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AFDELING NUMERIEKE WISKUNDE NW 142/82 NOVEMBER
(DEPARTMENT OF NUMERICAL MATHEMATICS)

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THE CONVERGENCE RATE OF MULTI-LEVEL ALGORITHMS
APPLIED TO THE CONVECTION-DIFFUSION EQUATION

Preprint

kruislaan 413 1098 SJ amsterdam

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65Nxx

The convergence rate of multi-level algorithms applied to the convection-diffusion equation *)

by

P.M. de Zeeuw & E.J. van Asselt

ABSTRACT

We consider the solution of the convection-diffusion equation in two dimensions by various multi-level algorithms (MLAs).

We study the convergence rate of the MLAs and the stability of the coarse-grid operators, depending on the choice of artificial viscosity at the different levels. Four strategies are formulated and examined. A method to determine the convergence rate is described and applied to the MLAs, both in a problem with constant and in one with variable coefficients.

As relaxation procedures the 7-point ILU and Symmetric Gauss Seidel (SGS) methods are used.

KEY WORDS & PHRASES: *artificial viscosity, convection-diffusion equation, multi-level algorithm, asymptotic stability, Galerkin approximation*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

We consider the convection-diffusion equation

$$(1.1) \qquad L_\varepsilon u \equiv -\varepsilon \, \Delta \, u + b_1(x,y) \frac{\partial u}{\partial x} + b_2(x,y) \frac{\partial u}{\partial y} = f(x,y)$$

for $(x,y) \in \Omega \subset \mathbb{R}^2$, $\varepsilon > 0$, with Dirichlet and Neumann boundary conditions on different parts of $\delta\Omega$.

In case of a small diffusion coefficient ε in comparison with the mesh-width h, the stability of discretizations of (1.1) by central differences (CD) or the finite element method (FEM) can be improved by augmenting ε with an artificial viscosity of O(h). This rather crude way of stabilizing the discrete problem may form part of more subtle iterative methods for solving (1.1) with small ε. (See e.g. HEMKER [3]).

In section 2 we introduce four strategies for choosing the artificial viscosity on the coarse grids in the multi-level algorithm (MLA) (cf. VAN ASSELT [1]).

In section 3 we describe the method which is used to determine the convergence behaviour of the multi-level algorithm for these strategies.

In section 4 we compare the convergence rates as measured by the method described in section 3.

Finally, some conclusions are formulated in section 5.

2. ARTIFICIAL VISCOSITY, STRATEGIES, STABILITY AND ASYMPTOTIC CONVERGENCE RATE

In this section we introduce various strategies for choosing the coarse-grid operators in the MLA. We give a motivation for the choice of these strategies, and analyze their stability (cf. (2.14), (2.18),(2.19),(2.24)). Further we formulate some important properties of the different strategies (cf.(2.25),(2.26),(2.27)). In the case of FEM discretization we also consider the Galerkin-coarse-grid-approximation. In this paper we only consider the FEM based on a uniform triangulation of Ω with rectangular triangles (cf. Figure 2). The trial - and test-space is spanned by the set of piecewise-linear "hat-functions" ϕ_{ij} which take the value 1 at x_{ij} and 0 at all other vertices of triangles.

We consider the MLA (cf. HEMKER [4]) with $\ell+1$ levels : $0,\dots,\ell$ and uniform square meshes on each level with meshwidths h_0 and $h_k = h_{k-1}/2$ for $k = 1,\dots,\ell$.

Let $\{L_\varepsilon^{k,\ell}\}_{k=0,\dots,\ell}$ be a sequence of discretizations of L_ε. For the constant-coefficient equation we denote by $\hat{L}_\varepsilon(\omega)$, $\omega \in \mathbb{R}^2$ the symbol (or characteristic form) of the continuous operator L_ε.
By $\hat{L}_\varepsilon^{k,\ell}(\omega)$, $\omega \in T_k \equiv [-\pi/h_k, \pi/h_k]^2$, we denote the symbol of the discrete operator $L_\varepsilon^{k,\ell}$.

(2.1) DEFINITION. The ε-*asymptotic stability of* L_ε *with respect to the mode* $e^{i\omega x}$ is the quantity $\lim_{\varepsilon \downarrow 0} |\hat{L}_\varepsilon(\omega)|$.

(2.2) DEFINITION. The δ-*domain of* L_ε is the set of all $\omega \in \mathbb{R}^2$ for which $\lim_{\varepsilon \downarrow 0} |\hat{L}_\varepsilon(\omega)| > \delta > 0$.

(2.3) DEFINITION. The ε-*asymptotic stability of* $L_\varepsilon^{k,\ell}$ *with respect to the mode* $e^{i\omega x}$ is the quantity $\lim_{\varepsilon \downarrow 0} |L_\varepsilon^{k,\ell}(\omega)|$.

(2.4) DEFINITION. The δ-*domain of* $L_\varepsilon^{k,\ell}$ is the set of all $\omega \in T_k$ for which

$$\lim_{\varepsilon \downarrow 0} |\hat{L}_\varepsilon^{k,\ell}(\omega)| > \delta > 0.$$

(2.5) DEFINITION. A *strategy for coarse-grid operators* is a set $\{L_\varepsilon^0, L_\varepsilon^1, \dots, L_\varepsilon^\ell, \dots\}$ with $L_\varepsilon^\ell \equiv \{L_\varepsilon^{0,\ell}, \dots, L_\varepsilon^{\ell,\ell}\}$.

(2.6) DEFINITION. Let S be a strategy for coarse-grid operators then S is ε-*asymptotically stable with respect to* L_ε if for every $\delta_0 > 0$ there exists a $\delta_1 > 0$ such that for all $0 \le k \le \ell$ we have δ_1-domain of $L_\varepsilon^{k,\ell} \supset \delta_0$ - domain of $L_\varepsilon \cap T_k$.

2.7. REMARK. In order to avoid useless residual transfers in the multi-level algorithm due to oscillating solutions we require that a strategy is ε-asymptotically stable with respect to L_ε. Besides we need a relaxation-method for which the smoothing factors on all grids are less than 1. The usual arguments show that these two requirements guarantee convergence of the MLA.

Another approach would be to admit ε-asymptotically unstable strategies and to require that the relaxation-method is such that bad components in the residuals are sufficiently smoothed. This makes very strong demands upon the relaxation method.

For fixed h_0 and $\gamma > 0$ (independent of ε,k and ℓ) we define four strategies for coarse-grid operators. By $L_{\varepsilon+\beta_k^\ell,h_k}$ we denote a discretization of (1.1) with artificial viscosity β_k^ℓ and meshwidth h_k.

(2.8) DEFINITION. *Strategy* 1, denoted by S_1, is the set $\{L_\varepsilon^0, L_\varepsilon^1, \ldots, L_\varepsilon^\ell, \ldots\}$ with $L_\varepsilon^\ell = \{L_\varepsilon^{0,\ell}, L_\varepsilon^{1,\ell}, \ldots, L_\varepsilon^{\ell,\ell}\}$ where

$$L_\varepsilon^{k,\ell} = L_{\varepsilon+\beta_k^\ell, h_k} \text{ and } \beta_k^\ell = \gamma\, h_\ell, \qquad k = 0,\ldots,\ell.$$

(2.9) DEFINITION. *Strategy* 2, denoted by S_2, is the set

$$\{L_\varepsilon^0, L_\varepsilon^1, \ldots, L_\varepsilon^\ell, \ldots\} \text{ with } L_\varepsilon^\ell = \{L_\varepsilon^{0,\ell}, L_\varepsilon^{1,\ell}, \ldots, L_\varepsilon^{\ell,\ell}\} \text{ where}$$

$$L_\varepsilon^{k,\ell} = L_{\varepsilon+\beta_k^\ell, h_k} \text{ and}$$

$$\begin{cases} \beta_\ell^\ell = \gamma\, h_\ell \\ \beta_k^\ell = \gamma\, h_{k+1}, \qquad k = 0,\ldots,\ell-1. \end{cases}$$

(2.10) DEFINITION. *Strategy* 3, denoted by S_3 is the set

$$\{L_\varepsilon^0, L_\varepsilon^1, \ldots, L_\varepsilon^\ell, \ldots\} \text{ with } L_\varepsilon^\ell = \{L_\varepsilon^{0,\ell}, L_\varepsilon^{1,\ell}, \ldots L_\varepsilon^{\ell,\ell}\} \text{ where}$$

$$L_\varepsilon^{k,\ell} = L_{\varepsilon+\beta_k^\ell, h_k} \text{ and}$$

$$\beta_k^\ell = \gamma\, h_k, \qquad k = 0,\ldots,\ell.$$

(2.11) DEFINITION. *Strategy* 4, denoted by S_4, is the set

$$\{L_\varepsilon^0, L_\varepsilon^1, \ldots, L_\varepsilon^\ell, \ldots\} \text{ with } L_\varepsilon^\ell = \{L_\varepsilon^{0,\ell}, L_\varepsilon^{1,\ell}, \ldots, L_\varepsilon^{\ell,\ell}\} \text{ where}$$

$$\begin{cases} L_\varepsilon^{\ell,\ell} \equiv L_{\varepsilon+\beta_\ell^\ell, h_\ell} \text{ with } \beta_\ell^\ell = \gamma\, h_\ell \\ L_\varepsilon^{k,\ell} \equiv R_{k,k+1}\, L_\varepsilon^{k+1,\ell}\, P_{k+1,k}, \quad k = \ell-1,\ldots,0. \end{cases}$$

($R_{k,k+1}$ and $P_{k+1,k}$ are the restriction and the prolongation which are consistent with the FEM used.)

<u>(2.12) REMARK</u>. If we consider a constant-coefficient problem and neglect the boundaries, then a coarse-grid operator constructed with the FEM according to S_1, is identical with the Galerkin approximation of the fine-grid discretization (cf. 2.11).

<u>(2.13) REMARK</u>. It follows from (2.8) - (2.10) that

$$\text{for } S_1\text{:}\quad \lim_{\ell\to\infty} \beta_0^\ell/h_k = \lim_{\ell\to\infty} \gamma/2^\ell = 0.$$

$$\text{for } S_2\text{:}\quad \beta_k^\ell/h_k \geq \gamma/2 \text{ uniformly for all } k,\ell.$$

$$\text{for } S_3\text{:}\quad \beta_k^\ell/h_k = \gamma \text{ uniformly for all } k,\ell.$$

In (2.14), (2.18) and (2.24) we will prove respectively that S_1 and S_4 are not ε-asymptotically stable and S_2 and S_3 are. Further we will point out that the convergence rate of the MLA with S_2 is better than with S_3.

<u>(2.14) THEOREM</u>. *Consider the* CD- *or* FEM- *discretizations of* (1.1) *with artificial viscosity* β_k^ℓ *and constant coefficients, then* S_1 *is not* ε*-asymptotically stable with respect to* L_ε.

<u>PROOF</u>. We give the proof only for the CD-discretizations; the proof for the FEM-discretizations is similar.

The CD-discretization of (1.1) with artificial viscosity β_k^ℓ and constant coefficients b_1 and b_2, $b_1^2+b_2^2 = 1$, reads

(2.15) $$L_{\varepsilon+\beta_k^\ell,h_k} u \equiv$$

$$\left(-\frac{\varepsilon+\beta_k^\ell}{h_k^2} - \frac{b_2}{2h_k}\right) u_{i,j-1}^{h_k} + \left(-\frac{\varepsilon+\beta_k^\ell}{h_k^2} + \frac{b_2}{2h_k}\right) u_{i,j+1}^{h_k} +$$

$$\left(-\frac{\varepsilon+\beta_k^\ell}{h_k^2} - \frac{b_1}{2h_k}\right) u_{i-1,j}^{h_k} + \left(-\frac{\varepsilon+\beta_k^\ell}{h_k^2} + \frac{b_1}{2h_k}\right) u_{i+1,j}^{h_k} +$$

$$4\left(\frac{\varepsilon+\beta_k^\ell}{h_k^2}\right) u_{i,j}^{h_k} = f_{i,j}^{h_k}.$$

Its characteristic form reads:

(2.16) $$\hat{L}_{\varepsilon+\beta_k^\ell,h_k}(\omega) =$$

$$-2(\varepsilon+\beta_k^\ell)(\cos\omega_1 h_k + \cos\omega_2 h_k - 2)/h_k^2$$

$$+\underline{i}(b_1 \sin\omega_1 h_k + b_2 \sin\omega_2 h_k)/h_k.$$

The characteristic form of L_ε reads:

(2.17) $$\hat{L}_\varepsilon(\varepsilon) = \varepsilon(\omega_1^2+\omega_2^2) + \underline{i}(b_1\omega_1 + b_2\omega_2),$$

hence the δ_0-domain of L_ε is the set of all $\omega \in \mathbb{R}^2$ for which $|b_1\omega_1+b_2\omega_2| > \delta_0 > 0$. We have to show that a $\delta_0 > 0$ exists such that for all $\delta_1 > 0$ there exist $k, \ell \in \mathbb{Z}$, $0 \le k \le \ell$, such that for an $\tilde{\omega} \in \mathbb{R}^2$ with $\tilde{\omega} \in (\delta_0\text{-domain of } L_\varepsilon) \cap T_k$ we have $\tilde{\omega} \notin \delta_1$- domain of $L_{\varepsilon+\beta_k^\ell,h_k}$.
For that purpose we proceed as follows.
Take $\delta_0 = 0.1\,\pi/h_0$ and let $\delta_1 > 0$ be arbitrary.
Take $k = 0$ and $\ell > \log(4\gamma/h_0\delta_1)$, then for either

$\tilde{\omega} = (\pi/h_0, 0) \in T_0$ or $\tilde{\omega} = (0, \pi/h_0) \in T_0$ both $|b_1\tilde{\omega}_1+b_2\tilde{\omega}_2| > \delta_0$ and

$$\lim_{\varepsilon\downarrow 0} |\hat{L}_{\varepsilon+\beta_0^\ell,h_0}(\tilde{\omega})| = 4\gamma/(h_0 2^\ell) < \delta_1 \text{ hold.}$$

Hence S_1 is not ε-asymptotically stable with respect to L_ε. QED.
This leads us to

(2.18) COROLLARY. *Consider* L_ε *with constant coefficients* b_1 *and* b_2, *then* S_4 *is not* ε*-asymptotically stable with respect to* L_ε.

PROOF. The proof follows immediately from (2.12) and (2.14). QED.

(2.19) THEOREM. *Consider the* CD*-discretizations of* (1.1) *with artificial viscosity* β_k^ℓ *and constant coefficients. Let* S *be a strategy with* $\beta_k^\ell/h_k \geq C > 0$ *uniformly for all* $k,\ell (k \leq \ell) \in \mathbb{Z}$, *then* S *is* ε*-asymptotically stable.*

PROOF. Again we use (2.15)-(2.17).
We have to proof: $\forall\ \delta_0 > 0\ \exists\ \delta_1 > 0\ \forall\ k, \ell\ 0 \leq k \leq \ell \Rightarrow \delta_0$ - domain of $L_\varepsilon \cap T_k \subset \delta_1$ - domain of $L_{\varepsilon+\beta_k^\ell, h_k}$.
Take $\delta_1 \equiv \min(1/2, 2C/5)\ \delta_0$. In the case $\delta_0 > 2^{1/2}\pi/h_k$ the inclusion is trivially satisfied because δ_0 - domain of $L_\varepsilon \cap T_k = \emptyset$.
If $0 < \delta_0 \leq 2^{\frac{1}{2}}\pi/h_k$ then $\omega \in \delta_0$ - domain of $L_\varepsilon \cap T_k$ implies

$$\delta_0 h_k < |b_1\omega_1 h_k + b_2\omega_2 h_k|.$$

The normalization $b_1^2 + b_2^2 = 1$ and the inequality $|\sin x - x| \leq |x^3|/4$ for all $x \in \mathbb{R}$ yield

$$\delta_0 h_k < |b_1 \sin \omega_1 h_k + b_2 \sin \omega_2 h_k| + |\omega_1 h_k|^3/4 + |\omega_2 h_k|^3/4. \tag{2.20}$$

We distinguish the two complementary cases:

$$\left[\begin{array}{l} \text{(i)}\ |\omega_1 h_k|^3 \leq \delta_0 h_k \text{ and } |\omega_2 h_k|^3 \leq \delta_0 h_k \\ \\ \text{(ii)}\ |\omega_1 h_k|^3 > \delta_0 h_k \text{ or } |\omega_2 h_k|^3 > \delta_0 h_k. \end{array}\right.$$

Because of (2.16) and (2.20) case (i) implies:

$$\lim_{\varepsilon \downarrow 0} |\hat{L}_{\varepsilon+\beta_k^\ell, h_k}(\omega)| \geq |b_1 \sin \omega_1 h_k + b_2 \sin \omega_2 h_k|/h_k > \delta_0/2 \geq \delta_1. \tag{2.21}$$

To complete the proof we now consider case (ii). It follows from (2.16) and $\beta_k^{\ell}/h_k \geq C$ that

(2.22) $$\lim_{\varepsilon\downarrow 0}\left|\hat{L}_{\varepsilon+\beta_k^{\ell},h_k}(\omega)\right| \geq 2C(1-\cos\omega_1 h_k + 1 - \cos\omega_2 h_k)/h_k,$$

and from condition (ii) and $0 < \delta_0 h_k \leq 2^{\frac{1}{2}}\pi$ it follows that the right-hand-side of (2.22) is greater than or equal to

$$2C\,\delta_0(1-\cos((\delta_0 h_k)^{1/3}))/(\delta_0 h_k),\text{ hence}$$

(2.23) $$\lim_{\varepsilon\downarrow 0}\left|\hat{L}_{\varepsilon+\beta_k^{\ell},h_k}(\omega)\right| > 2C\,\delta_0/5 \geq \delta_1 > 0.$$

Both (2.21) and (2.23) hold uniformly for all k, ℓ so S is ε-asymptotically stable with respect to L_ε.
QED.

(2.24) COROLLARY. *Consider the* CD-*discretizations of* (1.1) *with artificial viscosity* β_k^{ℓ} *and constant coefficients, then* S_2 *and* S_3 *are* ε-*asymptotically stable with respect to* L_ε.

PROOF. The proof follows immediately from (2.13) and (2.19). QED.

It is obvious that the ε-asymptotic stability of the operators belonging to S_2 is larger than in case of S_1. Moreover for decreasing γ the smoothing factors for S_1 become worse (cf. Table 2). We formulate this in the following

(2.25) SUPPOSITION. For a fixed number of levels the set of γ - values for which the MLA with S_2 converges, is larger than that for which the MLA with S_1 converges.

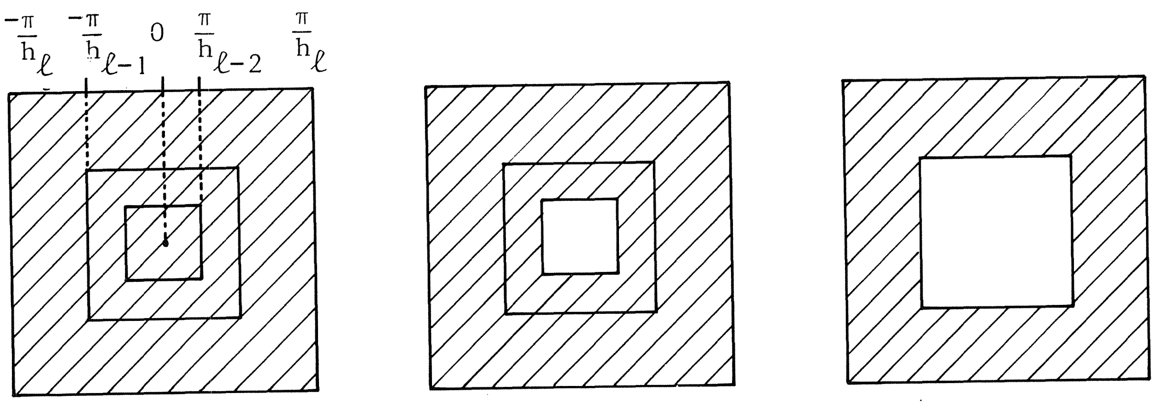
In case of a two-level algorithm (TLA), $\ell = 1$, a two-level analysis shows that the asymptotic rate of convergence for S_1 or S_2, for which the artificial viscosity is equal on both levels is better than for S_3, where the artificial viscosity corresponds to the meshwidth. (cf. VAN ASSELT [1]). Hence we consider in S_1 an equal artificial viscosity on all levels. For this strategy however on coarser grids stability problems may occur (cf. 2.14).

S_3 is ε-asymptotically stable (cf. 2.24), but the two-level analysis indicates that the convergence rate is slower. S_2 is an intermediate strategy where on level ℓ and $\ell-1$ the artificial viscosity is the same, and it is also ε-asymptotically stable (cf. 2.24). These arguments lead to the following

(2.26) SUPPOSITION. S_2 combines the rapid convergence rate of S_1 with the stability of S_3.

At level ℓ the discrete operators $L^{\ell}_{\varepsilon+\beta_\ell,h_\ell}$ using S_1,S_2,S_3 are equal.

At level $\ell-1$ the discrete operators $L^{\ell}_{\varepsilon+\beta_{\ell-1},h_{\ell-1}}$ using S_1,S_2 are equal (S_3 is not), and the relative order of consistency of the S_1 and S_2 operators on level ℓ and $\ell-1$ is the same and higher than that of S_3.
Further consider the part of T_ℓ where the smoothing effect of the relaxation-method applied to S_2 and S_3 is the same as in case of S_1. For S_2 this part is larger than for S_3 (cf. Figure 1).



S_1 S_2 S_3

Figure 1. Parts of T_ℓ where for S_2 and S_3 the smoothing-effect is the same as for S_1.

This leads us to formulate the following

(2.27) SUPPOSITION. For a finite number of levels and γ sufficiently large the difference between the asymptotic rate of convergence of the MLAs using S_1 and S_2 is smaller than that between S_2 and S_3.

The properties stated in (2.14), (2.18) and (2.24) - (2.27) will be confirmed by numerical experiments in section 4.

3. NUMERICAL APPROXIMATION OF THE CONVERGENCE RATE

In this section we give a description of the method that is used to determine the asymptotic rate of convergence of the MLA. Let

(3.1) $A_h \bar{v}_h = f_h$ be a discretization of (1.1). The MLA to solve (3.1) can be described as a defect-correction-process (cf. HEMKER [4]:

$$(3.2)\quad \begin{cases} \bar{v}_h^{(0)} & \text{given start approximation} \\ \bar{v}_h^{(i+1)} = M_h \bar{v}_h^{(i)} + B_h^{-1} f_h & i = 0,1,\ldots \end{cases}$$

with amplification-matrix $M_h = I_h - B_h^{-1} A_h$. I_h is the identity-matrix, and B_h^{-1} is an approximate inverse of A_h, determined by coarse-grid- and smoothing operators, prolongation and restriction. We suppose A_h and B_h to be non-singular. For the error $\bar{e}_h^{(i)} = \bar{v}_h$, $i = 0,1,\ldots$ the following relation holds:

$$\bar{e}_h^{(i+1)} = M_h \bar{e}_h^{(i)}.$$

The convergence behaviour of the MLA is considered in the following way:

(3.3) DEFINITION. The *asymptotic rate of convergence* of the MLA (3.2) is $-{}^{10}\log \rho\ (M_h)$ where $\rho(M_h) \equiv \max_j |\lambda_j|$ is the spectral-radius of M_h; λ_j are the eigenvalues of M_h.

(3.4) THEOREM.

$$\sup_{x\neq 0} \lim_{k\to\infty} (\| M_h^k x \| / \| x \|)^{1/k} = \rho(M_h),\ \textit{with}\ \|\cdot\|\ \textit{an arbitrary}$$

norm.

PROOF. See STOER, BULIRSCH [6], Satz (8.2.4).QED.

Because of (3.4) we compute an approximation $\rho_{m,k}(M_h, \bar{e}_h^0)$ of $\rho\ (M_h)$ defined by

(3.5) $\rho_{m,k}(M_h,\bar{e}_h^0) \equiv (\| M_h^{m+k}\ \bar{e}_h^0 \|_2 / \| M_h^m\ \bar{e}_h^0 \|_2)^{1/k}$ where $\|\cdot\|_2$ is

the Euclidean norm. Note that

(3.6) $$\sup_{\bar{e}_h^0 \neq 0} \lim_{m,k\to\infty} \rho_{m,k}(M_h,\bar{e}_h^0) = \rho(M_h).$$

In numerical computations $v_h^j, j = m,\ldots,m + k$ are obtained by the iterative method under consideration. When for increasing m and k, $\|\bar{e}_h^j\|_2$ reaches values near the square root of the machine accuracy, we replace $\bar{e}_h^j$ by $\bar{e}_{h,\eta}^j$:

(3.7) $\bar{e}_{h,\eta}^{(j)} \equiv \eta\ \bar{e}_h^j\ (\eta>>1)$ and replace

$\bar{v}_h^j$ by $\bar{v}_{h,\eta}^j$:

(3.8) $\bar{v}_{h,\eta}^j \equiv \bar{v}_h + \bar{e}_{h,\eta}^j$

Thus $\|\bar{e}_{h,\eta}^{j+1}\|_2 / \|\bar{e}_{h,\eta}^j\|_2 = \|\bar{e}_h^{j+1}\|_2 / \|\bar{e}_h^j\|_2$, and as

(3.9) $\rho_{m,k}(M_h,\bar{e}_h^0) = \prod_{j=m}^{m+k-1} (\|\bar{e}_h^{(j+1)}\|_2 / \|\bar{e}_h^j\|_2))^{1/k}$,

in this way values of $\rho_{m,k}(M_h,\bar{e}_h^0)$ can be computed for large m and k. By this method ultimately the eigenfunctions of M_h corresponding to non-dominant eigenvalues will decrease exponentially relative to the dominant eigenfunctions. Note that for small m and k $\rho_{m,k}$ depends strongly on f_h while ρ does not.

4. NUMERICAL RESULTS

In this section we give the results of numerical experiments to compare the strategies S_1,S_2,S_3 and S_4 and to verify the properties stated in (2.14), (2.18), and (2.24) -(2.27). We take three testproblems. Testproblem 1 with constant coefficients closely resembles the problem analysed by two-level analysis in VAN ASSELT [1]. Testproblem 2 has variable coefficients. Although a strict application of Fourier-analysis-arguments does not hold for these

variable-coefficients-problems, the experiments for the latter testproblem show that globally the same properties hold as for the constant-coefficients-case. For the second problem we also show to what extent the strategies $S_1,\dots,S_4$ are better than relaxation alone (i.e. without coarse-grid-correction).
Testproblem 3 differs from testproblem 1 by discretization (FEM), relaxation (ILU) and number of levels.

Testproblem 1. We consider the convection-diffusion equation

(4.1) $$-(\varepsilon+\gamma h)\ \Delta u + \frac{\partial}{\partial y} u = 0 \text{ on } \Omega = [0,1] \times [-1,1],$$

$$\varepsilon = 10^{-6},\ h = 1/16 \text{ (cf. Figure 2).}$$

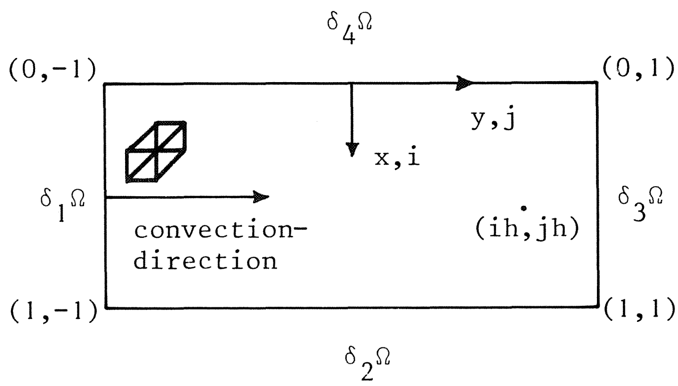


Figure 2. The domain Ω

The boundary conditions are:

(4.2) $$u\Big|_{\delta_1\Omega} = \begin{cases} 1 & , \quad x < 1/2 - 10^{-6}\ ; \\ -10^{6}(x-1/2), & \quad 1/2 - 10^{-6} \le x \le 1/2 + 10^{-6}\ ; \\ -1 & , \quad x > 1/2 + 10^{-6}\ ; \end{cases}$$

$$\frac{\partial u}{\partial n}\Big|_{\delta_2\Omega} = \frac{\partial u}{\partial n}\Big|_{\delta_3\Omega} = \frac{\partial u}{\partial n}\Big|_{\delta_4\Omega} = 0.$$

Equation (4.1) is discretized by CD on levels $k = 0,\dots,\ell = 3$ with meshsize $h_k = 1/2^{k+1}$.

The Neumann boundary conditions are discretized as follows:

$$\delta_2\Omega : u(1,y) - u(1-h_k,y) = 0,$$

$$\delta_3\Omega : u(x,1) - u(x,1-h_k) = 0,$$

$$\delta_4\Omega : u(0,y) - u(h_k,y) = 0, \quad k = 0,\ldots, \ell = 3.$$

For different values of γ the discretized equation is solved with the W-cycle MLA (i.e. the application of 2 multi-level-iteration steps to approximate the solution of the coarse-grid equation).

We perform one pre- and one post-relaxation-step consisting of SGS-relaxation in the y-direction. We use 7-point prolongation and 7-point restriction (cf. HEMKER [5],WESSELING [7]).

A random initial approximation of the solution is used. The values for m and k in (3.9) are 30 and 10 respectively.

Testproblem 2. We consider the convection-diffusion equation:

$$(4.3) \qquad -(\varepsilon+\gamma h)\ \Delta u + b_1 \frac{\partial}{\partial x} u + b_2 \frac{\partial}{\partial y} u = 0, \text{ on } \Omega = [0,1] \times [-1,1],$$

$$\varepsilon = 10^{-6},\ h = 1/16,\ b_1 = y(1-x^2),\ b_2 = - x(1-y^2) \text{(cf. Figure 3).}$$

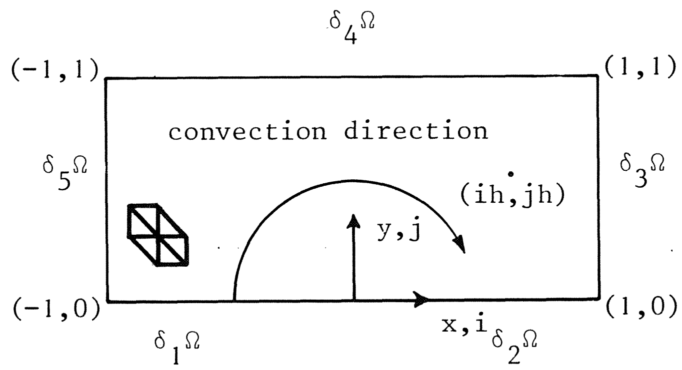


Figure 3. The domain of Ω.

The boundary conditions are

$$(4.4) \qquad u\Big|_{\delta_1\Omega} = 1 + \tanh(10+20x),$$

$$\frac{\partial u}{\partial n}\Big|_{\delta_2\Omega} = \frac{\partial u}{\partial n}\Big|_{\delta_3\Omega} = \frac{\partial u}{\partial n}\Big|_{\delta_4\Omega} = \frac{\partial u}{\partial n}\Big|_{\delta_5\Omega} = 0.$$

Equations (4.3) and (4.4) are discretized by the FEM on levels $k = 0,\ldots,$ $\ell = 4$ with mesh-size $h_k = 1/2^k$.
For different values of γ, and S_1-S_4 the discretized equation is solved with the W-cycle MLA. We perform one pre - and one post-relaxation-step by means of 7 point-ILU-relaxation, (cf. WESSELING AND SONNEVELD [8]). The ILU-decomposition is ordered lexicographically (cf. Figure 3). Again we use 7-point prolongation and 7-point restriction (that are consistent with the FEM discretization), and a random initial approximation. In (3.9) m and k are again 30 and 10.

Testproblem 3. For $\ell = 4,5,6$ we consider (4.1) and (4.2) discretized by the FEM on levels $k = 0,\ldots, \ell$, with mesh-size $h_k = (1/2)^{k+1}$, $\gamma = 1/2$.

The discretized equation is solved with the W-cycle MLA. We perform one pre- and post-relaxation-step by means of 7-point-ILU relaxation (on the coarsest level we do not solve directly, but perform relaxation-sweeps). The LU-decomposition is ordered lexicographically (cf. Figure 3). We use 7-point prolongation and 7-point restriction. A random initial approximation of the solution is used. The values for m and k in (3.9) are 20 and 10 respectively.

Figures 4 and 5 show the properties stated in (2.25)-(2.27) for testproblem 1 and 2 respectively. Figure 5 also shows that all strategies S_1 - S_4 are better than relaxations without coarse-grid-corrections. In table 2 for S_1,S_2 and S_3 the smoothing factors of SGS are given at different levels, and for different γ. We notice that for ${}^2\log\gamma > 0$ the big difference in the asymptotic rate of convergence of S_2 and S_3 (cf. Figure 4) is mainly caused by the order of consistency and for a small part by the relaxation method.

In order to verify (2.14),(2.18) and (2.24) we take testproblem 3. Table 1 reports the convergence rates as measured (cf. 3.3). Note that S_1 and S_4 show similar stability- and convergence-behaviour (cf. 2.12).

(4.5) REMARK. With respect to (2.7) we notice that in many cases a decreasing stability coincides with a worsening smoothing factor (cf. Table 2).

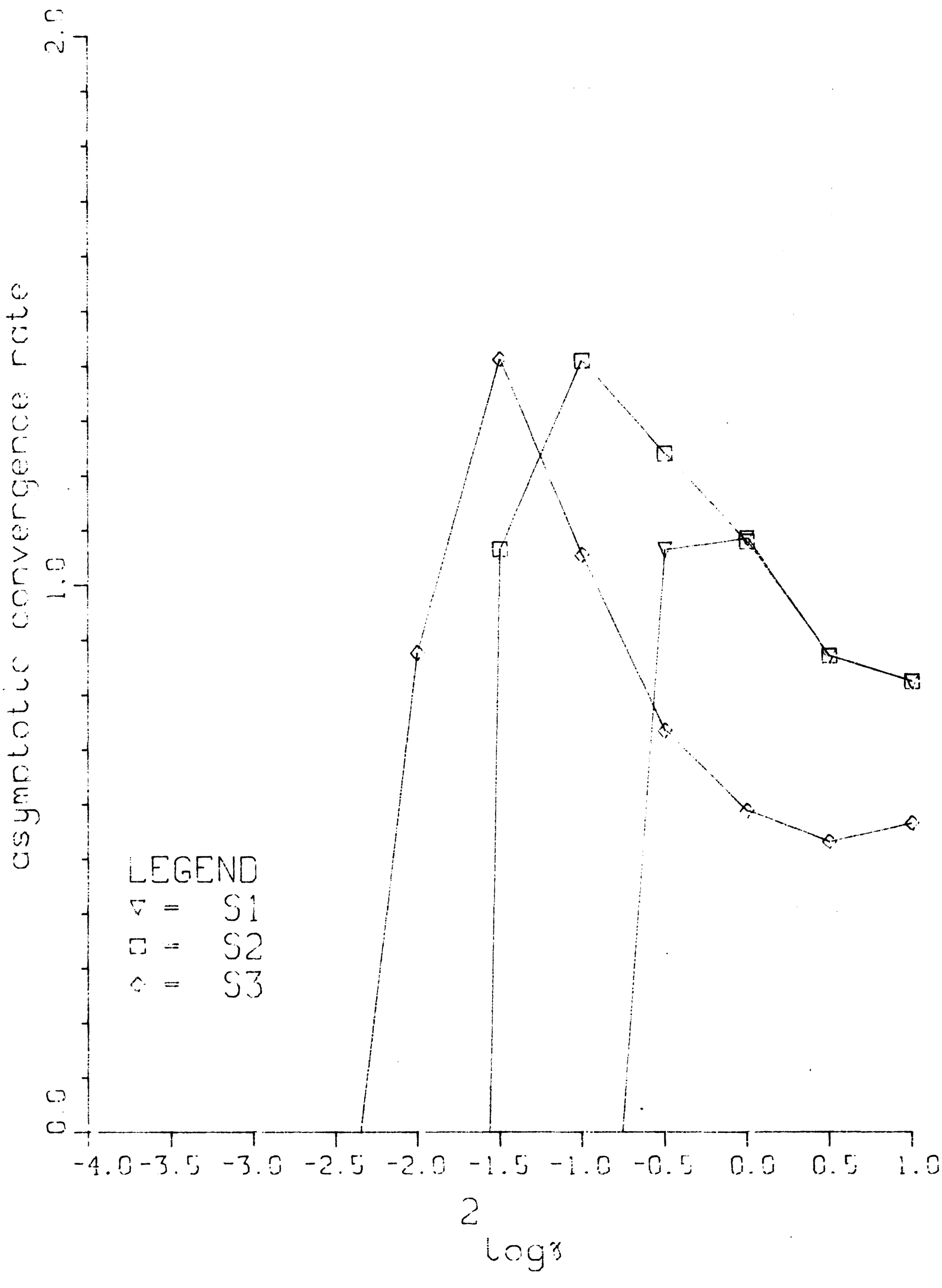


Figure 4. Asymptotic convergence rates for testproblem 1. Only the part of the figure with positive asymptotic convergence rate is drawn.

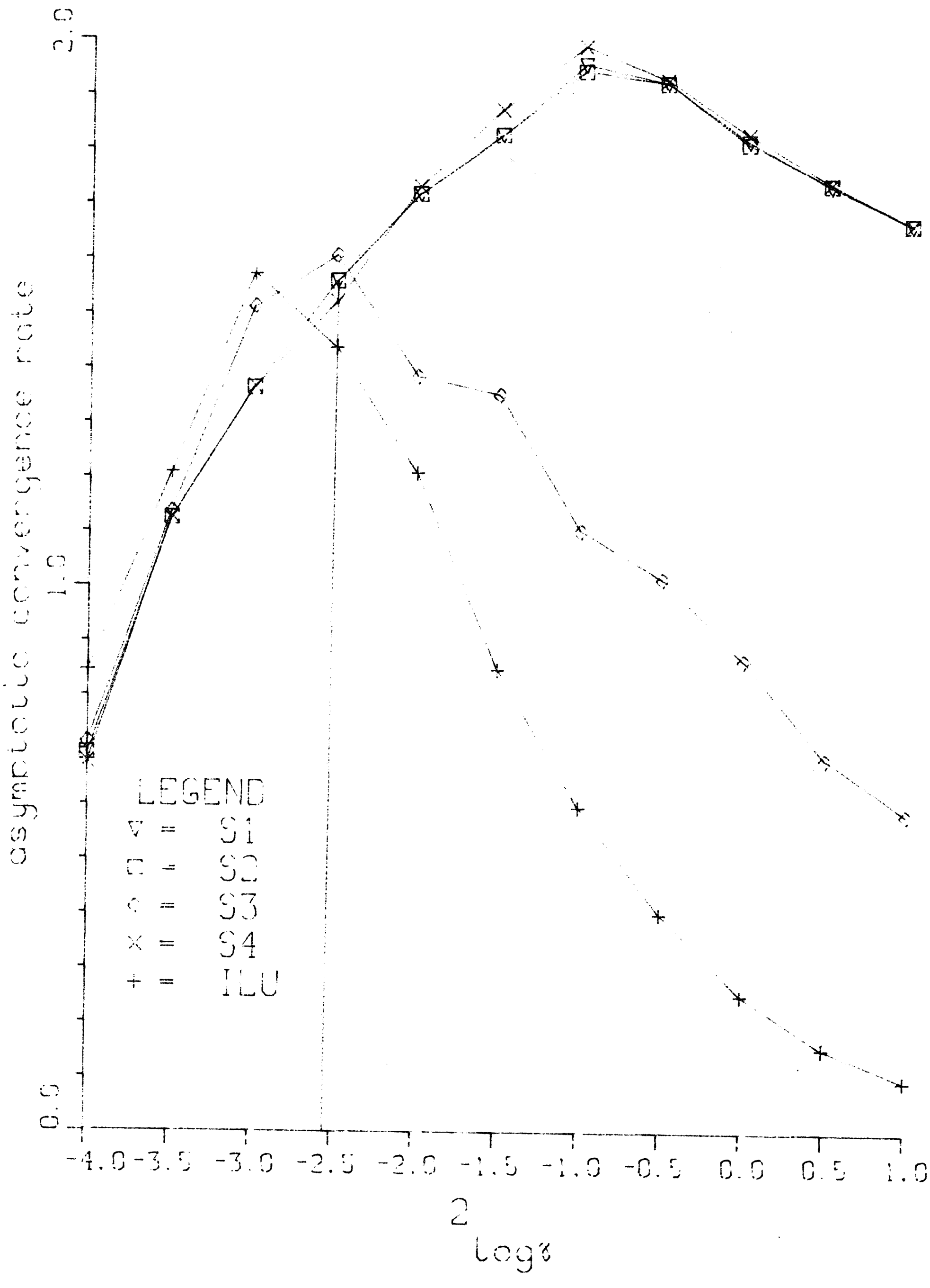


Figure 5. Asymptotic convergence rates for testproblem 2. The graph depicted by "+" represents two ILU-relaxation-sweeps in one iteration step without coarse-grid correction. Only the part of the figure with positive asymptotic convergence rate is drawn.

level ℓ	h_ℓ.	strategy			
		S_1	S_2	S_3	S_4
4	1/32	2.01	1.78	1.61	2.01
5	1/64	<<0	1.70	1.33	<<0
6	1/128	<<0	1.17	0.87	<<0

Table 1. Convergence rates for testproblem 3, $S_1 - S_4$, and increasing ℓ.

k \ S	S_1	S_2	S_3
3	0.36	0.36	0.36
2	4.84	4.84	0.36
1	186.	4.84	0.36

$^2\log \gamma = -1.5$

k \ S	S_1	S_2	S_3
3	0.24	0.24	0.24
2	0.80	0.80	0.24
1	15625.	0.80	0.24

$^2\log \gamma = -1.0$

k \ S	S_1	S_2	S_3
3	0.23	0.23	0.23
2	0.36	0.36	0.23
1	4.84	0.36	0.23

$^2\log \gamma = -0.5$

k \ S	S_1	S_2	S_3
3	0.24	0.24	0.24
2	0.24	0.24	0.24
1	0.80	0.24	0.24

$^2\log \gamma = 0.0$

k \ S	S_1	S_2	S_3
3	0.24	0.24	0.24
2	0.23	0.23	0.24
1	0.36	0.23	0.24

$^2\log \gamma = 0.5$

k \ S	S_1	S_2	S_3
3	0.25	0.25	0.25
2	0.24	0.24	0.25
1	0.24	0.24	0.25

$^2\log \gamma = 1.0$

Table 2. Smoothing-factors for one SGS sweep, testproblem 1, different γ, levels and strategies (local mode analysis, cf. BRANDT [2])

5. CONCLUSIONS

In order to solve the convection-diffusion equation in two dimensions by a multi-level algorithm (MLA), we consider 4 strategies for coarse-grid operators:

S_1 : on each coarse grid the same artificial viscosity as on the finest grid,

S_3 : on each coarse grid the artificial viscosity corresponding to the meshwidth,

S_2 : an intermediate choice, with the same artificial viscosity on the two finest grids,

S_4 : Galerkin approximation for the coarse-grid operators.

In case of S_1 and S_4 the artificial viscosity may become too small on coarse grids and hence stability problems and bad smoothing-factors may occur. S_1 and S_4 are not ε-asymptotically stable, S_2 and S_3 are. (cf. (2.6),(2.14), (2.18),(2.24), Table 1).

If the finest-grid-artificial viscosity is sufficiently large the asymptotic rate of convergence of the MLA according to S_2 is far better than that of S_3. (cf. (2.26), Figure 4,5).

ACKNOWLEDGEMENTS

The authors like to thank Dr. P.W. Hemker (M.C., Amsterdam) and Prof. Dr. Ir. P. Wesseling (Department of Mathematics, Delft University) for their constructive comments and careful reading of the manuscript.

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