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ON THE BERRY-ESSEEN THEOREM FOR MULTIVARIATE U-STATISTICS

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On the Berry-Esseen theorem for multivariate U-statistics*

by

R. Helmers & P. Janssen**

ABSTRACT

A Berry-Esseen bound of order $N^{-1/2}$ is established for multivariate U-statistics. The bound is uniform w.r.t. wide classes of Borelsets. The result is extended to multivariate L-estimators.

KEY WORDS & PHRASES : *order of normal approximation, Berry-Esseen bounds, multivariate U-statistics, multivariate L-estimators*

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1. Introduction and result.

The theory of Berry-Esseen bounds for sums of independent random variables and vectors is a well established part of probability theory. In recent years the problem to extend these results to statistics of a more general form received considerable attention. One such extension would be a theory of Berry-Esseen bounds for univariate and multivariate U-statistics, similar to the one for sums of independent random variables and vectors. A number of authors have contributed to the univariate case. Bickel (1974) showed that the distribution function (d.f.) of a standardized univariate U-statistic tends to its normal limit at the rate $N^{-1/2}$ (N being the sample size) for the case of bounded kernels. His result was subsequently generalized by Chan & Wierman (1977), Callaert & Janssen (1978) and Helmers & van Zwet (1982). However these authors restrict attention to univariate U-statistics. In a review paper on refinements of the multivariate central limit theorem, Bhattacharya (1977) suggested to consider also the rate of convergence problem for multivariate U-statistics. It is the purpose of the present paper to establish Berry-Esseen bounds for multivariate U-statistics. For simplicity we restrict attention to U-statistics of degree 2.

Let $X_i = (X_{1i},\ldots,X_{pi})$, $i = 1,\ldots,N$ be independent and identically distributed (i.i.d.) random vectors (r.v.) with values in $\mathbb{R}^p$ (or more generally in a suitable product space S^p). Define, for $N = 2,3,\ldots$, the p-variate U-statistic by

$$(1.1) \qquad U_N = (U_{1N},\ldots,U_{pN})$$

where, for $\lambda = 1,\ldots,p$,

$$U_{\lambda N} = \binom{N}{2}^{-1} \sum_{1\leq i<j\leq N} h_\lambda (X_{\lambda i}, X_{\lambda j})$$

where h_λ is a symmetric real-valued functions of two variables with

$$(1.2) \qquad Eh_\lambda (X_{\lambda 1}, X_{\lambda 2}) = 0$$

for $\lambda = 1,\ldots,p$. Let g_λ, $\lambda = 1,\ldots,p$, be given by

(1.3) $$g_\lambda(x_\lambda) = E[h_\lambda(X_{\lambda 1},X_{\lambda 2})|X_{\lambda 1} = x_\lambda]$$

and the covariance matrix of the random vector

(1.4) $$(g_1(X_{11}),\ldots,g_p(X_{p1}))$$

by V. Note that $Eg_\lambda(X_{\lambda 1}) = 0$ for $\lambda = 1,\ldots,p$. Throughout we will write $h(X_1,X_2)$ to denote $(h_1(X_{11},X_{12}),\ldots,h_p(X_{p1},X_{p2}))$ and $g(X_1)$ to indicate $(g_1(X_{11}),\ldots,g_p(X_{p1}))$. Let $\Phi(\cdot;0,V)$ denote the p-variate normal distribution with mean zero and (positive definite) covariance matrix V.

Hoeffding (1948) has shown that, as $N \to \infty$, the d.f. F_N^* of $U_N^* = 2^{-1}N^{1/2}U_N$ converges weakly to $\Phi(\cdot;0,V)$ provided $Eh_\lambda^2(X_{\lambda 1},X_{\lambda 2}) < \infty$, for $\lambda = 1,\ldots,p$, and V is positive definite. Here we investigate the rate of this convergence. For any Borelset A in the euclidean p-space $(\partial A)^\varepsilon$ will denote the ε-neighborhood of the boundary ∂A of A; i.e. $(\partial A)^\varepsilon$ is the set of all points at distances less than ε from ∂A. Let $\|x\|$ denote the euclidean norm of a vector $x = (x_1,\ldots,x_p)$; $\|x\| = (\Sigma_{\lambda=1}^p x_\lambda^2)^{1/2}$.

Theorem 1.1. If $E\|h(X_1,X_2)\|^{p+4} < \infty$ *and* V *is positive definite, then*

(1.5) $$\sup_{A\in \mathcal{A}} |P(\{U_N^*\in A\}) - \Phi(A;0,V)| = O(N^{-1/2})$$

where $\mathcal{A}$ *is any class of Borelsets in* $\mathbb{R}^p$, *which is closed under translations and which satisfies the assumption*

(1.6) $$\sup_{A\in \mathcal{A}} \Phi((\partial A)^\varepsilon;0,V) = O(\varepsilon) \quad \textit{as} \quad \varepsilon \downarrow 0 .$$

It is well-known (see for instance Bhattacharya & Rao (1976), corollary 3.2, p.24) that relation (1.6) is fullfilled for the important class $\mathcal{C}$ of all Borel-measurable convex subsets of $\mathbb{R}^p$. Obviously this implies that

$$\sup_{C\in\mathcal{C}} |P(\{U_N^* \in C\}) - \Phi(C;0,V)| = O(N^{-1/2})$$

and consequently also

(1.7) $$\sup_{x} |F_N^*(x) - \Phi(x;0,V)| = O(N^{-1/2})$$

is true under the moment assumptions of theorem 1.1.

Theorem 1.1 is the first general theorem establishing a Berry-Esseen bound of order $N^{-1/2}$ for multivariate U-statistics of the form (1.1). The strength of this result is that the optimal rate of convergence is obtained, uniformly valid, for wide classes of Borelsets. To establish this a rather stringent moment assumption is required. The restriction to translation invariant classes $\mathcal{A}$ seems rather harmless for applications. However, this condition can easily be dispensed with at cost of some technicalities. Theorem 1.1 and its corollary (1.7) immediately imply the earlier result of Bickel (1974) for the one-dimensional case p = 1. It fails, however, to yield the stronger results of Chan & Wierman (1977), Callaert & Janssen (1978) and Helmers & van Zwet (1982) for univariate U-statistics.

After the result of this paper was obtained we found that Berry-Esseen bounds for multivariate U-statistics of a different type than (1.1) were recently obtained by Carmichael (1981). In his (unpublished) Ph.D. thesis Carmichael derived a Berry-Esseen type theorem for multivariate U-statistics where each of the component U-statistics is based on the same univariate sample of N observations. Another related paper is that of Hušková (1978). She obtained Berry-Esseen type bounds for multivariate rank statistics. Both Carmichael and Hušková's proofs resemble ours as these authors also employ smoothing techniques.

We conclude this section with an application to multivariate L-estimators. Define, for N = 1,2,...,

(1.8) $$T_N = (T_{1N},\dots,T_{pN})$$

where, for $\lambda = 1,\dots,p$,

(1.9) $$T_{\lambda N} = N^{-1} \sum_{i=1}^{N} J_\lambda\left(\frac{i}{N+1}\right) X_{i:N}^{(\lambda)}$$

where $X_{i:N}^{(\lambda)}$, $i = 1,\ldots,N$, denotes the ith order statistic of $X_{\lambda 1},\ldots,X_{\lambda N}$ and, for $\lambda = 1,\ldots,p$, J_λ is a bounded measurable function on $(0,1)$. Let F_λ denote the marginal d.f. of $X_{\lambda 1}$ and 1_A the indicator of a set A. Let Λ denote the $p\times p$ correlation matrix whose (λ,ν) element is the correlation between $\bar{g}_\lambda(X_{\lambda 1})$ and $\bar{g}_\nu(X_{\nu 1})$ where $\bar{g}_\lambda(x_1) = -\int_{-\infty}^{+\infty} J_\lambda(F_\lambda(z))(1_{(-\infty,z]}(x_1) - F_\lambda(z))dz$. Finally let $G_N^*(x) = P(\{T_N^* \leq x\})$ for $x \in \mathbb{R}^p$, where $T_N^* = (T_{1N}^*,\ldots,T_{pN}^*)$ with $T_{\lambda N}^* = (T_{\lambda N} - ET_{\lambda N})/\sigma(T_{\lambda N})$ for $\lambda = 1,\ldots,p$.

Corollary 1.2. Let $E\|X_1\|^{p+4} < \infty$. *If, for* $\lambda = 1,\ldots,p$, *the function* J_λ *satisfies a Lipschitz condition of order* 1 *on* $(0,1)$ (*i.e.* $|J_\lambda(u) - J_\lambda(v)| \leq K|u-v|$ *for* $0 < u,v < 1$ *and some fixed* $K > 0$) *and* Λ *is positive definite, then*

(1.10) $$\sup_x |G_N^*(x) - \Phi(x;0,\Lambda)| = O(N^{-1/2}) .$$

Proof. It follows directly from Helmers (1981) that the components $T_{\lambda N}$, $\lambda = 1,\ldots,p$, can be related to appropriate U-statistics $U_{\lambda N+}$ and $U_{\lambda N-}$, $\lambda = 1,\ldots,p$, defined by

$$U_{\lambda N\pm} = \binom{N}{2}^{-1} \sum_{1\leq i<j\leq N} \bar{h}_{\lambda\pm}(X_{\lambda i},X_{\lambda j}) \quad \text{where}$$

$$\bar{h}_{\lambda\pm}(X_{\lambda i},X_{\lambda j}) = -\int_{-\infty}^{+\infty} J_\lambda(F_\lambda(z))\{1_{(-\infty,z]}(X_{\lambda i}) + 1_{(-\infty,z]}(X_{\lambda j}) - 2F_\lambda(z)\}dz$$

$$\pm\, 2K \int_{-\infty}^{+\infty} (1_{(-\infty,z]}(X_{\lambda i}) - F_\lambda(z))(1_{(-\infty,z]}(X_{\lambda j}) - F_\lambda(z))dz$$

with K as in the corollary, such that, with $U_{N\pm}^* = (U_{1N\pm}^*,\ldots,U_{pN\pm}^*)$, (cf. (3.34) and (3.35) of Helmers (1981))

$$(1.11)\qquad G_N^*(x) \leqslant P(\{U_{N-}^* \leqslant x_{N+}\}) + O(N^{-2/3})$$

$$(1.12)\qquad G_N^*(x) \geqslant P(\{U_{N+}^* \leqslant x_{N-}\}) + O(N^{-2/3})$$

for appropriate sequences x_{N+}, $N = 1,2,3,\ldots$ and x_{N-}, $N = 1,2,3,\ldots$ in $\mathbb{R}^p$ satisfying

$$(1.13)\qquad x_{N\pm} = x(1 + O(N^{-1/2})) + O(N^{-1/2})$$

uniformly in x.
We easily check that (1.7) can be applied to the first term on the right hand side of (1.11) and (1.12). It follows that these terms are equal to $\Phi(x_{N+};0,\Lambda) + O(N^{-1/2})$ and $\Phi(x_{N-};0,\Lambda) + O(N^{-1/2})$ respectively, uniformly in x. As these two expressions are easily seen to be equal to $\Phi(x;0,\Lambda) + O(N^{-1/2})$, uniformly in x, the proof of the corollary is complete. □

2. Proof of the theorem.

We introduce some more notation. For vectors $x = (x_1,\ldots,x_p)$, $y = (y_1,\ldots,y_p)$ in $\mathbb{R}^p$, $\langle x,y\rangle$ denotes the usual inner product between x and y, $\|x\| = \langle x,x\rangle^{1/2}$ and recall that 1_A denotes the indicator of a set A.

Define, for $N = 2,3,\ldots$, the p-variate r.v. S_N by

$$(2.1)\qquad S_N = (S_{1N},\ldots,S_{pN})$$

where, for $\lambda = 1,\ldots,p$,

$$(2.2)\qquad S_{\lambda N} = N^{-1/2} \sum_{i=1}^{N} g_\lambda(X_{\lambda i})$$

with g_λ as in (1.3). Define, for $\lambda = 1,\ldots,p$, functions ψ_λ by

$$(2.3)\qquad \psi_\lambda(X_{\lambda i},X_{\lambda j}) = [h_\lambda(X_{\lambda i},X_{\lambda j}) - g_\lambda(X_{\lambda i}) - g_\lambda(X_{\lambda j})]/2$$

then the random variable $\psi_\lambda(X_{\lambda i},X_{\lambda j})$ has the property

$$(2.4)\qquad E[\psi_\lambda(X_{\lambda i},X_{\lambda j})|X_{\lambda i}] = 0 \qquad \text{a.s.}$$

whenever $i \neq j$. Define, for $N = 2,3,\ldots$, the p-variate r.v. R_N by

(2.5) $$R_N = (R_{1N},\ldots,R_{pN})$$

where, for $\lambda = 1,\ldots,p$,

(2.6) $$R_{\lambda N} = N^{-1/2}(N-1)^{-1} \sum_{1 \leq i < j \leq N} \psi_\lambda(X_{\lambda i}, X_{\lambda j}) \quad .$$

It is easily checked that

(2.7) $$U_N^* = S_N + R_N$$

where S_N is a sum of independent and identically distributed random vectors and R_N is a remainder term. It will be convenient to decompose R_N as follows :

(2.8) $$R_N = R_N' + R_N''$$

where $R_N' = (R_{1N}',\ldots,R_{pN}')$ and $R_N'' = (R_{1N}'',\ldots,R_{pN}'')$ are p-vectors given by

(2.9) $$R_{\lambda N}' = N^{-1/2}(N-1)^{-1} \sum_{1 \leq i < j \leq C_N} \psi_\lambda(X_{\lambda i}, X_{\lambda j})$$

and

(2.10) $$R_{\lambda N}'' = R_{\lambda N} - R_{\lambda N}', \quad \lambda = 1,\ldots,p,$$

where

(2.11) $$N - C_N = N^{2/3} \quad .$$

The starting point of our proof will be the following simple inequality (F_Z denotes the d.f. of a r.v. Z). For any Borelset A in $\mathbb{R}^p$ we have

$$(2.12)\quad |P(\{U_N^* \in A\}) - \Phi(A;0,V)|$$
$$\leqslant |\int 1_A(x)d(F_N^*(x) - F_{Q_N}(x))|$$
$$+ |\int 1_A(x)d(F_{Q_N}(x) - F_{S_N}(x))|$$
$$+ |\int 1_A(x)d(F_{S_N}(x) - \Phi(x;0,V))|$$
$$= I_{1N} + I_{2N} + I_{3N},$$

where $Q_N = S_N + R'_N$. Thus the theorem is proved if we can show that each of the terms I_{1N}, I_{2N} and I_{3N} is of the required order of magnitude $N^{-1/2}$, uniformly for all Borelsets A in any class $\mathcal{A}$ which is closed under translations and which satisfies the assumption (1.6). The order bound $O(N^{-1/2})$ for I_{3N} follows from :

Lemma 2.1. If the assumptions of theorem 1.1 *are satisfied, then*

$$(2.13)\quad \sup_{A\in\mathcal{A}} I_{3N} = O(N^{-1/2})$$

for any class $\mathcal{A}$ *as in theorem* 1.1.

Proof. Immediate from theorem 17.1 of Bhattacharya & Rao (1976), p.165. □

The next step is to reduce the problem of estimating I_{1N} to one of estimating I_{2N}.

Lemma 2.2. If the assumptions of theorem 1.1 *are satisfied, then*

$$(2.14)\quad \sup_{A\in\mathcal{A}} I_{1N} \leqslant \sup_{B\in\mathcal{B}_N} I_{2N} + O(N^{-1/2})$$

where $\mathcal{B}_N = \{(\partial A)^{N^{-1/2}} : A \in \mathcal{A}\}$ *for any class* $\mathcal{A}$ *as in theorem* 1.1.

Proof. Since $U_N = Q_N + R''_N$ we easily see that :

$$|I_{1N}| = |\int 1_A(x)d(F^*_N(x) - F_{Q_N}(x))|$$

$$= |E[1_A(U^*_N) - 1_A(Q_N)]|$$

$$\leq E[|1_A(U^*_N) - 1_A(Q_N)| 1_{\{\|R''_N\| < N^{-1/2}\}}]$$

$$+ E[|1_A(U^*_N) - 1_A(Q_N)| 1_{\{\|R''_N\| \geq N^{-1/2}\}}]$$

$$\leq P(\{Q_N \in (\partial A)^{N^{-1/2}}\}) + P(\{\|R''_N\| \geq N^{-1/2}\})$$

$$\leq |\int 1_{(\partial A)^{N^{-1/2}}}(x)d(F_{Q_N}(x) - F_{S_N}(x))|$$

$$+ |\int 1_{(\partial A)^{N^{-1/2}}}(x)d(F_{S_N}(x) - \Phi(x;0,V))|$$

$$+ |\int 1_{(\partial A)^{N^{-1/2}}}(x)d\Phi(x;0,V)|$$

$$+ P(\{\|R''_N\| \geq N^{-1/2}\})$$

$$= I_{1N1} + I_{1N2} + I_{1N3} + I_{1N4} .$$

Note that $\sup_{A \in \mathcal{A}} I_{1N1} = \sup_{B \in \mathcal{B}_N} I_{2N}$. Furthermore $\sup_{A \in \mathcal{A}} I_{1N2} = O(N^{-1/2})$ using lemma 2.1, whereas $\sup_{A \in \mathcal{A}} I_{1N3} = O(N^{-1/2})$ as a simple consequence of condition (1.6). Finally we consider I_{1N4}. Application of the Bonferroni inequality and the Markov inequality yields

$$I_{1N4} \leq \sum_{\lambda=1}^{p} P(\{|R''_{\lambda N}| \geq (pN)^{-1/2}\})$$

$$\leq (pN)^{3/2} \sum_{\lambda=1}^{p} E|R''_{\lambda N}|^3 .$$

The last expression is of the order $(N-C_N)^{3/2}N^{-3/2} = N^{-1/2}$ by (2.11) and the lemma on p.419 of Callaert & Janssen (1978). The proof is now complete. □

In view of the preceding results it remains to show that $\sup_{A \in \mathcal{A}} I_{2N}$ as well as $\sup_{B \in \mathcal{B}_N} I_{2N}$ are of the order $N^{-1/2}$. This is a rather delicate matter involving characteristic functions (ch.f.) arguments. We first reduce the problem of estimating I_{2N} to one of estimating derivatives up to order p+1 of the difference of the ch.f. of $Q_N = S_N + R'_N$ and S_N for values of the arguments inside a ball in $\mathbb{R}^p$ with radius of the order $N^{1/2}$. This transition is carried out in the following lemma. For any nonnegative integer p-vector $\gamma = (\gamma_1, \ldots, \gamma_p)$ let $|\gamma| = \sum_{\lambda=1}^{p} \gamma_\lambda$. For any r.v. X let $\varphi_X(t)$, $t \in \mathbb{R}^p$, denote the ch.f. of X. Finally we write $(\frac{\partial}{\partial t_1})^{\gamma_1} \ldots (\frac{\partial}{\partial t_p})^{\gamma_p}$ as D^γ.

Lemma 2.3. *If the assumptions of theorem* 1.1 *are satisfied then there exist positive constants* $C_1 > 0$ *and* $0 < C_2 < 1$ *such that*

$$(2.15) \qquad \sup_{A \in \mathcal{A}} I_{2N} \leq C_1 \max_{|\beta|=0,\ldots,p+1} \left\{ \int_{\|t\| \leq C_2 N^{1/2}} |D^\beta(\varphi_{Q_N}(t) - \varphi_{S_N}(t))| dt \right\} + O(N^{-1/2})$$

for any class $\mathcal{A}$ *as in theorem* 1.1. *Inequality* (2.15) *remains valid with* $\mathcal{A}$ *replaced by* $\mathcal{B}_N$.

Proof. Application of an appropriate smoothing inequality (lemma 1.8 of Bhattacharya (1977))and of a well-known lemma (lemma 1.9 of Bhattacharya (1977)), by means of which one can translate the L_1-norm of an integrable function to the L_1-norm of certain derivatives of its Fourier transform, we find that there exists positive constants $C_1 > 0$ and $0 < C_2 < 1$, such that

$$\sup_{A\in\mathcal{A}} I_{2N} \leq C_1 \max_{|\beta|=0,\dots,p+1} \{ \int_{\|t\|\leq C_2 N^{1/2}} |D^\beta(\varphi_{Q_N}(t) - \varphi_{S_N}(t))|dt\}$$

$$+ \sup_{A\in\mathcal{A}} P(\{S_N \in (\partial A)^{2N^{-1/2}}\}).$$

Application of lemma 2.1 implies that

$$\sup_{A\in\mathcal{A}} P(\{S_N \in (\partial A)^{2N^{-1/2}}\}) = O(N^{-1/2})$$

and the lemma is proved. □

It remains to show that the first term on the right hand side of (2.15) is of order $N^{-1/2}$. Let μ denote the smallest eigenvalue of V. Note that μ is positive. To begin with we consider 'large' values of $\|t\|$.

Lemma 2.4. *If the assumptions of theorem* 1.1 *are satisfied, then*

$$(2.16) \qquad \int_{d_N < \|t\|\leq C_2 N^{1/2}} |D^\beta(\varphi_{Q_N}(t) - \varphi_{S_N}(t)|dt = O(N^{-1/2})$$

with $0 \leq |\beta| \leq p+1$, C_2 *as in lemma* 2.3 *and*

$$d_N = \frac{p+7}{(3-\sqrt{2})\mu} N^{1/6}(\log N)^{1/2}.$$

Proof. Decompose S_N as $S'_N + S''_N$ where $S'_N = (S'_{1N},\dots,S'_{pN})$ and $S''_N = (S''_{1N},\dots,S''_{pN})$ are p-vectors given by

$$S'_{\lambda N} = N^{-1/2} \sum_{i=1}^{C_N} g_\lambda(X_{\lambda i})$$

and consequently

$$S''_{\lambda N} = S_{\lambda N} - S'_{\lambda N}.$$

By independence and Leibniz rule for differentiation

$$(2.17)\qquad D^{\beta}(\varphi_{Q_N}(t) - \varphi_{S_N}(t))$$

$$= D^{\beta}[E\{e^{i<t,S''_N>}\}E\{e^{i<t,S'_N>}(e^{i<t,R'_N>} - 1)\}]$$

$$= \Sigma^* D^{\alpha}E[e^{i<t,S''_N>}]D^{\delta}E[e^{i<t,S'_N>}(e^{i<t,R'_N>} - 1)]$$

where Σ^* denotes (finite) summation over all p-vectors $\alpha = (\alpha_1,\ldots,\alpha_p)$ and $\delta = (\delta_1,\ldots,\delta_p)$ with nonnegative integer components such that $|\alpha| + |\delta| = |\beta|$. The second factor in a summand of Σ^* can be bounded in absolute value by terms of the form

$$(2.18)\qquad E|\prod_{\lambda=1}^{p} (S'_{\lambda N})^{\delta'_\lambda} (R'_{\lambda N})^{\delta''_\lambda}|$$

with $|\delta'| + |\delta''| = |\delta|$. Application of Hölder's inequality yields that (2.18) can be bounded by

$$(2.19)\qquad \prod_{\lambda=1}^{p} [\{E|S'_{\lambda N}|^{|\delta|}\}^{\frac{\delta'_\lambda}{|\delta|}} \{E|R'_{\lambda N}|^{|\delta|}\}^{\frac{\delta''_\lambda}{|\delta|}}] .$$

The inequality of Marcinkievitz, Zygmund and Chung (Chung (1951)) yields that $E|S'_{\lambda N}|^{|\delta|}$, $\lambda = 1,\ldots,p$, is $O(1)$, whereas the lemma on p.419 of Callaert & Janssen (1978) implies that $E|R'_{\lambda N}|^{|\delta|}$, $\lambda = 1,\ldots,p$, is $O(N^{-|\delta|/2})$. Together these order bounds imply that (2.19) is at most $O(1)$. It follows that, uniformly for all t,

$$|D^{\delta}E[e^{i<t,S'_N>}(e^{i<t,R'_N>} - 1)]| = O(1)$$

for any $0 \leq |\delta| \leq |\beta|$.

To approximate the first factor in a summand of Σ^* we note that

$$|D^{\alpha}\, \varphi_{g(X_1)}^{N-C_N}(tN^{-1/2})|$$

$$\leqslant (N-C_N)^{|\alpha|} N^{-|\alpha|/2} E\|g(X_1)\|^{|\alpha|} |\varphi_{g(X_1)}(tN^{-1/2})|^{N-C_N-|\alpha|} .$$

By a slight modification of theorem 8.7 of Bhattacharya & Rao (1976) (see also the remark following their theorem 9.12) we find for all $\|t\| \leqslant C_2 N^{1/2}$

$$|\varphi_{g(X_1)}(tN^{-1/2})| \leqslant \exp[-(\tfrac{1}{2} - \tfrac{\sqrt{2}}{6}) \frac{<t,Vt>}{N}]$$

$$\leqslant \exp[-(\tfrac{1}{2} - \tfrac{\sqrt{2}}{6}) \mu \frac{\|t\|^2}{N}]$$

Combining these results, we easily obtain (2.16) with the choice of d_N as given in the statement of the lemma. This completes the proof. □

Next we investigate the first term on the right hand side of (2.15) for 'small' values of $\|t\|$. In the first place we remark that

$$(2.20) \qquad D^{\beta}(\varphi_{Q_N}(t) - \varphi_{S_N}(t))$$

$$= D^{\beta}[E\, e^{i<t,Q_N>} - E\, e^{i<t,S_N>}]$$

$$= i^{|\beta|} E[e^{i<t,Q_N>} \prod_{\lambda=1}^{p} (Q_{\lambda N})^{\beta_\lambda} - e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\beta_\lambda}] .$$

Taylor expansion of $e^{i<t,R'_N>}$ (recall that $Q_N = S_N + R'_N$) yields

$$(2.21) \qquad D^{\beta}(\varphi_{Q_N}(t) - \varphi_{S_N}(t))$$

$$= i^{|\beta|} E[e^{i<t,S_N>} \{ \prod_{\lambda=1}^{p} (Q_{\lambda N})^{\beta_\lambda} (1 + i<t,R'_N> + \frac{i^2}{2} <t,R'_N>^2$$

$$+ \frac{\theta i^3}{6} <t,R'_N>^3) - \prod_{\lambda=1}^{p} (S_{\lambda N})^{\beta_\lambda}\}]$$

where the modulus of θ is an appropriate random point in [0,1]. Clearly

$$\prod_{\lambda=1}^{p} (Q_{\lambda N})^{\beta_\lambda} = \prod_{\lambda=1}^{p} (S_{\lambda N} + R'_{\lambda N})^{\beta_\lambda}$$

$$= \prod_{\lambda=1}^{p} (S_{\lambda N})^{\beta_\lambda} + \sum_{\substack{|\alpha|+|\delta|\leqslant p+1 \\ |\delta|\geqslant 1}} \sum_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda}$$

where $\alpha = (\alpha_1,\ldots,\alpha_p)$, $\beta = (\beta_1,\ldots,\beta_p)$ and $\delta = (\delta_1,\ldots,\delta_p)$ again denote p-vectors with nonnegative integer components. It follows that (2.21) can be written as

(2.22) $$D^{\beta}(\varphi_{Q_N}(t) - \varphi_{S_N}(t))$$

$$= i^{|\beta|} E[e^{i<t,S_N>} \sum_{\substack{|\alpha|+|\delta|\leqslant|\beta| \\ |\delta|\geqslant 1}} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda}]$$

$$+ i^{|\beta|} E[e^{i<t,S_N>} \sum_{|\alpha|+|\delta|\leqslant|\beta|} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda} \cdot$$

$$(i<t,R'_N> + \frac{i^2}{2} <t,R'_N>^2 + \frac{\theta i^3}{6} <t,R'_N>^3)] \quad .$$

Appropriate bounds for the terms appearing on the right hand side of (2.22) are given in the next two lemma's.

<u>*Lemma 2.5.*</u> *If the assumptions of theorem* 1.1 *are satisfied then, with* d_N *as in lemma* 2.4,

(2.23) $$\int_{\|t\|\leqslant d_N} |E[e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda}]| dt = O(N^{-1/2})$$

with $0 \leqslant |\alpha| + |\delta| \leqslant p+1$ *and* $|\delta| \geqslant 2$, (2.24) $$\int_{\|t\|\leq d_N} \left| E\left[e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda} <t,R'_N>^k\right]\right| dt = O(N^{-1/2})$$

with $0 \leq |\alpha| + |\delta| \leq p+1$, $|\delta| \geq 1$ *and* $k = 1,2$,

(2.25) $$\int_{\|t\|\leq d_N} E\left| \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda} <t,R'_N>^3\right| dt = O(N^{-1/2})$$

with $0 \leq |\alpha| + |\delta| \leq p+1$.

Proof. We first prove (2.25). It is easily seen that the integrand in (2.25) can be bounded by a finite sum of terms of the form

$$\|t\|^3 \; E \left| \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda+\nu_\lambda}\right|$$

where $\nu = (\nu_1,\dots,\nu_p)$ is a non-negative integer p-vector satisfying $|\nu| = 3$. Application of Hölder's inequality yields, with $A = |\alpha| + |\delta| + 3$,

$$\|t\|^3 \; E \left| \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} (R'_{\lambda N})^{\delta_\lambda+\nu_\lambda}\right|$$

$$\leq \|t\|^3 \prod_{\lambda=1}^{p} [E|S_{\lambda N}|^A]^{\frac{\alpha_\lambda}{A}} [E|R'_{\lambda N}|^A]^{\frac{\delta_\lambda+\nu_\lambda}{A}}$$

$$= O(\|t\|^3 \, N^{-\frac{|\delta|}{2} - \frac{3}{2}})$$

where we have used arguments similar to those applied after (2.19) to obtain the orderbound in the last line. Hence the integral on the left-hand side of (2.25) is $O(d_N^4 N^{-3/2}) = O(N^{-5/6}(\log N)^2)$. This completes the proof of (2.25).

The proofs of (2.23) and (2.24) are similar. To establish (2.23) we first bound the integrand by $E|\prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}(R'_{\lambda N})^{\delta_\lambda}|$ and apply Hölder's inequality once more to find that the integral in (2.23) is of the order $O(d_N N^{-1}) = O(N^{-5/6}(\log N)^{1/2})$.
Similarly we bound the integrand in (2.24) by $\|t\|^k\ E|\prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}(R'_{\lambda N})^{\delta_\lambda}\|R'_N\|^k|$ and we obtain the order bound $O(d_N^{k+1}N^{-(|\delta|+k)/2}) = O(N^{1/6-k/3-|\delta|/2}(\log N)^{(k+1)/2}$ $= O(N^{-2/3}\log N)$ for the integral (2.24). This completes the proof. □

Our final lemma deals with the remaining terms in (2.22). As these terms are of a larger order of magnitude then those treated in the previous lemma, a much more careful analysis will be needed to obtain adequate order bounds.

Lemma 2.6. *If the assumptions of theorem* 1.1 *are satisfied then, with* d_N *as in lemma* 2.4,

$$\text{(2.26)} \qquad \int_{\|t\|\leqslant d_N} |E[e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}(R'_{\lambda N})^{\delta_\lambda}]|dt = O(N^{-1/2})$$

with $0 \leqslant |\alpha| \leqslant p$ *and* $|\delta| = 1$,

$$\text{(2.27)} \qquad \int_{\|t\|\leqslant d_N} |E[e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} <t,R'_N>]|dt = O(N^{-1/2})$$

with $0 \leqslant |\alpha| \leqslant p+1$,

$$\text{(2.28)} \qquad \int_{\|t\|\leqslant d_N} |E[e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} <t,R'_N>^2]|dt = O(N^{-1/2})$$

with $0 \leqslant |\alpha| \leqslant p+1$.

Proof.
We first prove (2.26). Without loss of generality we shall assume $\delta_1 = 1,\ \delta_2 = \ldots = \delta_p = 0$. First note that

$$\text{(2.29)}\qquad E[e^{i<t,S_N>} R'_{1N} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}]$$

$$= N^{-1/2}(N-1)^{-1} \sum_{1 \leqslant j < k \leqslant C_N} E[e^{i<t,S_N>} \psi_1(X_{1j},X_{1k}) \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}]$$

$$= 2^{-1} N^{-1/2}(N-1)^{-1} C_N(C_N-1)\ E[e^{i<t,S_N>} \psi_1(X_{11},X_{12}) \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}]$$

$$= O(N^{(1-|\alpha|)/2} \sum_{i_{11}=1}^{N} \cdots \sum_{i_{1\alpha_1}=1}^{N} \cdots \sum_{i_{p1}=1}^{N} \cdots \sum_{i_{p\alpha_p}=1}^{N} E[e^{i<t,N^{-1/2} \sum_{\ell=1}^{N} g(X_\ell)>} \cdot$$

$$g_1(X_{1i_{11}})\cdots g_1(X_{1i_{1\alpha_1}})\cdots g_p(X_{pi_{p1}})\cdots g_p(X_{pi_{p\alpha_p}})\psi_1(X_{11},X_{12})]).$$

To evaluate the order of magnitude of the summands we shall have to distinguish three cases. For a fixed summand, let Q denote the set of different integers among the indices $i_{11},\ldots,i_{1\alpha_1},\ldots,i_{p1},\ldots,i_{p\alpha_p}$ appearing in this summand and let m be the cardinality of Q. Obviously $1 \leqslant m \leqslant |\alpha|$. We either have (i) $Q \cap \{1,2\} = \phi$, (ii) $Q \cap \{1,2\}$ is $\{1\}$ or $\{2\}$ or (iii) $Q \cap \{1,2\} = \{1,2\}$. We first consider case (i). Using the independence present we have for a fixed value of m :

$$\text{(2.30)}\qquad E[e^{i<t,N^{-1/2} \sum_{\ell=1}^{N} g(X_\ell)>} \cdot$$

$$g_1(X_{1i_{11}})\cdots g_1(X_{1i_{1\alpha_1}})\cdots g_p(X_{pi_{p1}})\cdots g_p(X_{pi_{p\alpha_p}})\ \psi_1(X_{11},X_{12})]$$

$$= \varphi^{N-m-2}_{g(X_1)}(tN^{-1/2})\ E[\prod_{\lambda=1}^{p} \prod_{\ell=1}^{\alpha_\lambda} e^{it_\lambda N^{-1/2} g_\lambda(X_{\lambda i_{\lambda\ell}})} g_\lambda(X_{\lambda i_{\lambda\ell}})] \cdot$$

$$E[e^{i<t,N^{-1/2}(g(X_1)+g(X_2))>} \psi_1(X_{11},X_{12})]\ .$$

Similarly as in the proof of lemma 2.4 we find that for all $\|t\| \leq C_2 N^{1/2}$

$$\varphi^{N-m-2}_{g(X_1)}(tN^{-1/2}) = O(\exp(-(\tfrac{1}{2}-\tfrac{\sqrt{2}}{6})\,\mu\,\tfrac{N-p-3}{N}\,\|t\|^2)) \quad . \tag{2.31}$$

Moreover, by the moment assumption,

$$|E[\prod_{\lambda=1}^{p}\prod_{\ell=1}^{\alpha_\lambda} e^{it_\lambda N^{-1/2} g_\lambda(X_{\lambda i_{\lambda\ell}})}\, g_\lambda(X_{\lambda i_{\lambda\ell}})]| = O(1) \tag{2.32}$$

uniformly for all $\|t\|$. Finally by a well known argument (cf. Callaert & Janssen (1978), p.418) involving conditional expectations (see also relation (2.4)), we find, uniformly for all t,

$$|E[e^{i\langle t,N^{-1/2}(g(X_1)+g(X_2))\rangle}\,\psi_1(X_{11},X_{12})]| \tag{2.33}$$

$$= |E[(e^{i\langle t,N^{-1/2}g(X_1)\rangle}-1)(e^{i\langle t,N^{-1/2}g(X_2)\rangle}-1)\,\psi_1(X_{11},X_{12})]|$$

$$= O(N^{-1}\|t\|^2) \quad .$$

Combining now (2.30) - (2.33) we see that, for a fixed value of m, the contribution of a single summand of type (i) in (2.29) is of the order

$$O(N^{-1/2-|\alpha|/2}\|t\|^2\,\exp(-(\tfrac{1}{2}-\tfrac{\sqrt{2}}{6})\,\mu\,\tfrac{N-p-3}{N}\,\|t\|^2))$$

for all $\|t\| \leq C_2 N^{1/2}$. Since the number of terms in (2.29) with Ω satisfying (i) and with cardinality m is of order N^m we obtain that the contribution of all such terms is

$$O(N^{-1/2-|\alpha|/2+m}\|t\|^2\,\exp(-(\tfrac{1}{2}-\tfrac{\sqrt{2}}{6})\,\mu\,\tfrac{N-p-3}{N}\,\|t\|^2)) \tag{2.34}$$

for all $\|t\| \leq C_2 N^{1/2}$. It follows from (2.34) that the total contribution of summands in (2.29) with Q satisfying (i) and with cardinality $m \leq |\alpha|/2$ is of the order

(2.35) $$O(N^{-1/2}\|t\|^2 \exp(-(\tfrac{1}{2}-\tfrac{\sqrt{2}}{6})\,\mu\,\frac{N-p-3}{N}\,\|t\|^2)$$

which will be sufficient for our purposes. For summands in (2.29) with Q satisfying (i) and with cardinality $|\alpha|/2 < m \leq |\alpha|$ a more refined argument is needed to show that the total contribution of all such terms is also of a required order of magnitude. In this case the estimation of (2.32) is too crude. If $|\alpha|$ is even and $m = \frac{|\alpha|}{2} + k$ for some $1 \leq k \leq |\alpha|/2$ there are exactly 2k different factors of the form $\exp(it_\lambda N^{-1/2} g_\lambda(X_{\lambda i_{\lambda\ell}}))g_\lambda(X_{\lambda i_{\lambda\ell}})$ in the product appearing in (2.32). Thus, by independence, we have

(2.36) $$\Big|E\Big[\prod_{\lambda=1}^{p}\prod_{\ell=1}^{\alpha_\lambda} e^{it_\lambda N^{-1/2} g_\lambda(X_{\lambda i_{\lambda\ell}})}\, g_\lambda(X_{\lambda i_{\lambda\ell}})\Big]\Big|$$

$$= O(\Pi^*|E[e^{it_\lambda N^{-1/2}\, g_\lambda(X_{\lambda i_{\lambda\ell}})}\, g_\lambda(X_{\lambda i_{\lambda\ell}})]|$$

where Π^* denotes the product over the 2k different factors. Since

$$|E[e^{it_\lambda N^{-1/2}\, g_\lambda(X_{\lambda i_{\lambda\ell}})}\, g_\lambda(X_{\lambda i_{\lambda\ell}})]|$$

$$= |E[(e^{it_\lambda N^{-1/2}\, g_\lambda(X_{\lambda i_{\lambda\ell}})} - 1)\, g_\lambda(X_{\lambda i_{\lambda\ell}})]| = O(N^{-1/2}\|t\|),$$

we find that in the present case (2.32) is $O(\|t\|^{2k}\, N^{-k})$ uniformly for all t. Combining this with (2.31) and (2.33) yields that the contribution of all summands with Q satisfying (i) and with cardinality $m = \frac{|\alpha|}{2} + k$, $1 \leq k \leq |\alpha|/2$, is of order (cf.(2.35))

$$(2.37) \qquad O(N^{-1/2}\|t\|^{2k+2} \exp(-(\tfrac{1}{2}-\tfrac{\sqrt{2}}{6})\mu \frac{N-p-3}{N} \|t\|^2)).$$

If $|\alpha|$ is odd a similar argument yields (2.37) with $\|t\|^{2k+2}$ replaced by $\|t\|^{2k+1}$. Thus we have obtained adequate order bounds for the summands in (2.29) corresponding to case (i). The other two cases (ii) and (iii) can be treated similarly. We indicate the modifications. In case (ii) (2.31) remains valid with the exponent N-m-2 replaced by N-m-1, whereas instead of (2.33) we use the similar bound $|E[e^{i<t,N^{-1/2}g(X_1)>}\psi_1(X_{11},X_{12})]| = O(N^{-1/2}\|t\|)$. Note that this bound is of higher order than the corresponding bound (2.33) in case (i). This, however, is compensated by the fact that the number of summands of type (ii) appearing in (2.29) is of a lower order. Finally in case (iii) the order bound corresponding to (2.33) is only of order O(1), but now the number of summands of type (iii) appearing in (2.29) is even lower than those of type (ii). We can conclude that the order of the contribution of summands of type (ii) and (iii) in (2.29) is of a required order $O(N^{-1/2}\|t\|^{\ell}\exp(-(\tfrac{1}{2}-\tfrac{\sqrt{2}}{6})\mu \frac{N-p-3}{N} \|t\|^2))$, with ℓ only depending on Ω.

This together with (2.35) and (2.37) shows that the integral (2.26) is $O(N^{-1/2})$. This completes the proof of (2.26).

Proof of (2.27). This is a simple consequence of (2.26), since the integrand in (2.27) can be bounded by

$$\|t\| \sum_{\lambda=1}^{p} |E[e^{i<t,S_N>} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda} R'_{\lambda N}]| \quad .$$

Sketch of the proof of (2.28). To begin with we note that

$$(2.38)\qquad \left|E\left[e^{i<t,S_N>} <t,R'_N>^2 \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}\right]\right|$$

$$\leq \|t\|^2 \sum_{\nu_1=1}^{p} \sum_{\nu_2=1}^{p} \left|E\left[e^{i<t,S_N>} R'_{\nu_1 N} R'_{\nu_2 N} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}\right]\right|$$

where

$$(2.39)\qquad E\left[e^{i<t,S_N>} R'_{\nu_1 N} R'_{\nu_2 N} \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}\right]$$

$$= N^{-1}(N-1)^{-2} \sum_{1 \leq j_1 < k_1 \leq C_N} \sum_{1 \leq j_2 < k_2 \leq C_N} \cdot$$

$$E\left[e^{i<t,S_N>} \psi_{\nu_1}(X_{\nu_1 j_1}, X_{\nu_1 k_1}) \psi_{\nu_2}(X_{\nu_2 j_2}, X_{\nu_2 k_2}) \prod_{\lambda=1}^{p} (S_{\lambda N})^{\alpha_\lambda}\right]$$

$$= O(N^{1-|\alpha|/2} \Sigma^{(1)} E[e^{i<t,S_N>} g_1(X_{1i_{11}}) \ldots g_1(X_{1i_{1\alpha_1}}) \ldots$$

$$\ldots g_p(X_{pi_{p1}}) \ldots g_p(X_{pi_{p\alpha_p}}) \psi_{\nu_1}(X_{\nu_1 1}, X_{\nu_1 2}) \psi_{\nu_2}(X_{\nu_2 3}, X_{\nu_2 4})])$$

$$+ O(N^{-|\alpha|/2} \Sigma^{(1)} E[e^{i<t,S_N>} g_1(X_{1i_{11}}) \ldots g_1(X_{1i_{1\alpha_1}}) \ldots$$

$$\ldots g_p(X_{pi_{p1}}) \ldots g_p(X_{pi_{p\alpha_p}}) \psi_{\nu_1}(X_{\nu_1 1}, X_{\nu_1 2}) \psi_{\nu_2}(X_{\nu_2 2}, X_{\nu_2 3})])$$

$$+ O(N^{-1-|\alpha|/2} \Sigma^{(1)} E[e^{i<t,S_N>} g_1(X_{1i_{11}}) \ldots g_1(X_{1i_{1\alpha_1}}) \ldots$$

$$\ldots g_p(X_{pi_{p1}}) \ldots g_p(X_{pi_{p\alpha_p}}) \psi_{\nu_1}(X_{\nu_1 1}, X_{\nu_1 2}) \psi_{\nu_2}(X_{\nu_2 1}, X_{\nu_2 2})]) ,$$

with $\Sigma^{(1)} = \sum_{i_{11}=1}^{N} \ldots \sum_{i_{1\alpha_1}=1}^{N} \ldots \sum_{i_{p1}=1}^{N} \ldots \sum_{i_{p\alpha_p}=1}^{N}$. The remaining part of the proof of (2.38) can be carried out along the lines of the proof of (2.26). Again we distinguish a number of different cases, use the independence present in each case and establish order bounds for the expectations appearing in the resulting expression. Al together this leads to the desired order bound for the integrand of (2.28). □

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