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ASYMPTOTIC ANALYSIS OF A QUEUEING SYSTEM WITH A TWO-DIMENSIONAL STATE SPACE

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Asymptotic analysis of a queueing system with a two-dimensional state space*)

by

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ABSTRACT

The asymptotic behaviour on the long run of a queueing system with two types of customers, a Poissonian arrival stream, paired services and a general service time distribution is considered. The generating function of the time-dependent joint queue length distribution can be obtained with the aid of the theory of boundary value problems for regular functions and of the theory of conformal mappings of the unit disk onto a given domain. In the asymptotic analysis of this generating function an extensive use is made of theorems on the boundary behaviour of conformal mappings.

KEY WORDS & PHRASES: *asymptotic behaviour, queueing system, two-dimensional state space, paired service, conformal mapping*

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*) This report will be submitted for publication elsewhere.

1. Introduction

The asymptotic behaviour of queueing systems with a one-dimensional state space has been extensively studied in the past, see e.g. [2], §III.7.3. In the performance analysis of computer networks the study of queueing models with a two-dimensional state space, especially of their transient behaviour, is of great importance. The analysis of such queueing processes often leads to the problem of solving a functional equation for a bivariate generating function of a probability distribution. Until recently the analysis of the asymptotic behaviour of queueing systems with a two- (or more) dimensional state space was not possible because of the lack of analytical tools, which are powerful enough for the solution of these functional equations. However, during the last few years a method has been developed for the solution of an important class of these functional equations with the aid of the theory of boundary value problems for regular functions and by using the conformal mapping of the unit disk onto a given domain, see [5],[3]. In these papers only stationary distributions are considered. The aim of this paper is to provide a first step in the asymptotic analysis of queueing processes with a two-dimensional state space to which the above mentioned method of analysis is applicable.
For this purpose a relatively simple model, related to the model studied in [3], see also [4], has been chosen; see §2 for its description. First the derivation of the generating function of the time-dependent queue length distribution will be sketched. Then this paper is concerned with the question on which conditions the process is ergodic, null-recurrent, or transient. The answer to this question is not always obvious, see e.g. the ergodic conditions in [5]. The relaxation time, a mesure for the speed at which the stationary distribution is approached, for queueing systems with a two-dimensional state space is still a topic of further research.

2. The model, definitions

The following queueing model will be considered. Customers arrive at a single service facility according to a Poisson process with mean interarrival time α. With equal probabilities an arriving customer is of type 1 or of type 2. An arriving customer who finds the system empty is immediately taken into service; otherwise he joins queue 1 or 2 depending on his type. As soon as a service has been completed, a new service is started if any customers are present. In general a couple of two customers of different type is simultaneously served. If after the completion of a service only customers of one type are present, a customer of this type is individually served. In each queue customers are served in order of their arrival. Successive service times are independent random variables with a common distribution function B(t), for paired services as well as for individual services.

Let $\underline{y}_j(t)$, $t \geq 0$, $j = 1,2$, be the number of type j customers present in the system at time t, and let $\underline{y}_j(0) = 0$. Our aim is to study the time-dependent behaviour of the process $\{(\underline{y}_1(t), \underline{y}_2(t)), t \geq 0\}$, especially its asymptotic behaviour as $t \to \infty$.

In [1] the process $\{(\underline{y}_1(t), \underline{y}_2(t)), t \geq 0\}$ has been studied with the aid of two supplementary variables. Here we shall use another approach. It should be noted that the model that we consider is related to the M/G/1 model. Therefore, in order to obtain the distribution of the process $\{(\underline{y}_1(t), \underline{y}_2(t)), t \geq 0\}$ first the imbedded process at departure instants will be analysed, and then the continuous time distribution will be derived with the aid of renewal functions, in analogy with the analysis of the M/G/1 model, cf. [2], §II.4.3. Denote by $\underline{d}_n$, $n = 0,1,..$, the n^{th} departure instant and by $\underline{x}_j(n)$, $n = 0,1,..$, $j = 1,2$, the number of type j customers left behind in the system at the n^{th} departure instant. Because it was assumed that the process starts at $t=0$ with an empty system, we take $\underline{d}_0=0$, $\underline{x}_1(0)=\underline{x}_2(0)=0$. It is readily seen that the process $\{(\underline{x}_1(n),\underline{x}_2(n),\underline{d}_n)n=0,1,..\}$ is an imbedded Markov chain which is irreducible and aperiodic. This Markov chain will be studied in the next section.

For the analysis of the queueing system the following functions and quantities are defined: for $|r|<1$, $|p_1|\le 1$, $|p_2|\le 1$, Re $\rho\ge 0$,

(1) $$\Phi(r;p_1,p_2,\rho) := \sum_{n=0}^{\infty} r^n\, E\{p_1^{\underline{x}_1(n)}\, p_2^{\underline{x}_2(n)}\, e^{-\rho \underline{d}_n}\},$$

(2) $$\Psi(\rho;p_1,p_2) := \int_0^\infty e^{-\rho t}\, E\{p_1^{\underline{y}_1(t)}\, p_2^{\underline{y}_2(t)}\}\, dt;$$

(3) $$\beta(s) := \int_0^\infty e^{-st}\, d\,B(t), \qquad \text{Re } s\ge 0;$$

(4) $$\beta_k := \int_0^\infty t^k\, d\,B(t), \qquad k=1,2,..;$$

(5) $$a := \beta_1/\alpha.$$

It is assumed that $\beta_3<\infty$ (see the remark in §6).

3. The imbedded Markov chain

For the transform (1) the following functional equation can be derived in a similar way as in [2], §II.4.3, for the M/G/1 queueing system: for $|r|<1$, $|p_1|\le 1$, $|p_2|\le 1$, Re $\rho\ge 0$,

(6) $$\left[p_1p_2 - r\,\beta\!\left(\rho+\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\right)\right]\Phi(r;p_1,p_2,\rho) = p_1p_2 + r\,\beta\!\left(\rho+\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\right)\times$$
$$\times\ [(p_2-1)\Phi(r;p_1,0,\rho) + (p_1-1)\Phi(r;0,p_2,\rho) + (1-p_1-p_2+\frac{p_1p_2}{1+\alpha\rho})\Phi(r;0,0,\rho)].$$

In the solution of this functional equation given below it is assumed that r *and* ρ *are real,* $0<r<1$, $\rho\ge 0$.

The functional equation (6) can be reduced by considering zeros (p_1,p_2) of the *kernel*

$$(7) \qquad p_1 p_2 - r\,\beta\Big(\rho + \frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\Big),$$

inside the region $|p_1| \le 1$, $|p_2| \le 1$, where the generating function (1) is finite. In fact, we choose $p_1 = w$, $p_2 = \bar{w}$, and w in the set

$$(8) \qquad L(r;\rho) := \{w;\ |w| < 1,\ |w|^2 = r\,\beta(\rho + \frac{1-\mathrm{Re}\ w}{\alpha})\};$$

then equation (6) reduces to: for $w \in L(r;\rho)$,

$$(9) \quad \frac{\Phi(r;w,0,\rho)}{1-w} + \frac{\Phi(r;0,\bar{w},\rho)}{1-\bar{w}} = \frac{1}{|1-w|^2} + \left[1 - \frac{|w|^2}{|1-w|^2}\,\frac{\alpha\rho}{1+\alpha\rho}\right]\Phi(r;0,0,\rho).$$

From the properties of the Laplace-Stieltjes transform (3) it is easily derived that $L(r;\rho)$ is a contour (i.e. it does not intersect itself) which has the real axis as an axis of symmetry.

Next, the conformal mapping $g(r;\rho;z)$ of the unit disk $|z| < 1$ onto the domain $L^+(r;\rho)$, the interior of the contour $L(r;\rho)$, is introduced. This conformal mapping is uniquely determined by the conditions (cf. [7], vol. III, theorem 1.2,1.3):

$$(10) \qquad g(r;\rho;0) = 0, \qquad \frac{\partial}{\partial z}\,g(r;\rho;z)\Big|_{z=0} > 0.$$

By [7], vol.III,§8, theorem 2.24, the conformal mapping $g(r;\rho;z)$ is continuous in $|z| \le 1$, and maps the unit circle $|z| = 1$ one-to-one onto the contour $L(r;\rho)$. Moreover, the symmetry of the contour $L(r;\rho)$ leads to the property: for $|z| \le 1$,

$$(11) \qquad g(r;\rho;\bar{z}) = \overline{g(r;\rho;z)}.$$

By inserting $w = g(r;\rho;t)$, $|t| = 1$, so that $\bar{w} = g(r;\rho;\frac{1}{t})$ by (11), we obtain from (9) the equation: for $|t| = 1$,

$$(12)\qquad \frac{\Phi(r;g(r;\rho;t),0,\rho)}{1-g(r;\rho;t)}+\frac{\Phi(r;0,g(r;\rho;1/t),\rho)}{1-g(r;\rho;1/t)}=$$

$$=\frac{1}{|1-g(r;\rho;t)|^2}+\left[1-\frac{|g(r;\rho;t)|^2}{|1-g(r;\rho;t)|^2}\,\frac{\alpha\rho}{1+\alpha\rho}\right]\Phi(r;0,0,\rho).$$

Because $L^+(r;\rho)\subset\{w;\ |w|<1\}$, the first term at the lefthand side of (12) is regular for $|t|<1$, the second term for $|t|>1$. Relation (12) forms the boundary condition of a coupling problem (or Hilbert problem), cf. [8], §37. It is easily solved by applying the operator

$$(13)\qquad \frac{1}{2\pi i}\int_C \ldots\ldots \frac{dt}{t-z},\qquad\qquad C:=\{t;\ |t|=1\},$$

on both sides of equation (11), for $|z|<1$ as well as for $|z|>1$. The last unknown $\Phi(r;0,0,\rho)$ is obtained by taking $z=0$:

$$(14)\quad \Phi(r;0,0,\rho)=\frac{\dfrac{1}{2\pi i}\displaystyle\int_C \frac{1}{|1-g(r;\rho;t)|^2}\frac{dt}{t}}{1+\dfrac{\alpha\rho}{1+\alpha\rho}\dfrac{1}{2\pi i}\displaystyle\int_C\frac{|g(r;\rho;t)|^2}{|1-g(r;\rho;t)|^2}\frac{dt}{t}}\,.$$

By introducing the inverse conformal mapping of $g(r;\rho;z)$ the functions $\Phi(r;p_1,0,\rho)$ and $\Phi(r;0,p_2,\rho)$ are obtained. Substitution of these functions and of (14) in the functional equation (6) leads to an expression for the generating function $\Phi(r;p_1,p_2,\rho)$. This expression is deleted here because of its length (see [1], chapter III).

4. The continuous time process

By using a similar relation as [2], formula (II.4.45), between the distribution of $(\underline{y}_1(t),\underline{y}_2(t))$, $t\geq 0$, and that of $(\underline{x}_1(n),\underline{x}_2(n),\underline{d}_n)$, $n=0,1,\ldots$, we obtain the following relation between the generating functions (1) and (2): for $|p_1|\leq 1$, $|p_2|\leq 1$, $\mathrm{Re}\ \rho>0$,

$$(15)\qquad \Psi(\rho;p_1,p_2) = \frac{\alpha}{1+\alpha\rho}\,\beta\!\left(\rho+\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\right)\Phi(1;0,0,\rho) +$$

$$+\ \alpha\,\frac{1-\beta\!\left(\rho+\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\right)}{\alpha\rho+1-\frac{1}{2}p_1-\frac{1}{2}p_2}\,\Phi(1;p_1,p_2,\rho).$$

This determines the function $\Psi(\rho;p_1,p_2)$. In particular, (15) leads with (14) to: for real ρ, $\rho > 0$,

$$(16)\qquad \Psi(\rho;0,0) = \frac{\dfrac{\alpha}{2\pi i}\displaystyle\int_C \frac{1}{|1-g(r;\rho;t)|^2}\frac{dt}{t}}{1+\alpha\rho+\dfrac{\alpha\rho}{2\pi i}\displaystyle\int_C \frac{|g(r;\rho;t)|^2}{|1-g(r;\rho;t)|^2}\frac{dt}{t}}.$$

In the next sections the asymptotic behaviour of the process $\{(\underline{y}_1(t),\underline{y}_2(t)), t \geq 0\}$ as $t \to \infty$ will be studied. Similarly as for the M/G/1 queueing system it can be proved with the key renewal theorem, cf. [2], p.102, p.246, that the limits

$$\lim_{t\to\infty} \Pr\{\underline{y}_1(t)=k_1,\ \underline{y}_2(t)=k_2\},\qquad k_1=0,1,..,\ k_2=0,1,..,$$

exist. Hence, the generating function of this limiting distribution as $t \to \infty$ can be obtained from $\Psi(\rho;p_1,p_2)$ with the aid of an Abelian theorem; in particular we have

$$(17)\qquad \psi_0 := \lim_{t\to\infty} \Pr\{\underline{y}_1(t)=0,\ \underline{y}_2(t)=0\} = \lim_{\rho\downarrow 0} \rho\,\Psi(\rho;0,0).$$

Before we start with the investigation of this limit we introduce for real ρ, $\rho > 0$, the abbreviations

$$(18)\qquad \gamma(\rho;z) := g(1;\rho;z),\qquad |z| \leq 1;\qquad\qquad \Lambda(\rho) := L(1;\rho).$$

5. Asymptotic analysis

In this section the limit (17) will be investigated. First we note that, cf. [1], p.225, for $\rho > 0$,

$$\frac{1}{2\pi i}\int_C \frac{|\gamma(\rho;t)|^2}{|1-\gamma(\rho;t)|^2}\frac{dt}{t} = \frac{1}{2\pi i}\int_C \frac{1}{|1-\gamma(\rho;t)|^2}\frac{dt}{t} - 1,$$

so that (16) can be rewritten as, cf. (18), for $\rho > 0$,

$$\Psi(\rho;0,0) = \frac{\dfrac{\alpha}{2\pi i}\displaystyle\int_C \frac{1}{|1-\gamma(\rho;t)|^2}\frac{dt}{t}}{1+\dfrac{\alpha\rho}{2\pi i}\displaystyle\int_C \frac{1}{|1-\gamma(\rho;t)|^2}\frac{dt}{t}}. \tag{19}$$

Hence, in order to obtain the limit (17) we have to determine

$$\lim_{\rho\downarrow 0}\frac{\alpha\rho}{2\pi i}\int_C \frac{1}{\{1-\gamma(\rho;t)\}\{1-\gamma(\rho;\frac{1}{t})\}}\frac{dt}{t}. \tag{20}$$

For the determination of this limit we shall first investigate the behaviour of the contour $\Lambda(\rho)$ and of the conformal mapping $\gamma(\rho;z)$, cf. (18), as $\rho \downarrow 0$. In order to obtain a parametric equation for the contour $\Lambda(\rho)$ it is proved:

Lemma 1. *For* $\rho > 0$, $u \le 1$, *and for* $\rho \ge 0$, $u < 1$, *the equation*

$$\sigma^2 = \beta\left(\rho + \frac{1-\sigma u}{\alpha}\right), \tag{21}$$

has exactly one root $\sigma = \sigma(\rho;u)$ *on the real interval* $0 < \sigma < 1$. *This root* $\sigma(\rho;u)$ *is an infinitely differentiable function of* ρ *and* u, *with*

$$\frac{\partial}{\partial\rho}\sigma(\rho;u) < 0, \qquad \rho > 0,\ u \le 1. \tag{22}$$

Further,

$$\sigma(u) := \lim_{\rho\downarrow 0}\sigma(\rho;u) \begin{cases} = 1, & \text{if } u = 1,\ \alpha \le 2, \\ \in (0,1), & \text{otherwise;} \end{cases} \tag{23}$$

$$(24)\qquad \sigma'(1) = \frac{a}{2-a}, \qquad \sigma''(1) = 2\,\frac{2\beta_2/\alpha^2 + a^2 - a^3}{(2-a)^3}, \qquad \textit{if } a < 2,$$

$$(25)\qquad \sigma(u) = 1 - \sqrt{\frac{2}{2\beta_2/\beta_1^2 - 1}}\,\sqrt{1-u} + O(|1-u|), \qquad u \uparrow 1,\ \textit{if } a = 2.$$

Proof. With Rouché's theorem, cf. [7], vol.II,§7, theorem 2.4, it is readily proved that for $\rho > 0$, $u \le 1$, as well as for $\rho \ge 0$, $u < 1$, equation (21) has exactly two roots in the unit disk $|\sigma| < 1$. From the properties of the Laplace-Stieltjes transform $\beta(s)$ it follows then that these roots are real, and that one of them is positive, the other one negative. Hence, for $\rho > 0$, $u \le 1$, and for $\rho \ge 0$, $u < 1$, the positive root $\sigma(\rho;u)$ of (21) is a simple root, so that the stated differentiability follows from the implicit function theorem and the regularity of $\beta(s)$, Re $s > 0$.
Differentiation of (21) as function of ρ gives

$$(26)\qquad \frac{\partial}{\partial\rho}\sigma(\rho;u) = \frac{\beta'\left(\rho + \frac{1-u\,\sigma(\rho;u)}{\alpha}\right)}{2\sigma(\rho;u) + \frac{u}{\alpha}\,\beta'\left(\rho + \frac{1-u\,\sigma(\rho;u)}{\alpha}\right)}, \qquad \rho > 0,\ u \le 1.$$

Because $\beta'(s) < 0$ for real s, $s > 0$, the above derivative clearly is negative for $u \le 0$, $\rho > 0$. But then this derivative must be negative for $u \le 1$, $\rho > 0$, because the denominator in (26) is continuous and non-vanishing for $\rho > 0$, $u \le 1$, since the root $\sigma(\rho;u)$ is simple.
In the case $\rho = 0$, $u = 1$, comparison of the derivatives of the functions σ^2 and $\beta((1-\sigma)/\alpha)$ at $\sigma = 1$ proves that the smallest positive root of (21) is smaller than one if $a > 2$, and that it is equal to one if $a \le 2$. In the case $a < 2$ the root $\sigma(1) = 1$ of (21) is simple, and $\sigma'(1)$, $\sigma''(1)$, can be found by repeated differentiation of (21) as function of u, for $\rho = 0$. In the case $a = 2$ the root $\sigma(1) = 1$ of (21) is a double root; the expansion (25) can be derived from (21) by using the assumption $\beta_3 < \infty$ (see §2). □

With the aid of the function $\sigma(\rho;u)$ the contour $\Lambda(\rho)$ can be described as, cf. (18), (21), for $\rho > 0$,

(27) $$\Lambda(\rho) = \{w; \quad w = \sigma(\rho;\cos\theta)\, e^{i\theta}, \quad -\pi \le \theta \le \pi\}.$$

As ρ, $\rho > 0$, decreases to zero the contour $\Lambda(\rho)$ expands, cf. (22), to the contour Λ,

(28) $$\Lambda := \{w; \quad w = \sigma(\cos\theta)\, e^{i\theta}, \quad -\pi \le \theta \le \pi\}.$$

Lemma 2. *The contour* $\Lambda(\rho)$, $\rho > 0$, *has a tangent at every point. The contour* Λ *has a tangent at every point, except in the case* $a = 2$ *at the point* $w = 1$; *it has then at* $w = 1$ *a corner point with inner angle* $\omega\pi$,

(29) $$\omega\pi = 2 \arctan \sqrt{2\, \beta_2/\beta_1^2 - 1}, \qquad \tfrac{1}{2} \le \omega < 1.$$

Proof. In a similar way as in the proof of lemma 1, cf. (26), it follows by differentiation of (21) as function of u, that

(30) $$\frac{\partial}{\partial u}\sigma(\rho;u) > 0, \quad \rho > 0,\ u \le 1; \qquad \frac{\partial}{\partial u}\sigma(u) > 0, \quad u < 1.$$

Because the smallest positive root of equation (21) is simple except in the case $a = 2$ for $\rho = 0$, $u = 1$, cf. the proof of lemma 1, it follows from (27) and (28) that the contours $\Lambda(\rho)$, $\rho > 0$, and Λ possess a tangent at every point, with the possible exception for Λ at the point $w = 1$ when $a = 2$. In the case $a = 2$ it is obtained from (25) that

(31) $$\lim_{\theta\uparrow 0} \frac{d}{d\theta} \cos\theta\ \sigma(\cos\theta) = -\lim_{\theta\downarrow 0} \frac{d}{d\theta} \cos\theta\ \sigma(\cos\theta) = \{2\beta_2/\beta_1^2 - 1\}^{-\frac{1}{2}},$$
$$\lim_{\theta\downarrow 0} \frac{d}{d\theta} \sin\theta\ \sigma(\cos\theta) = \lim_{\theta\uparrow 0} \frac{d}{d\theta} \sin\theta\ \sigma(\cos\theta) = 1.$$

Hence, the lefthand and righthand tangents at $w = 1$ ($\theta = 0$) have different directions, with an inner angle given by (29). In general, $\beta_2 \ge \beta_1^2$, so that $\frac{1}{2} \le \omega < 1$. □

Next we introduce the conformal mapping $\gamma(z)$ of the unit disk $|z| < 1$ onto the interior Λ^+ of the contour Λ, satisfying $\gamma(0) = 0$, $\gamma'(0) > 0$, cf. (10). Because the contours $\Lambda(\rho)$ expand continuously to the contour Λ as ρ decreases to zero, cf. (22), (27), it follows with Carathéodory's mapping theorem, cf. [7], vol.III, §4, theorem 2.1, that

(32) $$\lim_{\rho\downarrow 0} \gamma(\rho;z) = \gamma(z),$$

uniformly for $|z| < 1$; because $\gamma(\rho;z)$, $\rho > 0$, and $\gamma(z)$ are continuous for $|z| \leq 1$, cf. §3, this limit also holds for $|z| = 1$.

Corollary 1. *In the case* $a > 2$ *the limit* (20) *vanishes.*

Proof. As noted above, $\gamma(\rho;t)$, $t \in C$, tends to a point on Λ as $\rho \downarrow 0$. From lemma 1, cf. (23), and (28) it is clear that $1 \notin \Lambda$ in the case $a > 2$, so that in this case the integrand, and hence also the integral, in (20) remains finite as $\rho \downarrow 0$. □

In the case $a \leq 2$ the integrand in (20) tends to infinity at $t = 1$ as $\rho \downarrow 0$, but only at $t = 1$, cf. (23).

Lemma 3. *For* $\rho > 0$ *the derivative* $\frac{\partial}{\partial z}\gamma(\rho;z)$ *is continuous and non-vanishing for* $|z| \leq 1$. *The derivative* $\gamma'(z)$ *is continuous and non-vanishing for* $|z| \leq 1$, *except in the case* $a = 2$ *at* $z = 1$.
In the case $a < 2$, *for every* δ, $0 < \delta < 1$,

(33) $$\gamma(z) = 1 + (z-1)\,\gamma'(1) + O(|1-z|^{2-\delta}), \qquad z \to 1,\ |z| \leq 1;$$

in the case $a = 2$ *there exist positive constants* N_1, N_2, *such that*

(34) $$N_1\,|1-z| \leq |1-\gamma(z)| \leq N_2\,\sqrt{|1-z|}, \qquad |z| \leq 1.$$

Proof. First consider the case $a < 2$. Let $s = s(\theta)$ denote the arc length of Λ at the point $w = \sigma(\cos\theta)e^{i\theta}$ counted from the point $w = -\sigma(-1)$, cf. (28), then for $-\pi \le \theta \le \pi$,

$$(35) \qquad s(\theta) = \int_{-\pi}^{\theta} \sqrt{[\sigma(\cos\phi)]^2 + \sin^2\phi\,[\sigma'(\cos\phi)]^2}\,d\phi.$$

Further, let $w = w(s)$ be the parametric equation of Λ with its arc length as parameter, so that for $-\pi \le \theta \le \pi$,

$$(36) \qquad w(s(\theta)) = \sigma(\cos\theta)\,e^{i\theta},$$

$$\frac{d}{d\theta}\,w(s(\theta)) = w'(s(\theta))\,s'(\theta) = [i\,\sigma(\cos\theta) - \sin\theta\,\sigma'(\cos\theta)]\,e^{i\theta},$$

so that

$$(37) \qquad w'(s(\theta)) = \frac{i\,\sigma(\cos\theta) - \sin\theta\,\sigma'(\cos\theta)}{\sqrt{[\sigma(\cos\theta)]^2 + \sin^2\theta\,[\sigma'(\cos\theta)]^2}}\,e^{i\theta}.$$

The denominator in (37) is non-vanishing on $[-\pi,\pi]$, because, cf. (30), (23), $\sigma(\cos\theta) \ge \sigma(-1) > 0$ for $-\pi \le \theta \le \pi$. Further, the second derivative of $\sigma(\cos\theta)$ is continuous on $[-\pi,\pi]$ by lemma 1, if $a < 2$. This proves that there exists a constant such that for every θ_1, θ_2, $-\pi \le \theta_1 \le \theta_2 \le \pi$,

$$(38) \qquad |w'(s(\theta_1)) - w'(s(\theta_2))| < \text{const.}\ |\theta_1 - \theta_2|.$$

Moreover, it follows from (35) and (30), that for $-\pi \le \theta_1 \le \theta_2 \le \pi$,

$$(39) \qquad |s(\theta_1) - s(\theta_2)| \ge \sigma(-1)\ |\theta_1 - \theta_2|.$$

Together, (38) and (39) prove that there exists a constant such that for every s_1, s_2, $0 \le s_1 \le s_2 \le s_0$ (s_0 is the length of Λ),

$$(40) \qquad |w'(s_1) - w'(s_2)| < \text{const.}\ |s_1 - s_2|.$$

By Kellogg's theorem, cf. [10], theorem IX.7, it follows from (40) that $\gamma'(z)$ exists and is non-vanishing for $|z| \le 1$, and that for every δ,

$0 < \delta < 1$, there exists a constant such that for every θ_1, θ_2, $-\pi \le \theta_1 \le \theta_2 \le \pi$,

$$(41) \qquad |\gamma'(e^{i\theta_1}) - \gamma'(e^{i\theta_2})| < \text{const. } |\theta_1 - \theta_2|^{1-\delta}.$$

By a theorem of Hardy and Littlewood, cf. [6], §IX.5, Satz 4, it then follows that for every z, $|z| \le 1$,

$$(42) \qquad |\gamma'(z) - \gamma'(1)| < \text{const. } |1-z|^{1-\delta}.$$

This inequality implies (33).
A similar argument shows that $\frac{\partial}{\partial z}\gamma(\rho;z)$, $\rho > 0$, exists and does not vanish for $|z| \le 1$, for every value of a.
Next consider the case $a = 2$. By lemma 2 the contour Λ has a corner point at $w = 1$ with inner angle $\omega\pi$, cf. (29). Therefore, we introduce the mapping, regular in $\mathbb{C}\backslash[1,\infty)$,

$$(43) \qquad \xi(w) = 1 - (1-w)^{1/\omega}, \qquad \xi(0) = 0.$$

The function $\xi(w)$ maps the domain Λ^+ conformally onto a domain X^+, and it maps the contour Λ onto a contour X which is the boundary of X^+, and which has a parametric equation given by

$$(44) \qquad \xi = 1 - (1-\sigma(\cos\theta)e^{i\theta})^{1/\omega}, \qquad -\pi \le \theta \le \pi.$$

Let $\xi = x(\theta)$ denote the arc length of X at the corresponding point ξ given by (44), counted from the point where $\theta = -\pi$; and let $v(x)$ be the parametric equation of X with its arc length as parameter. In a similar way as before, cf. (37), it is obtained that for $-\pi \le \theta \le \pi$,

$$(45) \qquad v'(x(\theta)) = \frac{[1-\sigma(\cos\theta)e^{i\theta}]^{\frac{1}{\omega}-1}}{|1-\sigma(\cos\theta)e^{i\theta}|^{\frac{1}{\omega}-1}} \; \frac{[i\sigma(\cos\theta) - \sin\theta\ \sigma'(\cos\theta)]}{\sqrt{[\sigma(\cos\theta)]^2 + \sin^2\theta\,[\sigma'(\cos\theta)]^2}}\; e^{i\theta}.$$

From (25) it is readily derived that

(46) $$\sigma(\cos\theta) = 1 - \{2\beta_2/\beta_1^2 - 1\}^{-\frac{1}{2}}\ |\theta| + O(|\theta|^2), \qquad \theta \to 0.$$

From (45) and (46) it follows by straightforward calculation that

(47) $$\lim_{\theta\uparrow 0} v'(x(\theta)) = \lim_{\theta\downarrow 0} v'(x(\theta)) = i.$$

This implies that the contour X has a tangent at $\xi = 1$, and hence by lemma 2 and the properties of the mapping $\xi(w)$, cf. (43), at every point. Further, it follows from (45) and (46) that for $\theta \downarrow 0$ and for $\theta \uparrow 0$,

$$v'(x(\theta)) = i + O(|\theta|).$$

This leads to the inequality: for every θ_1, θ_2, $-\pi \le \theta_1 \le \theta_2 \le \pi$,

(48) $$|v'(x(\theta_1)) - v'(x(\theta_2))| < \text{const.}\ |\theta_1 - \theta_2|.$$

From (44) it follows that

(49) $$x'(\theta) = O(|\theta|^{\frac{1}{\omega}-1}), \qquad \theta \to 0,$$

so that, by using [8], §5, for every θ_1, θ_2, $-\pi \le \theta_1 \le \theta_2 \le \pi$,

(50) $$|\theta_1 - \theta_2| \le \text{const.}\ |x(\theta_1) - x(\theta_2)|^{\omega}.$$

Together, (48) and (50) imply that for every x_1, x_2, $0 \le x_1 \le x_2 \le x_0$ (x_0 is the length of X), cf. (40),

(51) $$|v'(x_1) - v'(x_2)| < \text{const.}\ |x_1 - x_2|^{\omega}.$$

Now let $f(z)$ be the conformal mapping of the unit disk $|z| < 1$ onto the domain X^+ with $f(0) = 0$, $f'(0) > 0$. Then again by Kellogg's theorem it follows that $f'(z)$ exists and is non-vanishing in $|z| \le 1$, and that for every θ_1, θ_2, $-\pi \le \theta_1 \le \theta_2 \le \pi$,

$$|f'(e^{i\theta_1}) - f'(e^{i\theta_2})| < \text{const.}\ |\theta_1 - \theta_2|^{\omega}.$$

As in the case $a < 2$, cf. (42), this leads to: for $|z| \le 1$,

(52) $$f(z) = 1 + (z-1)\ f'(1) + O(|1-z|^{1+\omega}), \qquad z \to 1.$$

By using the inverse mapping of (43) and the uniqueness theorem for conformal mapping, cf. [7], vol.III, §2, theorem 1.3, we have

(53) $$\gamma(z) = 1 - [1 - f(z)]^{\omega}, \qquad |z| \le 1,$$

so that (52) leads to: for $|z| \le 1$,

(54) $$\gamma(z) = 1 - [f'(1)]^{\omega}\ (1-z)^{\omega} + o(|1-z|^{\omega}), \qquad z \to 1.$$

Because $\frac{1}{2} \le \omega < 1$, cf. lemma 2, (54) proves (34). The existence of $\gamma'(z) \ne 0$ for $|z| = 1$, $z \ne 1$, follows from (53) and the existence of $f'(z) \ne 0$ for $|z| = 1$. □

With the aid of the foregoing lemmas we are able to prove the following theorems on the limit (20).

Theorem 1. *In the case* $a > 2$ *the limit*

(55) $$\lim_{\rho\downarrow 0} \left|\frac{1}{2\pi i} \int_C \frac{1}{\{1-\gamma(\rho;t)\}\{1-\gamma(\rho;\frac{1}{t})\}} \frac{dt}{t}\right|,$$

is finite; in the case $a = 2$ *this limit is infinite.*

Proof. See for the case $a > 2$ the proof of corollary 1. Consider the case $a = 2$. Let ε be a constant, $0 < \varepsilon < \frac{1}{4}\pi$, and write the integral in (55) as follows:

(56) $$\frac{1}{2\pi} \int_{\varepsilon}^{2\pi-\varepsilon} \frac{d\theta}{|1-\gamma(\rho;e^{i\theta})|^2} + \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{d\theta}{|1-\gamma(\rho;e^{i\theta})|^2}\ .$$

Because $\gamma(\rho;1) = \sigma(\rho;1)$, cf. (10),(11),(27), it follows from lemma 1 and (32) that $\gamma(\rho;t) \to 1$ $(\rho \downarrow 0)$ if and only if $t = 1$. Hence, the first integral in (56) remains finite as $\rho \downarrow 0$. Now consider the second integral in (56). From lemma 3 it follows that there exists a constant M (independent of ρ), such that for $\rho > 0$ and $-\varepsilon \leq \theta \leq \varepsilon$,

$$(57) \qquad |\gamma(\rho;1) - \gamma(\rho;e^{i\theta})| < M\,|\theta|^{\frac{1}{2}}.$$

This implies the inequality: for $\rho > 0$, $-\varepsilon \leq \theta \leq \varepsilon$,

$$(58) \qquad |1-\gamma(\rho;e^{i\theta})| \leq |1-\gamma(\rho;1)| + M\,|\theta|^{\frac{1}{2}}.$$

This inequality leads to the following lower bound for the second integral in (56): for ε and M independent of ρ, $\rho > 0$,

$$(59) \qquad \int_{-\varepsilon}^{\varepsilon} \frac{d\theta}{|1-\gamma(\rho;e^{i\theta})|^2} \geq \int_{-\varepsilon}^{\varepsilon} \frac{d\theta}{\{1-\gamma(\rho;1)+M|\theta|^{\frac{1}{2}}\}^2} =$$

$$= \frac{4}{M^2}\left[\log\{1-\gamma(\rho;1)+M\sqrt{\varepsilon}\} - \log\{1-\gamma(\rho;1)\} - \frac{M\sqrt{\varepsilon}}{1-\gamma(\rho;1)+M\sqrt{\varepsilon}}\right].$$

Because ε and M are positive and because $\gamma(\rho;1) \uparrow 1$ as $\rho \downarrow 0$, it is obvious that this lower bound tends to infinity as $\rho \downarrow 0$. This proves the assertion in the case $a = 2$. □

Theorem 2.

$$(60) \qquad \lim_{\rho\downarrow 0} \frac{\alpha\rho}{2\pi i} \int_C \frac{1}{\{1-\gamma(\rho;t)\}\{1-\gamma(\rho;\frac{1}{t})\}} \frac{dt}{t} = 0, \qquad \textit{if } a = 2,$$
$$= \frac{1-\frac{1}{2}a}{a\gamma'(1)}, \qquad \textit{if } a < 2.$$

Proof. First let $a = 2$. Again we split up the integral as in (56). As noted in theorem 1 the first integral in (56) remains finite as $\rho \downarrow 0$, so that multiplied by $\alpha\rho$ it vanishes as $\rho \downarrow 0$. From (34) it follows that for $0 < \varepsilon < \frac{1}{4}\pi$ and for, say, $0 < \rho < 1$, there exists a constant K such that for $-\varepsilon \le \theta \le \varepsilon$,

(61) $$|\gamma(\rho;1) - \gamma(\rho;e^{i\theta})| > K\,|\theta|.$$

Because the point $w = \gamma(\rho;1) = \sigma(\rho;1)$ is the point on $\Lambda(\rho)$ with the largest absolute value, cf. (27),(30), the angle which the line joining the points $\gamma(\rho;1)$ and $\gamma(\rho;e^{i\theta})$ makes with the positive direction on the real axis is obtuse, so that the cosine rule implies that for $\rho > 0$, $-\varepsilon \le \theta \le \varepsilon$,

(62) $$|1 - \gamma(\rho;e^{i\theta})| \ge |1 - \gamma(\rho;1)|^2 + |\gamma(\rho;1) - \gamma(\rho;e^{i\theta})|^2.$$

From (61),(62), the following upper bound for the second integral in (56) is obtained: for $0 < \rho < 1$,

(63) $$\int_{-\varepsilon}^{\varepsilon} \frac{d\theta}{|1-\gamma(\rho;e^{i\theta})|^2} \le \int_{-\varepsilon}^{\varepsilon} \frac{d\theta}{\{1-\gamma(\rho;1)\}^2 + K^2\theta^2} =$$

$$= \frac{1}{K\{1-\gamma(\rho;1)\}} \arctan\left[\frac{\theta K}{1-\gamma(\rho;1)}\right]\Bigg|_{\theta=-\varepsilon}^{\theta=\varepsilon}.$$

In a similar way as (25) it can be found that in the case $\alpha = 2$,

(64) $$\gamma(\rho;1) = \sigma(\rho;1) = 1 - \sqrt{\frac{\beta_1\rho}{2\beta_2/\beta_1^2 - 1}} + O(\rho), \qquad \rho \downarrow 0.$$

Consequently,

(65) $$\lim_{\rho\downarrow 0} \frac{\rho}{1-\gamma(\rho;1)} = 0; \qquad \lim_{\rho\downarrow 0} \arctan\left[\frac{\theta K}{1-\gamma(\rho;1)}\right]\Bigg|_{\theta=-\varepsilon}^{\theta=\varepsilon} = \pi.$$

Hence, the upper bound for the second integral in (56) given in (63), multiplied by ρ, tends to zero as $\rho \downarrow 0$. This proves the assertion for $\alpha = 2$.

Next consider the case $\alpha < 2$. Because for $\rho > 0$ the function $\sigma(\rho;u)$ is an infinitely differentiable function of u, $u \le 1$, cf. lemma 1, $\Lambda(\rho)$ is an analytic contour, cf. [9], p.186. This implies that the conformal mapping

$\gamma(\rho;z)$ is regular on the boundary $|z| = 1$, cf. [9], p.186, so that it can be continued analytically into (a part of) the region $|z| > 1$. Further, because by lemma 3 the derivative $\frac{\partial}{\partial z}\gamma(\rho;z)$ is non-vanishing at $z = 1$, and since $\gamma(\rho;1) \uparrow 1$ as $\rho \downarrow 0$, it follows (see [1], section II.5 for a more rigorous proof) that for ρ close to zero there exists a $t_0(\rho) > 1$ such that

(66) $$\gamma(\rho;t_0(\rho)) = 1; \qquad \text{and} \quad t_0(\rho) \downarrow 1 \quad \text{as} \quad \rho \downarrow 0.$$

With this $t_0(\rho)$ we write for ρ close to zero,

(67) $$\int_C \frac{1}{|1-\gamma(\rho;t)|^2}\frac{dt}{t} = \int_C K(\rho;t)\frac{1}{\{t-t_0(\rho)\}\{t\,t_0(\rho)-1\}}\frac{dt}{t},$$

here

(68) $$K(\rho;t) := \frac{t-t_0(\rho)}{1-\gamma(\rho;t)}\,\frac{t\,t_0(\rho)-1}{1-\gamma(\rho;1/t)}.$$

From (66) it follows by differentiation and by using $\gamma(\rho;1) = \sigma(\rho;1)$ and (26), that

(69) $$\lim_{\rho\downarrow 0}\frac{d}{d\rho}t_0(\rho) = -\lim_{\rho\downarrow 0}\frac{d}{d\rho}\gamma(\rho;1)/\gamma'(1) = \frac{1}{2-a}\frac{\beta_1}{\gamma'(1)}.$$

Hence,

(70) $$\lim_{\rho\downarrow 0} K(\rho;1) = -\{\gamma'(1)\}^{-2}.$$

Moreover, it follows from (33) and the fact that for $\rho > 0$ the conformal mapping $\gamma(\rho;z)$ is regular at $z = 1$, that for every δ, $0 < \delta < 1$, there exists a constant such that for $|t| = 1$ and ρ close to zero,

(71) $$|K(\rho;t) - K(\rho;1)| < \text{const.}\ |t-1|^{\delta}.$$

This implies that by splitting up the second integral in (67) as

(72) $$\frac{1}{t_0^2(\rho)-1}\left[\int_C K(\rho;t)\frac{1}{t-t_0(\rho)}\frac{dt}{t} - \int_C K(\rho;t)\frac{t_0(\rho)}{t\,t_0(\rho)-1}\frac{dt}{t}\right],$$

we may apply on both integrals an extended version of the Sochozki-Plemelj formulas (cf. [8], §16, [1], lemma I.3.6), which leads to

(73) $$\lim_{\rho\downarrow 0}\frac{1}{2\pi i}\int_C K(\rho;t)\,\frac{1}{t-t_0(\rho)}\,\frac{dt}{t} = -\tfrac{1}{2}K(0;1) + \frac{1}{2\pi i}\int_C K(0;t)\,\frac{dt}{t-1},$$

$$\lim_{\rho\downarrow 0}\frac{1}{2\pi i}\int_C K(\rho;t)\,\frac{t_0(\rho)}{t\,t_0(\rho)-1}\,\frac{dt}{t} = \tfrac{1}{2}K(0;1) + \frac{1}{2\pi i}\int_C K(0;t)\,\frac{dt}{t-1}.$$

Finally, by using, cf. (69), that

(74) $$\lim_{\rho\downarrow 0}\frac{\alpha\rho}{t_0^2(\rho)-1} = \frac{(2-a)\,\gamma'(1)}{2a},$$

the assertion for $a < 2$ follows from (67),(72),(73) and (70). □

6. Asymptotic behaviour of the queueing process

With the aid of the analysis of the preceding section we are able to formulate the main theorem on the ergodic properties of the queueing system.

Theorem 3. *The queueing system with two types of customers and paired services described in section* 2 *is transient if* $a > 2$, *it consists of null states if* $a = 2$, *and it is ergodic if* $a < 2$. *Further,*

(75) $$\psi_0 = \lim_{t\to\infty}\Pr\{\underline{y}_1(t) = 0,\ \underline{y}_2(t) = 0\} = \frac{1-\frac{1}{2}a}{1-\frac{1}{2}a + a\gamma'(1)}, \qquad \textit{if } a < 2.$$

Proof. For the queueing process defined in section 2 the state space $\{0,1,2,..\}\times\{0,1,2,..\}$ is irreducible, and the process is aperiodic. From theorem 1, (19) and (2), it follows that in the case $a > 2$,

$$\int_0^\infty \Pr\{\underline{y}_1(t) = 0,\ \underline{y}_2(t) = 0\}\ dt < \infty,$$

so that in this case the queueing process is transient.

In the case $a = 2$ we have by theorem 1:

$$\int_0^\infty \Pr\{\underline{y}_1(t) = 0,\ \underline{y}_2(t) = 0\}\, dt = \infty.$$

However, from theorem 2, (17) and (19) we obtain in this case

$$\lim_{t\to\infty} \Pr\{\underline{y}_1(t) = 0,\ \underline{y}_2(t) = 0\} = 0.$$

Hence, in the case $a = 2$ the queueing system consists of null states. Finally, in the case $a < 2$ theorem 2 leads with (17) and (19) to (75), thus showing that the process is ergodic. □

Remark. The above result has been proved here under the assumption that $\beta_3 < \infty$. Theorem 3 also holds without this assumption, but the proof becomes more tedious. Because the expansion (25) is not valid if $\beta_3 = \infty$, more general theorems than Kellogg's theorem have to be applied in order to prove the inequalities (34), cf. [1], theorem II.8.2, [10], chapter IX, part I. □

With the same technique as applied in the proof of theorem 2 also the generating function of the limiting distribution of the process $\{(\underline{y}_1(t), \underline{y}_2(t)), t \geq 0\}$ as $t \to \infty$ can be obtained in the ergodic case from the function $\Psi(\rho;p_1,p_2)$, cf. (2),(15). We only state here the result: if $a < 2$, for $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\text{(76)}\qquad \Psi(p_1,p_2) := \lim_{t\to\infty} E\{p_1^{\underline{y}_1(t)}\, p_2^{\underline{y}_2(t)}\} = \psi_0\, \beta\Big(\frac{1-\tfrac{1}{2}p_1-\tfrac{1}{2}p_2}{\alpha}\Big)\Big[1 + \frac{(1-p_1)(1-p_2)}{1-\tfrac{1}{2}p_1-\tfrac{1}{2}p_2}\, \frac{1-\beta\Big(\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\Big)}{\beta\Big(\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\Big) - p_1p_2}\, \frac{1-\gamma_0(p_1)\,\gamma_0(p_2)}{\{1-\gamma_0(p_1)\}\{1-\gamma_0(p_2)\}}\Big],$$

here ψ_0 is given by (75) and $\gamma_0(w)$ stands for the inverse conformal mapping of $\gamma(z)$, which is first defined for $w \in \Lambda^+ \cup \Lambda$, and then continued analytically to the region $|w| \leq 1$. For the first moments of the process we find:

$$\text{(77)}\qquad \lim_{t\to\infty} E\{\underline{y}_j(t)\} = \tfrac{1}{2}a\Big[1 + \frac{1-\psi_0}{1-\tfrac{1}{2}a}\,\frac{\beta_2}{2\beta_1^2}\Big], \qquad j = 1,2.$$

The queueing model as described in §2 can be generalized by the assumption that an arriving customer is with probability c_j of type j, j = 1,2, $c_1 + c_2 = 1$. For this case a similar theorem as theorem 3 can be proved. In fact, the analysis becomes simpler when $c_1 \neq \frac{1}{2}$, because the first term in the asymptotic expansion of $\Psi(\rho;0,0)$ at $\rho = 0$ depends only on the Laplace-Stieltjes transform of a busy period in an M/G/1 system. The result is that in the general case the system is ergodic if and only if $\max\{c_1, c_2\}a < 1$, cf. [1], theorem II.8.5.

An interesting subject for further research is the relaxation time for this type of queueing systems with a two-dimensional state space. The relaxation time is a measure for the speed at which $\Pr\{\underline{y}_1(t) = 0, \underline{y}_2(t) = 0\}$ tends to its limiting value as $t \to \infty$, cf. [2], §III.7.3.

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