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RIGOROUS HIGH SPEED SEPARATION OF ZEROS OF RIEMANN'S ZETA FUNCTION, III

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Rigorous high speed separation of zeros of Riemann's zeta function, III

by

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ABSTRACT

This is an intermediate report for the purpose of announcing that the first 300,000,001 zeros of the Riemann zeta function $\zeta(s)$ in the critical strip are simple and lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This extends our previous result that the first 250,000,000 zeros have this property.

KEY WORDS & PHRASES: *Riemann hypothesis, Riemann zeta function, Riemann-Siegel formula, Rosser's rule*

1. INTRODUCTION

This note is a continuation of our report NN 26/82 [2]. The purpose of the present note is to announce that *the first* 300,000,001 *zeros of the Riemann zeta function* $\zeta(s)$ *in the critical strip are simple and lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$. This result was obtained by extending the computations (up to zero # 250,000,000) described in [2]. Extending the computations a little further we found 4 Gram blocks in $[g_{300,000,000}, g_{300,000,004})$, all of them satisfying Rosser's rule. By applying Theorem 3.2 of BRENT [1] we completed the proof of our claim.

In Section 2 we present tables similar to those given in Section 2 of [2], this time for the range $[g_{249,999,999}, g_{300,000,000})$. The FORTRAN/COMPASS program by which the zeros in this range were separated is the same as that explicitly presented in [2].

We hope to extend our computations in the near future.

2. THE TABLES

In Table 2.1 we list the number of Gram blocks of type (L,k), $1 \le L \le 8$, $1 \le k \le L$, in the interval $[g_{249,999,999}, g_{300,000,000})$. Similarly as in NN 26/82 these counts are *exact*.

On the lines with $L = 2$ and $L = 3$ in Table 2.1 we also list the number of exceptions to Rosser's rule of length 2 (with "0 0"-zero pattern) and those of length 3 (with "0 1 0"-zero pattern), respectively. Moreover, on the line with $L = 2$ we also mention the three blocks with "2 2"-zero pattern found in relation to the exceptions of type 5 and 6 (cf. Table 2.3). One exception to Rosser's rule of length $L = 3$ was found. Note the occurrence of two blocks of type (7,1), not observed before, and one block of type (7,7), which is the second one observed (the first one being $B_{195,610,937}$). The entries in parentheses are the approximate percentages with respect to the total number of blocks of length L, given in the final column.

For all Gram blocks our strategy of finding the "missing two zeros" was the same as in [2]. Table 2.1 shows the justification of this strategy.

Table 2.1

Number of Gram blocks of type (L,k), $1 \le L \le 8$, $1 \le k \le L$, *in the interval* $[g_{249,999,999}, g_{300,000,000})$

L ↓ \ k →	1	2	3	4	5	6	7	8	Total
1	34,764,288								34,764,288
2	2,643,484	2,643,854 + 63 blocks with 0 0 zero-pattern							5,287,404
	(50)	(50) + 3 blocks with 2 2 zero-pattern							
3	565,054	62,319	564,937 + 1 block with 0 1 0 zero-pattern						1,192,311
	(47)	(5)	(47)						
4	107,657	9,992	9,979	107,578					235,206
	(46)	(4)	(4)	(46)					
5	10,511	2,159	1,123	2,195	10,388				26,376
	(40)	(8)	(4)	(8)	(39)				
6	284	442	137	153	448	256			1,720
	(17)	(26)	(8)	(9)	(26)	(15)			
7	2*)	44	21	4	26	34	1*)		132
8	0	0	1**)	0	0	2**)	0	0	3

*) viz. B_n, for n = 258,779,892 282,307,390 299,608,968 respectively.

**) viz. B_n, for n = 264,680,348 258,666,950 262,831,140 respectively.

In Table 2.2 we present the 64 exceptions to Rosser's rule in the range $[g_{249,999,999}, g_{300,000,000})$, including the local extreme values of S(t). (The definitions of the various types are implicitly given in Table 2.3).

Table 2.2

The exceptions to Rosser's rule of length 2 and 3 in the interval $[g_{249,999,999}, g_{300,000,000})$

63 exceptions of length 2 with zero pattern "0 0".

Notation: n (type) local extreme S(t), where n is the index of the Gram block $B_n = [g_n, g_{n+2})$ containing no zeros

250076378(1)	-2.036397	272096379(2)	2.001554	284764536(2)	2.001422
252442158(2)	2.085094	272583009(1)	-2.032037	286172640(2)	2.042925
252904232(2)	2.112235	274190882(2)	2.008416	286688824(1)	-2.046407
255145220(1)	-2.002286	274268747(1)	-2.018420	287222173(2)	2.048065
255285972(2)	2.034861	275297430(2)	2.014738	287235535(2)	2.024894
256713230(1)	-2.015377	275545477(2)	2.032087	287304862(2)	2.003208
257992082(1)	-2.042307	275898480(2)	2.005068	287433571(1)	-2.021945
258447957(6)	2.005655	275953000(1)	-2.007296	287823551(1)	-2.038399
259298046(2)	2.091955	277117197(5)	-2.069283	287872423(2)	2.016959
262141503(1)	-2.006009	277447311(2)	2.058999	288766616(2)	2.024072
263681744(2)	2.006016	279059658(2)	2.037126	290122964(2)	2.039001
266617122(1)	-2.046423	279259145(2)	2.033129	290450849(5)	-2.068090
266628045(2)	2.048158	279513637(2)	2.000375	291426142(2)	2.075533
267305763(1)	-2.028836	279849070(2)	2.048163	292810354(2)	2.048278
267388405(2)	2.012716	280291419(1)	-2.021221	293109862(2)	2.013978
267441673(2)	2.085691	281449426(2)	2.000609	293398055(2)	2.042772
267464886(1)	-2.006418	281507954(2)	2.001841	294134427(2)	2.043302
267554908(2)	2.112706	281825600(1)	-2.033191	294216438(1)	-2.005490
269787480(1)	-2.080890	282547094(2)	2.002833	295367142(2)	2.049246
270881434(1)	-2.026487	283120964(2)	2.028096	297834112(2)	2.022351
270997584(2)	2.021752	283323493(1)	-2.032511	299099970(2)	2.030191

1 exception of length 3 with zero pattern "0 1 0".

$B_{266,527,881}$, *followed* by a Gram block of length 1 containing 3 zeros; local extreme S(t) = -2.008550.

Table 2.3 contains the frequencies of occurrence of the various types of exceptions to Rosser's rule in the range $[g_{249,999,999}, g_{300,000,000})$. Note that the number of exceptions of type 2 is twice the number of exceptions of type 1. There are neither exceptions of type 3 nor of type 4 in the range under consideration.

Table 2.3

Various types of exceptions to Rosser's rule and their frequencies in $[g_{249,999,999}, g_{300,000,000})$

Exceptions of length 2

Gram block of length 2 without any zeros (from g_n to g_{n+2})

$[g_{n-2}, g_{n-1})$	$[g_{n-1}, g_n)$	$[g_n, g_{n+1})$	$[g_{n+1}, g_{n+2})$	$[g_{n+2}, g_{n+3})$	$[g_{n+3}, g_{n+4})$	type	frequency
		0	0	3		1	20
	3	0	0			2	40
		0	0	4	0	3	0
0	4	0	0			4	0
		0	0	2	2	5	2
2	2	0	0			6	1

Exceptions of length 3

Gram block of length 3 containing 1 zero (from g_n to g_{n+3})

$[g_{n-1}, g_n)$	$[g_n, g_{n+1})$	$[g_{n+1}, g_{n+2})$	$[g_{n+2}, g_{n+3})$	$[g_{n+3}, g_{n+4})$	frequency
3	0	1	0		0
	0	1	0	3	1

Finally, from Tables 2.1 and 2.2 we have counted the following numbers of Gram intervals in $[g_{249,999,999}, g_{300,000,000})$ containing exactly m $(0 \le m \le 4)$ zeros.

m	0	1	2	3	4
#	6,832,292	36,424,554	6,654,012	89,143*)	0
%	13.7	72.8	13.3	0.2	0.0

4. REFERENCES

[1] BRENT, R.P., *On the zeros of the Riemann zeta function in the critical strip*, Math. Comp., 33 (1979) pp. 1361-1372.

[2] LUNE, J. VAN DE & H.J.J. TE RIELE, *Rigorous high speed separation of zeros of Riemann's zeta function*, II, Report NN 26/82, June 1982, Mathematical Centre, Amsterdam.

*) We take the opportunity to correct a misprint in the corresponding table in [2]: the printed number 88,312 should read 88,314.