \subset R_{p-1}^* \\ [\bar{B}_i,R_{p-1}^*] \subset R_{p-1}^*, \quad i = 1,\dots,m. \end{cases}$$

Or, in our local coordinates

$$(3.8) \qquad \bar{A}(x) = \begin{bmatrix} \bar{A}^1(x_1,\dots,x_{p+1}) \\ \bar{A}^2(x_2,\dots,x_{p+1}) \\ \bar{A}^3(x_3,\dots,x_{p+1}) \end{bmatrix}, \quad \bar{B}_i(x) = \begin{bmatrix} \bar{B}_i^1(x_1,\dots,x_{p+1}) \\ \bar{B}_i^2(x_2,\dots,x_{p+1}) \\ \bar{B}_i^3(x_3,\dots,x_{p+1}) \end{bmatrix},$$

$i = 1,\dots,m$, where $\bar{A}^1(\bar{B}_i^1)$ is the first x_1-dimensional (=dim R_p^*) component of $\bar{A}(\bar{B}_i)$, $\bar{A}^2(\bar{B}_i^2)$ is the second x_2-dimensional (=dim R_{p-1}^*-dim R_p^*) component of $\bar{A}(\bar{B}_i)$ and $\bar{A}^3(\bar{B}_i^3)$ represents the remaining component of $\bar{A}(\bar{B}_i)$. Notice that this second local feedback law $v = \bar{\alpha}(x) + \bar{\beta}(x)\bar{v}$ is independent of x_1, i.e. $v = \bar{\alpha}(x_2,\dots,x_{p+1}) + \bar{\beta}(x_2,\dots,x_{p+1})\bar{v}$. Repetition of the above argument yields

$$(3.9)\qquad \bar{A}(x) = \begin{bmatrix} \bar{A}^1(x_1,\dots,x_{p+1}) \\ \bar{A}^2(x_2,\dots,x_{p+1}) \\ \vdots \\ \bar{A}^p(x_p,x_{p+1}) \\ \bar{A}^{p+1}(x_{p+1}) \end{bmatrix}, \quad \bar{B}_i(x) = \begin{bmatrix} \bar{B}_i^1(x_1,\dots,x_{p+1}) \\ \bar{B}_i^2(x_2,\dots,x_{p+1}) \\ \vdots \\ \bar{B}_i^p(x_{ps}x_{p+1}) \\ \bar{B}_i^{p+1}(x_{p+1}) \end{bmatrix}$$

$i = 1,\dots,m$, where $\bar{A}^j(\bar{B}_i^j)$ represents the j-th x_j-dimensional component of $\bar{A}(\bar{B}_i)$. That is, we have shown that the distributions R_i^* are compatible. Next we will use the fact that the R_i^*'s are regular local controllability distributions. Using this we see that (eventually after a permutation on the new input functions $(\bar{v}_1,\dots,\bar{v}_m)$) there exists a partitioning of the set $\{1,\dots,m\}$ into p subsets I_k, $k = 1,\dots,p$ such that $I_1 = \{1,\dots,m_1\}$, $I_2 = \{1,\dots,m_1,\dots,m_2\},\dots,I_p = \{1,\dots,m\}$ with the property $j \in I_k \Leftrightarrow R_{p-k+1}^*$ for $k = 1,\dots,p$. Therefore our system after applying feedback has the form

$$(3.10)\qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x} \\ \vdots \\ \dot{x}_p \\ \dot{x}_{p+1} \end{bmatrix} = \begin{bmatrix} \bar{A}^1(x_1,\dots,x_{p+1}) \\ \bar{A}^2(x_2,\dots,x_{p+1}) \\ \vdots \\ \bar{A}^p(x_p,x_{p+1}) \\ \bar{A}^{p+1}(x_{p+1}) \end{bmatrix} + \sum_{j\in I_1} \begin{bmatrix} \bar{B}_j^1(x_1,\dots,x_{p+1}) \\ \bar{B}_j^2(x_2,\dots,x_{p+1}) \\ \vdots \\ \bar{B}_j^p(x_p,x_{p+1}) \\ 0 \end{bmatrix} \bar{v}_j +$$

$$\sum_{j\in I_2\setminus I_1} \begin{bmatrix} \bar{B}_j^1(x_1,\dots,x_{p+1}) \\ \bar{B}_j^2(x_2,\dots,x_{p+1}) \\ 0 \\ \bar{B}_j^{p-1}(x_{p-1},x_p,x_{p+1}) \\ 0 \\ 0 \end{bmatrix} \bar{v}_j + \dots + \sum_{j\in I_p\setminus I_{p-1}} \begin{bmatrix} \bar{B}_j^1(x_1,\dots,x_{p+1}) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \bar{v}_j$$

Furthermore we obtain from $R_i^* \subset \bigcap_{j=1}^{i-1} \mathrm{Ker}\, H_{j*}$ for the output functions the following partitioning

$$
(3.11) \quad \begin{cases} z_1 = H_1(x_p, x_{p+1}) \\ z_2 = H_2(x_{p-1}, x_p, x_{p+1}) \\ \vdots \\ z_{p-1} = H_{p-1}(x_2, \ldots, x_{p+1}) \\ z_p = H_p(x_1, \ldots, x_{p+1}) \end{cases}
$$

Finally we note that the condition (3.1), $R_i^* + \mathrm{Ker} H_{i*} = TM$, automatically leads to the notion of output controllability. For example the matrix $(\partial H_1/\partial x_p(x_p, x_{p+1}))$ has full rank and so forth. □

REMARKS. (i) The system (1.1a) is strongly accessible, see [20], if R_i^*, the supremal controllability distribution, equals TM. If $R_1^* = TM$ we can skip the x_{p+1} component in (3.10) and (3.11). (ii) The decomposition given here is different from the cascade decomposition given in [19] (see also [9]). (iii) In some cases one can derive conditions for invertibility for the 'subsystems' with $\bar{v}_{p-j}$ as input function and z_j as output function; see [15] for a geometric interpretation of invertibility.

4. AN EXAMPLE; THE RIGID BODY

We will illustrate the Triangular Decoupling Problem by a simple example of controlling the rigid body. For a mathematical description of a control system on the rigid body together with various results on controllability of the system we refer to [1,2,4,5]. The setting used here is similar as in [18]. Consider the system on $SO(3) \times \mathbb{R}^3$

$$
(4.1) \quad \begin{cases} \dot{R} = S(\omega)R \\ \begin{bmatrix} a_1\dot{\omega}_1 \\ a_2\dot{\omega}_2 \\ a_3\dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} a_1\omega_1 \\ a_2\omega_2 \\ a3\omega_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_3 \end{cases}
$$

where $R \in SO(3)$ represents the position of a riged body with respect to an inertial set of axes in $\mathbb{R}^3$, $\omega = (\omega_1,\omega_2,\omega_3)^t \in \mathbb{R}^3$ is the angular velocity of the rigid body, $(u_1,u_2,u_3)^t$ are the controls of the system and

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

As output functions we consider

$$(4.2) \qquad \begin{cases} z_1 = H_1(t,\omega) = \text{last row of the matrix } R \\ z_2 = H_2(r,\omega) = \text{second row of } R, \end{cases}$$

i.e. $H_1: SO(3) \times \mathbb{R}^3 \to S^2$ and $H_2: SO(3) \times \mathbb{R}^3 \to S^2 \times S^2$. Similar as in [18] we will first solve a simpler T.D.P., namely let $r = (r_1,r_2,r_3)^t$ be the first column of R. Then (4.1) reduce to

$$(4.3) \qquad \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \omega_3 r_2 - \omega_2 r_3 \\ -\omega_3 r_1 + \omega_1 r_3 \\ \omega_2 r_1 - \omega_1 r_2 \\ b_1 \omega_2 \omega_3 \\ b_2 \omega_1 \omega_3 \\ b_3 \omega_1 \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1^{-1} \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_2^{-1} \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3^{-1} \end{bmatrix} u_3$$

where $b_1 = a_1^{-1}(a_2-a_3)$, $b_2 = a_2^{-1}(a_3-a_1)$ and $b_3 = a_3^{-1}(a_1-a_2)$. Instead of (4.2) we obtain:

$$(4.4) \qquad \begin{cases} z_1 = \tilde{H}_1(r,\omega) = r_3 \\ z_1 = \tilde{H}_2(r,\omega) = r_2 \end{cases}$$

According to theorem 3.1 we only have to compute the supremal regular controllabity distribution R_2^* in Ker H_{1*}. For this we first compute the supremal controlled invariant distribution D in Ker H_{1*}.

Then, see [18], $D = \text{Span}\{X_1, X_2\}$ where

(4.5) $$X_1(r,\omega) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad X_2(r,\omega) = \begin{bmatrix} r_2 \\ -r_1 \\ 0 \\ \omega_2 \\ -\omega_1 \\ 0 \end{bmatrix}$$

Now it is straighforward to show that D is also a regular controllability distribution and therefore we obtain $R_2^* = D$ (see also [10]). Note that the dimension of R_2^* is not fixed on $SO(3) \times \mathbb{R}^3$, but on the open submanifold of $SO(3) \times \mathbb{R}^3$ where $r_1 r_2 \omega_1 \omega_2 \neq 0$ we certainly have that R_2^* has fixed dimension and $R_2^* + \text{Ker}\tilde{H}_{2*} = T(SO(3) \times \mathbb{R}^3)$. Finally we note that the system (4.3) is strongly accessible, i.e. $R_1^* = T(SO(3) \times \mathbb{R}^3)$, see e.g [4,5], and thus $R_1^* + \text{Ker}\tilde{H}_{1*} = T(SO(3) \times \mathbb{R}^3)$. Therefore by theorem 3.1 the T.D.P. is solvable. The decoupling feedback law is given by, see [18],

(4.6) $$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a_1(1-b_1)\omega_2\omega_3 \\ -a_2(1+b_2)\omega_1\omega_3 \\ 0 \end{bmatrix} + \begin{bmatrix} \omega_2 & \omega_1 & 0 \\ -\omega_1 & \omega_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Finally we see that by the same *coupe de grâce* as in [18] this feedback law (4.6) also solves the Triangular Decoupling Problem for the system (4.1,2) on the open and dense submanifold of $SO(3) \times \mathbb{R}^3$ where $r_1 r_2 \omega_1 \omega_2 \neq 0$.

5. CONCLUSION

By generalizing the geometric approach to linear systems theory, we were able to solve the Triangular Decoupling Problem for nonlinear systems. Although it takes some more effort we think that several other 'geometric' synthesis problems can be formulated and solved - in a local fashion - by the same techniques used in this paper.

REFERENCES

[1] BAILLIEUL, J. & R.W.BROCKETT, *Controllability and Observability of Polynomial Dynamical Systems*, Nonl. Anal., Theory Methods and Applic., vol. 5, pp. 543-552.

[2] BAILLIEUL, J. & R.W.BROCKETT, *A Controllability Result with an Application to Rigid Body Orientation*, 21st Midwest Symposium on Circuits and Systems, H.W.Hale and A.N.Michel, Eds, Iowa State University E.E.Dept., pp. 114-117, 1978.

[3] BASILE, G. & G.MARRO, *A state space approach to noninteracting controls*, Ricerche di Automatica 1, pp. 68-77, 1970.

[4] CROUCH, P.E, B.BORNARD, A.J.PRITCHARD & N.CARMICHAEL, *An appraisal of nonlinear Analytic Systems, with Applications to Attitude Control of a Spacecraft*, Report to E.S.T.E.C., by Applied Systems Studies, 1980.

[5] CROUCH, P.E. & N.CARMICHAEL, *Application of linear analytic systems theory to attitude control*, Report to E.S.T.E.C., by Applied Systems Studies, 1981.

[6] HIRSCHORN, R.M., (A,B)-*Invariant distributions and disturbance decoupling of nonlinear systems*, SIAM J.Contr. and Opt., vol 19, pp. 1-19, 1981.

[7] ISIDORI, A., A.J.KRENER, C.GORI-GIORGI & S.MONACO, *Nonlinear decoupling via feedback: a differential geometric approach*, IEEE-Trans. Aut. Contr. vol. 26, pp. 331-345, 1981.

[8] ISIDORI, A., A.J.KRENER, C.GORI-GIORGI & S.MONACO, *Locally* (f,g)-*invariant distributions*, Syst. Contr. Lett., vol. 1, pp. 12-15, 1981.

[9] ISIDORI, A., A.J.KRENER, C.GORI-GIORGI & S.MONACO, *The observability of cascade connected nonlinear systems, paper presented at* IFAC *conference*, Kyoto, Japan, 1981.

[10] KRENER, A.J. & A.ISIDORI, (Adf, $\mathfrak{g}$) *Invariant and controllability distributions*, in *Feedback Control of Linear and Nonlinear Systems*, Lect. Notes in Control and Information Sciences 39, pp. 157-164.

[11] MORSE, A.S. & W.M.WONHAM, *Triangular Decoupling of Linear Multivariable Systems*, IEEE-Trans. Aut. Contr. vol. 15, pp. 447-449, 1970.

[12] MORSE, A.S. & W.M.WONHAM, *Status of Noninteracting Control*, IEEE-Trans. Aut. Contr., vol. 16, pp. 568-581, 1971.

[13] NIJMEIJER, H., *Controlled invariance for affine control systems*, Int. J. Contr, vol. 34, pp. 825-833, 1981.

[14] NIJMEIJER, H., *Controllability distributions for nonlinear control systems*, Syst. Contr. Lett., vol. 2, pp. 122-129, 1982.

[15] NIJMEIJER, H., *Invertibility of affine nonlinear control systems: a geometric approach*, to appear in Syst. Contr. Lett.

[16] NIJMEIJER, H., *Feedback decomposition of nonlinear control systems*, to appear in IEEE-Trans. Aut. Contr.

[17] NIJMEIJER, H., & A.J. VAN DER SCHAFT, *Controlled invariance for nonlinear systems*, IEEE-Trans. Aut. Contr., vol. 27, pp. 904-914, 1982.

[18] NIJMEIJER, H. & A.J. VAN DER SCHAFT, *Controlled invariance for nonlinear systems, two worked examples*, to appear in IEEE-Trans. Aut. Contr.

[19] RESPONDEK, W., *On decomposition of nonlinear control systems*, Syst. Contr. Lett., vol. 1, pp. 301-308, 1982.

[20] SUSSMANN, H.J., & V.JURDJEVIC, *Controllability of nonlinear systems*, J. Diff. Equat., vol. 12, pp. 95-116, 1972.

[21] WONHAM, W.M., *Linear multivariable: a geometric approach*, 2nd ed., Springer, 1979.

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