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ASYMPTOTIC ANALYSIS OF A SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

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Asymptotic analysis of a singular Sturm-Liouville boundary value problem *)

by

E.J.M. Veling **)

ABSTRACT

Asymptotic expansions are given for the eigenvalues λ_n and eigenfunctions u_n of the following singular Sturm-Liouville problem with indefinite weight:

$$-\frac{d}{dx}\left((1-x^2)\frac{d}{dx}u\right) = \lambda x u \quad \text{on} \quad (-1,1),$$

$$\lim_{|x|\to 1} u(x) \text{ finite.}$$

This eigenvalue problem arises if one separates variables in a partial differential equation which describes electron scattering in a one-dimensional slab configuration.

Asymptotic expansions of the normalization constants of the eigenfunctions are also given. The constants in these asymptotic expansions involve complete elliptic integrals. The asymptotic results are compared with the results of numerical calculations.

The results presented in this paper provide necessary information for the operator - theoretic analysis of certain types of boundary value problems in electron transport theory.

KEY WORDS & PHRASES: *singular Sturm-Liouville problem, turning point, indefinite weight function, asymptotic distribution of eigenvalues, asymptotic expansions of eigenfunctions, complete elliptic integral, electron scattering, transport theory*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

When electrons move through a metal strip, they carry mass, momentum and energy from one point of the strip to another. The equation which describes the electron density in phase space as a function of time is called a transport equation. In the case of a stationary transport process, the transport equation is simply a balance equation which balances the effect of the free streaming of the electrons against the effect of collisions. A simple model of a stationary transport equation is obtained when the strip is modeled as a homogeneous isotropic slab of finite thickness τ, which is infinite in both transverse directions, and all electrons are assumed to have the same speed (i.e., magnitude of the velocity vector). Then the phase space is two-dimensional; the relevant coordinates are x, the position inside the slab, and μ, the cosine of the angle between the velocity vector and the unit vector in the direction of increasing x, with $0 \le x \le \tau$ and $-1 \le \mu \le 1$. The following transport equation was first given by BOTHE [6] and, later, by BETHE, ROSE & SMITH [3]:

$$(1.1) \qquad -\frac{\partial}{\partial x}\mu\phi(x,\mu) = -\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial}{\partial \mu}\phi(x,\mu)\right), \quad (x,\mu)\in \Delta\times\bar{J},$$

where $\Delta = (0,\tau)$, $J = (-1,1)$ and ϕ is the electron density function. The left member represents the net effect of the free streaming of the electrons, it is the divergence of the electron current density; the right member represents the net effect of the collisions or interactions between the electrons and the atoms of the host medium.

The differential equation (1.1) is supplemented by boundary conditions of the following type:

$$(1.2) \qquad \lim_{x\downarrow 0} \mu\phi(x,\mu) = g_+(\mu), \qquad 0 \le \mu \le 1,$$

$$(1.3) \qquad \lim_{x\uparrow \tau} \mu\phi(x,\mu) = g_-(\mu), \qquad -1 \le \mu \le 0,$$

where g_+, g_- are given functions. Positive (negative) values of μ indicate motion towards increasing (decreasing) values of x, so equation (1.2) prescribes the incoming flux at the left endpoint of Δ, (1.3) at the right endpoint of Δ. The outgoing fluxes, both left and right, will be part of the solution of the problem. BETHE et al. [3] found a solution by a formal expansion method. BEALS [2] proved the existence and uniqueness of a solution in a weak formulation. KAPER, LEKKERKERKER & ZETTL [12] constructed the general solution of (1.1) using operator - theoretic techniques. In this paper we follow the notation of [12] whenever we refer to this operator - theoretic setting of the problem.

In section 2 we summarize part of the results of [12] and explain the motivation for the asymptotic analysis given in this paper.

In section 3 we study the singular Sturm-Liouville eigenvalue problem:

$$(1.4) \qquad -\frac{d}{dx}\left((1-x^2)\frac{d}{dx}u_n(x)\right) = \lambda_n x u_n(x), \; -1 < x < 1, \; n \in \mathbb{N}^+ \cup \mathbb{N}^-,$$

$$(1.5) \qquad \begin{array}{l} u_n(1) = 1, \; n \in \mathbb{N}^+, \; u_n(-1) = 1, \; n \in \mathbb{N}^-, \\ u_n \text{ bounded on } (-1,1), \; n \in \mathbb{N}^+ \cup \mathbb{N}^- . \end{array}$$

We give representations of the eigenfunctions u_n as a sum of Legendre polynomials, in which the coefficients in the expansion depend on the eigenvalue λ_n. Making the transformation $x' = -x$, we observe that, for every eigenfunction u_n and eigenvalue λ_n, the function u_{-n}, with $u_{-n}(x) = u_n(-x)$, satisfies (1.4), (1.5) at the eigenvalue $\lambda_{-n} = -\lambda_n$. We show that the first eigenvalues can be approximated by a continued fraction expansion. However, the expansions do not provide any information about the behaviour of λ_n as $|n| \to \infty$.

In section 4 we construct asymptotic expansions for the eigenfunctions (1.4), (1.5). The interval (-1,1) is subdivided into three regions; the matching conditions determine the eigenvalues.

In section 5 we give asymptotic results for the integrals $(1,u_n)$ and (xu_n,u_n), where $(\cdot,\cdot)$ denotes the inner product in $L^2(J)$, $J = (-1,1)$. These inner products play a role in the theory given by KAPER, LEKKERKERKER & ZETTL [12].

In section 6 we compare our asymptotic results with the results of numerical calculations of the eigenvalues and coefficients in the Legendre polynomial expansions of the eigenfunctions. Even for the first eigenvalue the numerical agreement is very good.

2. OPERATOR-THEORETIC APPROACH

In this section we summarize the so-called full-range theory developed in [12]. Let $J = (-1,1)$, and let $H = L^2(J)$ be the Hilbert space of complex-valued square integrable functions on J. Define the multiplicative operator T by the expression

$$(2.1) \qquad Tf(\mu) = \mu f(\mu), \quad \mu \in J,\ f \in H.$$

T is injective, bounded and selfadjoint, its inverse T^{-1} is unbounded and defined on the image of T. Let $p(\mu) = 1-\mu^2$, $\mu \in \bar{J}$; let N denote the differential expression

$$(2.2) \qquad N[f] = -\frac{d}{d\mu}\left(p(\mu)\frac{d}{d\mu} f(\mu)\right), \quad \mu \in J;$$

and let M be the maximal operator associated with N,

$$(2.3) \qquad \mathcal{D}(M) = \{f \mid f \in H;\ p(\mu)\frac{d}{d\mu} f(\mu) \text{ absolutely continuous on compact subintervals of } J;\ N[f] \in H\},$$

$$(2.4) \qquad Mf = N[f], \quad f \in \mathcal{D}(M).$$

Since the equation $N[f] = 0$ is singular at both endpoints, and both fundamental solutions ($f_1(\mu) = 1$, $f_2(\mu) = \ln((1+\mu)/(1-\mu))$) are elements of $\mathcal{D}(M)$, M is limit-circle at both endpoints. We recall that a differential equation $-(pf')' + qf = \lambda f$ on an interval $I = (a,b)$ with b a singular point, is called *limit-circle* at b if for some complex λ ($\mathrm{Im}\lambda \neq 0$) a solution f exists with $f \in L^2(I)$. According to the Weyl theory, all solutions are then elements of $L^2(I)$ for all real and complex λ. The equation is *limit-point* at b if, for some compex λ, a solution f exists with $f \notin L^2(I)$. Then all solutions for complex λ share this property. For real λ at most one of the two independent solutions belongs to $L^2(I)$ in that case. See e.g. CHAUDHURI & EVERITT [8].

To obtain a selfadjoint realization of M, boundary conditions at both endpoints are necessary. We quote from [12] (Theorem 2.1) that the following conditions are equivalent:

(2.5)
- (i) f is bounded on (-1,1),
- (ii) $\lim_{\mu\uparrow 1} f(\mu)$ and $\lim_{\mu\downarrow -1} f(\mu)$ exist and are finite,
- (iii) $\lim_{\mu\uparrow 1} p(\mu)f'(\mu) = \lim_{\mu\downarrow -1} p(\mu)f'(\mu) = 0$,
- (iv) $p^{\frac{1}{2}}f'(\mu) \in H$.

See also EVERITT [10] for an extensive discussion of these matters. We remark that another set of boundary conditions, which is not equivalent to any of those given in (2.5) can be constructed by means of the theory given in DUNFORD & SCHWARTZ ([9], Ch.13,

§8). According to this theory the full set of boundary operators is

(2.6) $A_+f = \lim_{\mu\uparrow 1} (1-\mu)f'(\mu)$, $A_-f = \lim_{\mu\uparrow 1} f(\mu)+(1-\mu)(\ln(1-\mu))f'(\mu)$,

(2.7) $B_+f = \lim_{\mu\downarrow -1} (1+\mu)f'(\mu)$, $B_-f = \lim_{\mu\downarrow -1} -f(\mu)+(1+\mu)(\ln(1+\mu))f'(\mu)$.

However, the boundary conditions $A_-f = B_-f = 0$ give rise to unbounded solutions which are still elements of $L^2(J)$. These solutions are not suitable in a physical sense. It follows from Theorem 2.2 of [12] that the operator A defined by

(2.8) $\mathcal{D}(A) := \{f \mid f \in \mathcal{D}(M),\ f \text{ satisfies } (2.5)(i)\}$,

(2.9) $Af = Mf, \quad f \in \mathcal{D}(A)$,

is selfadjoint in H, with a discrete spectrum $\sigma(A) = \{n(n+1) \mid n = 0,1,\ldots\}$. The eigenfunction corresponding to the eigenvalue $n(n+1)$ is the Legendre polynomial P_n.

The transport problem (1.1) leads to the study of the operator AT^{-1}. We quote some results from [12]. Let 1_J denote the function identical 1 on J.

THEOREM 1 [12]. (i) *The Hilbert space* H *admits a decomposition* $H = H_0 \oplus H_1$ *such that the pair* $\{H_0,H_1\}$ *reduces the operator* AT^{-1}. *In particular* $H_0 = \mathrm{sp}(T1_J,T^2 1_J)$ *and* $H_1 = \{f \mid f \in H;\ (f,1_J) = (f,T1_J) = 0\}$, *with projection operators* P *and* $P_0 = 1-P$, *where*

(2.10) $Pf = f - \frac{3}{2}(f,T1_J)T1_J - \frac{3}{2}(f,1_J)T^2 1_J, \quad f \in H.$

(ii) *The restriction* $AT^{-1}|H_1$ *is injective and* $(AT^{-1}|H_1)^{-1} = PTK|H_1$ *where* K *is the integral operator*

(2.11) $Kf = \int_{-1}^{1} k(\mu,\mu')f(\mu')d\mu' + 2(\ln 2-\frac{1}{2})(f,1_J)1_J, \mu \in J,\ f \in H,$

(2.12) $k(\mu,\mu') = -\frac{1}{2}\ln((1+\bar{\mu})/(1-\underline{\mu})), \quad \mu,\mu' \in J,$

(2.13) $\bar{\mu} = \max(\mu,\mu'), \quad \underline{\mu} = \min(\mu,\mu'), \quad \mu,\mu' \in J.$

(iii) K *is a compact selfadjoint operator in* H *with spectrum* $\sigma(K) = \{(n(n+1))^{-1} \mid n = 1,2,\ldots\}$ *and* $KP_n = (n(n+1))^{-1}P_n$, $n = 1,2,\ldots$, $K1_J = 2(\ln 2-\frac{1}{2})1_J$. *Furthermore,* K *maps* H_1 *into itself and*

(2.14) $KAf = f - \frac{1}{2}(f,1_J)1_J$, $f \in \mathcal{D}(A)$,

(2.15) $AKf = f$, $f \in H$,

(2.16) $(Kf,1_J) = 2(\ln 2-\frac{1}{2})(f,1_J)$, $f \in H$,

(2.17) $(Kf,T1_J) = \frac{1}{2}(f,T1_J)$, $f \in H$.

Let the operator B be defined by

(2.18) $Bf = PTK|H_1$, $f \in H_1$.

From this definition we learn that B is compact on H_1. Introduce the inner product

(2.19) $(f,g)_A = (K^{\frac{1}{2}}f,K^{\frac{1}{2}}g)$, $f,g \in H$.

We denote by H_A the Hilbert space which is obtained as the completion of the inner product space $(H,\|\cdot\|_A)$, and we define $H_{1,A}$ similarly. It is possible to extend B to $H_{1,A}$.

THEOREM 2 [12]. (i) $H_A = H_0 \oplus H_{1,A}$.
(ii) *The operator* B *is compact selfadjoint on* $H_{1,A}$.
(iii) *The operator* B *maps* $H_{1,A}$ *into* H_1.
(iv) *The spectrum* $\sigma(B)$ *of* B *on* $H_{1,A}$ *is simple and consists of a countably infinite sequence of real eigenvalues* $\{\lambda_n^{-1} \mid n = \pm 1, \pm 2,\ldots\}$ *with an accumulation point at the origin.*

Let χ_n denote the eigenfunctions of B in $H_{1,A}$:

(2.20) $B\chi_n = \lambda_n^{-1}\chi_n$, $n = \pm 1,\pm 2,\ldots$,

and define $\phi_n = K\chi_n$. We normalize the functions χ_n,ϕ_n by the condition

(2.21) $(P1_J,\phi_n) = 1$, $n = \pm 1,\pm 2,\ldots$.

THEOREM 3 [12]. *For all* $n = \pm 1,\pm 2,\ldots$
(i) $\chi_n,\phi_n \in H_1 \subset L^2(J)$.
(ii) $\chi_n \in \mathcal{D}(AT^{-1}) = \{f \mid f \in \mathcal{D}(T^{-1});\ T^{-1}f \in \mathcal{D}(A)\}$,
$\phi_n \in \mathcal{D}(T^{-1}A) = \{f \mid f \in \mathcal{D}(A);\ Af \in \mathcal{D}(T^{-1})\}$.
(iii) χ_n,ϕ_n *satisfy*

(2.22) $AT^{-1}\chi_n = \lambda_n\chi_n$,

(2.23) $T^{-1}A\phi_n = \lambda_n T^{-1}PT\phi_n$.

THEOREM 4 [12]. (i) *The eigenvectors* $\{\chi_n \mid n = \pm 1, \pm 2, \dots\}$ *form an orthogonal basis in* $H_{1,A}$.
(ii) *The eigenfunction expansion*

$$(2.24) \qquad f = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} (f,\chi_n)_A \, \|\chi_n\|_A^{-2} \, \chi_n, \qquad f \in H_{1,A},$$

converges in the topology of H_A.
(iii) *The eigenvectors* $\{\chi_n\}$ *and* $\{\phi_n\}$ *form a biorthogonal system in* H_1 *in the sense that* $(\chi_m,\phi_n) = 0$ *if* $m \neq n$ *and* $(\chi_n,\phi_n) \neq 0$ *for every* $n = \pm 1, \pm 2, \dots$.
(iv) *The eigenfunction expansion* (2.24) *can be written as*

$$(2.25) \qquad f = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} (f,\phi_n)(\chi_n,\phi_n)^{-1}\chi_n, \qquad f \in H_{1,A}.$$

THEOREM 5 [12]. (i) *The space* $H_{1,A}$ *is topologically isomorphic with the sequence space* ℓ^2_σ *of all square summable sequences* $c = [c_n \mid n = \pm 1, \pm 2, \dots]$, $c_n \in \mathbb{C}$, *with respect to the weight* σ: $\sigma_n = (\chi_n,\phi_n)^{-1}$, $n = \pm 1, \pm 2, \dots$. *There holds* $\sigma_n = \sigma_{-n}$. *The isomorphism* F *which maps* $H_{1,A}$ *onto* ℓ^2_σ *and its inverse* F^{-1} *are given by*

$$(2.26) \qquad Ff = [(f,\phi_n) \mid n = \pm 1, \pm 2, \dots], \quad f \in H_{1,A},$$

$$(2.27) \qquad F^{-1}c = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \sigma_n c_n \chi_n, \qquad c \in \ell^2_\sigma.$$

(ii) *The transformation* F *diagonalizes the operator* B *on* $H_{1,A}$:

$$(2.28) \qquad FBf = [\lambda_n^{-1}(f,\phi_n) \mid n = \pm 1, \pm 2, \dots], \qquad f \in H_{1,A}.$$

Let Λ denote the following multiplicative unbounded operator on ℓ^2_σ:

$$(2.29) \qquad \mathcal{D}(\Lambda) = \{c \in \ell^2_\sigma \mid \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \sigma_n |\lambda_n c_n|^2 < \infty\},$$

$$(2.30) \qquad \Lambda c = [\lambda_n c_n \mid n = \pm 1, \pm 2, \dots], \quad c \in \mathcal{D}(\Lambda).$$

By this definition (2.28) can be rewritten as

$$(2.31) \qquad FBf = \Lambda^{-1}Ff, \qquad f \in H_{1,A}.$$

Next we solve in H the differential equation (1.1) which we write in the form

(2.32) $\psi'(x) + AT^{-1}\psi(x) = 0, \quad x \in (0,\tau), \ ' = \frac{d}{dx},$

where $\psi(x) = T\phi(x)$ for all $x \in (0,\tau)$. Here we assume that ϕ,ψ are vector-valued functions: $[0,\tau] \to H$. It is possible to extend this equation into one in H_A. That means that we have to solve

(2.33) $(P_0\psi)'(x) + AT^{-1}P_0\psi(x) = 0,$

(2.34) $(P\psi)'(x) + B^{-1}P\psi(x) = 0.$

We define the decomposition $H_{1,A} = H_{1,p} \oplus H_{1,m}$ where $H_{1,p} = \overline{sp}\{\chi_n \mid n = 1,2,...\}$, $H_{1,m} = \overline{sp}\{\chi_n \mid n = -1,-2,...\}$ with the closure in the A-norm. Then it is evident that $\sigma(B|H_{1,p}) = \{\lambda_n^{-1} \mid n = 1,2,...\}$, $\sigma(B|H_{1,m}) = \{-\lambda_n^{-1} \mid n = 1,2,...\}$. Thus this decomposition reduces B to an accretive operator in $H_{1,p}$ and a dissipative operator in $H_{1,m}$. Let $P_{1,p}(P_{1,m})$ denote the projection operator which maps H_A onto $H_{1,p}(H_{1,m})$ along $H_0 \oplus H_{1,m}$ $(H_0 \oplus H_{1,p})$. The representations of $P_{1,p}$ and $P_{1,m}$ are

(2.35) $P_{1,p}f = \sum_{n=1}^{\infty} \sigma_n(Pf,\phi_n)\chi_n, \quad f \in H_A,$

(2.36) $P_{1,m}f = \sum_{n=1}^{\infty} \sigma_n(Pf,\phi_{-n})\chi_{-n}, \quad f \in H_A.$

The differential equation (2.34) is then equivalent with the following pair of differential equations

(2.37) $(P_{1,p}\psi)'(x) + B^{-1}P_{1,p}\psi(x) = 0,$

(2.38) $(P_{1,m}\psi)'(x) + B^{-1}P_{1,m}\psi(x) = 0.$

By means of semigroup methods it is possible to solve these equations with the following result.

THEOREM 6 [12]. *The general solution of* (2.32) *in* H_A *is given by*

(2.39) $\psi(x) = \exp((\tfrac{1}{2}\tau-x)AT^{-1})P_0h+\exp(-xB^{-1})P_{1,p}h +$
$\exp((\tau-x)B^{-1})P_{1,m}h, \ h \in H_A$ arbitrary,

where $P_0h = \alpha T1_J + \beta T^2 1_J$, α,β arbitrary and where the exponential operators are defined by

(2.40) $\exp((\frac{1}{2}\tau - x)AT^{-1})P_0 h = (\alpha + 2\beta(\frac{1}{2}\tau - x))T1_J + \beta T^2 1_J$,

(2.41) $\exp(-xB^{-1})P_{1,p}h = F^{-1}e^{-x\Lambda}FP_{1,p}h, \quad h \in H_A$,

(2.42) $\exp((\tau - x)B^{-1})P_{1,m}h = F^{-1}e^{(\tau - x)\Lambda}FP_{1,m}h, \quad h \in H_A$.

In this paper we study the eigenvalue problem (2.20) in the form

(2.43) $Av_n = \lambda_n Tv_n$,

so we identify $\chi_n = Tv_n$.

LEMMA 1. *The vectors* χ_n, ϕ_n *and* v_n *are related through the following identities*

(2.44) $\chi_n = Tv_n = \lambda_n T\phi_n + \frac{1}{2}\lambda_n T1_J$,

(2.45) $\phi_n = \lambda_n^{-1} T^{-1} \chi_n - \frac{1}{2}1_J = \lambda_n^{-1} v_n - \frac{1}{2}1_J$,

(2.46) $v_n = T^{-1}\chi_n = \lambda_n \phi_n + \frac{1}{2}\lambda_n 1_J$.

PROOF. From (2.21) we obtain the identity

(2.47) $$1 = (P1_J, \phi_n) = (1_J, \phi_n) - \frac{3}{2}(1_J, T1_J)(T1_J, \phi_n) - \frac{3}{2}(1_J, 1_J)(T^2 1_J, \phi_n).$$

Now, $(1_J, \phi_n) = 0$ because $\phi_n \in H_1$, $(1_J, T1_J) = 0$, and $(1_J, 1_J) = 2$, so $(T^2 1_J, \phi_n) = -\frac{1}{3}$ by (2.47). Evaluation of (2.23) gives

(2.48) $$T^{-1}A\phi_n = \lambda_n T^{-1}(T\phi_n - \frac{3}{2}(T\phi_n, T1_J)T1_J - \frac{3}{2}(T\phi_n, 1_J)T^2 1_J) = \lambda_n \phi_n + \frac{1}{2}\lambda_n 1_J,$$

since $(T\phi_n, T1_J) = -\frac{1}{3}$, and $(T\phi_n, 1_J) = 0$ because $\phi_n \in H_1$. If one applies the operator T on (2.48) and inserts $\phi_n = K\chi_n$, one finds, using (2.15), (2.44). The relations (2.45), (2.46) are equivalent with (2.44). □

Since it appears impossible to determine the functions v_n explicitly, we have studied their asymptotic behaviour for $|n| \to \infty$. We have also studied the asymptotic behaviour of λ_n and $\sigma_n = (\chi_n, \phi_n)^{-1}$ for $|n| \to \infty$. It turns out that these asymptotic results are very good approximations compared with numerical results.

We end this section with the following regularity result.

LEMMA 2. *The functions* v_n *are elements of* $C^\infty([-1,1])$ *for* $n = \pm 1, \pm 2, \ldots$.

PROOF. This result follows from the standard theory of differential equations with C^∞-coefficients. Since we select, by the boundary condition (2.5)(ii) that solution which is analytic in a neighborhood of $\mu = -1$, the solution is certainly an element of $C^\infty([-1,1))$. The regular singularity at $\mu = 1$ determines the radius of convergence of its expansion. Because of the boundary condition at $\mu = 1$, the expansion can be continued up to $\mu = 1$. □

3. EXPANSION OF THE EIGENFUNCTIONS IN LEGENDRE POLYNOMIALS

In this section and the next we study the eigenvalue problem

$$
\begin{aligned}
&-\frac{d}{dx}\left((1-x^2)\frac{d}{dx}u_n(x)\right) = \lambda_n x u_n(x), \quad -1 < x < 1, \\
&u_n \text{ bounded on } (-1,1),\ n \in \mathbb{N}^+ \cup \mathbb{N}^-,
\end{aligned} \tag{3.1}
$$

with the normalization

$$u_n(1) = 1, \quad n \in \mathbb{N}^+, \qquad u_n(-1) = 1, \quad n \in \mathbb{N}^-. \tag{3.2}$$

In the notation of section 2, problem (3.1) is written as

$$T^{-1}Av_n = \lambda_n v_n, \qquad v_n \in \mathcal{D}(T^{-1}A), \tag{3.3}$$

on which

$$v_n(\mu) = C_n u_n(x), \qquad \mu = x. \tag{3.4}$$

In section 2 the eigenfunctions v_n were normalized by (2.21); however, (3.2) turns out to be a more practical normalization.

Problem (3.1) is a singular Sturm-Liouville eigenvalue problem with an indefinite weight function. Both endpoints $x = -1$ and $x = 1$ are regular singularities, the midpoint $x = 0$ is a turning-point. Problem (3.1) admits the solution $u_0(x) = 1$ with $\lambda_0 = 0$. In addition, it follows from Theorem 2 that (3.1) admits a countable number of eigenvalues $\{\lambda_n \mid n = \pm 1, \pm 2, \ldots\}$. The corresponding eigenfunctions are elements of H_1.

LEMMA 3. *The eigenfunctions* u_n, $n = \pm 1, \pm 2, \ldots$ *satisfy the following orthogonality relation:*

$$\int_{-1}^{1} x u_n(x) u_m(x)\,dx = \delta_{nm}\, C_n^{-2} \lambda_n (\chi_n, \phi_n). \tag{3.5}$$

PROOF. Note that

(3.6) $$\int_{-1}^{1} x u_n(x)dx = 0,$$

by direct integration of (3.1) and by (2.5) (iii). Using the relation (3.4) and (2.46), we find that the left-hand side of (3.5) is equal to

(3.7) $$C_n^{-2} \int_{-1}^{1}\mu(\mu^{-1}\chi_n(\mu))(\lambda_m\phi_m(\mu)+\tfrac{1}{2})d\mu = C_n^{-2}(\lambda_n(\chi_n,\phi_m)+\tfrac{1}{2}(\chi_n,1_J)).$$

The last term in the right-hand side of (3.7) is zero, by (2.44) and (3.6); hence, (3.7) is equivalent with (3.5), because of the biorthogonality of χ_n and ϕ_n (Theorem 4). □

Since $u_n \in C^{\infty}([-1.1])$, we can write

(3.8) $$u_n(x) = \sum_{k=0}^{\infty} a_{k,n}P_k(x), \quad n = \pm1,\pm2,\ldots .$$

If one inserts (3.8) into (3.1) and (3.2), and uses two well-known properties of the Legendre polynomials, viz.,

(3.9) $$-((1-x^2)P_k')' = k(k+1)P_k, \quad k \geq 0,$$

(3.10) $$(2k+1)xP_k = (k+1)P_{k+1} + kP_{k-1}, \quad k \geq 1,$$

one finds the following identities for $n = \pm1,\pm2,\ldots$:

(3.11) $$a_{1,n} = 0,$$

(3.12) $$\frac{(k+1)}{(2k+3)}\lambda a_{k+1,n} - k(k+1)a_{k,n} + \frac{k}{(2k-1)}\lambda a_{k-1,n} = 0, \; k \geq 1,$$

subject to the normalization condition

(3.13) $$a_{0,n} + \sum_{k=2}^{\infty} a_{k,n} = 1, \quad n = 1,2,\ldots .$$

Explicitly,

(3.14) $$a_{2,n} = -\frac{5}{2}a_{0,n}, \quad a_{3,n} = -\frac{35}{\lambda}a_{0,n}, \quad n = \pm1,\pm2,\ldots .$$

It follows from the symmetry relation $u_{-n}(x) = u_n(-x)$ that

(3.15) $$a_{k,-n} = (-1)^k a_{k,n}, \quad n = 1,2,\ldots .$$

The unknown λ is still involved in the recurrence relation (3.12). Only for discrete values of λ is it possible to satisfy (3.13), as the following argument shows. The two independent solutions of any recurrence relation of the type

(3.16) $$y_{k+1} + A_k y_k + B_k y_{k-1} = 0, \qquad k \geq 1,$$

with $A_k \sim ak^{\alpha}$, $B_k \sim bk^{\beta}$, $b \to \infty$, $2\alpha > \beta$, $ab \neq 0$, exhibit the following asymptotic behaviour:

(3.17) $$y^+_{k+1}/y^+_k \sim -\, ak^{\alpha}, \qquad k \to \infty$$

(3.18) $$y^-_{k+1}/y^-_k \sim -\, (b/a)k^{\beta-\alpha}, \qquad k \to \infty,$$

see GAUTSCHI [11]. The general solution of (3.16) can be represented in the form

(3.19) $$y_k = C(y_0,y_1)y^+_k + D(y_0,y_1)y^-_k,$$

where the constants C and D depend on the initial values y_0, y_1. The particular solution $\{y^+_k\}$ is called *dominant*, $\{y^-_k\}$ *recessive*. Applying these results to (3.12), we obtain

(3.20) $$a^+_{k+1,n}/a^+_{k,n} \sim (2/\lambda)k^2, \qquad k \to \infty,$$

(3.21) $$a^-_{k+1,n}/a^-_{k,n} \sim (\lambda/2)k^{-2}, \qquad k \to \infty.$$

A solution of (3.12) for which (3.13) holds, must be recessive, so $C(a_{0,n},a_{1,n}) = C_\lambda(a_{0,n},0) = 0$. This equation depends only on λ; the value $a_{0,n}$ serves as a normalization constant. It is not possible to obtain an explicit expression for $C_\lambda(a_{0,n},0)$; however, it is possible to obtain approximations for the first few eigenvalues by means of a continued fraction expansion. The transformation

(3.22) $$b_{k,n} = \frac{\lambda^k}{2^{k-1}(2k+1)\Gamma(k)\Gamma(k+\frac{1}{2})}\, a_{k,n}, \qquad k \geq 1,$$

transforms the relation (3.12) into

(3.23) $$b_{k+1,n} - b_{k,n} - \frac{\lambda^2}{(4k^2-1)(k^2-1)}\, b_{k-1,n} = 0, \qquad k \geq 2,$$

with starting values $b_{1,n} = 0$, $b_{2,n} = -3^{-1}\lambda^2\pi^{-\frac{1}{2}}\, a_{0,n}$. Further, $b_{3,n} = b_{2,n}$. We define $\tau_k = b_{k+1}/b_k$, omitting the index n. Then τ_k satisfies

(3.24) $$\tau_{k-1} = \frac{\lambda^2}{(4k^2-1)(k^2-1)}\, \frac{1}{1-\tau_k}, \quad k \geq 3,\ \tau_2 = 1.$$

Since we look for the recessive solution $a^-_{k,n}$ of (3.12), we conclude from (3.21) and (3.22) that $\tau_k = O(k^{-4})$, $k \to \infty$. Hence, in order to find successive approximations of λ, we put $\tau_\ell = 0$ for some ℓ, calculate $\bar{\tau}_2^{(\ell)}$ and solve $\bar{\tau}_2^{(\ell)} = \tau_2 = 1$ for λ. The successive approximations become

$\ell = 3$, $\bar{\tau}_2^{(3)} = \lambda^2/280$,

$\ell = 4$, $\bar{\tau}_2^{(4)} = \lambda^2/(280(1-\lambda^2/945))$,

$\ell = 5$, $\bar{\tau}_2^{(5)} = \lambda^2/(280(1-\lambda^2/(945-\lambda^2/2376)))$,

$\ell = 6$, $\bar{\tau}_2^{(6)} = \lambda^2/(280(1-\lambda^2/(945-\lambda^2/(2376-\lambda^2/5005))))$,

and the corresponding eqations $\bar{\tau}_2^{(\ell)} = 1$ become

(3.25) $$\lambda^2 - 280 = 0 \Rightarrow \lambda_{\pm 1}^{(3)} = \pm 16.733,$$

(3.26) $$\lambda^2 - 216 = 0 \Rightarrow \lambda_{\pm 1}^{(4)} = \pm 14.697,$$

(3.27) $$\lambda^4 - 3369\ ^2 + 665280 = 0 \Rightarrow \lambda_{\pm 1}^{(5)} = \pm\ 14.536, \quad \lambda_{\pm 2}^{(5)} = \pm\ 56.113,$$

(3.28) $$7\lambda^4 - 15136\lambda^2 + 2882880 = 0 \Rightarrow \lambda_{\pm 1}^{(6)} = \pm 14.5282, \quad \lambda_{\pm 2}^{(6)} = \pm 44.174.$$

The values of λ can be compared with the values obtained from numerical calculations in section 5. There we find $\lambda_{\pm 1} = \pm 14.5280$, $\lambda_{\pm 2} = \pm 42.049$, so $\lambda_{\pm 1}^{(5)}$ and $\lambda_{\pm 2}^{(6)}$ give already good approximations. However, this approach does not give any insight into the location of the eigenvalues.

The next lemma gives the representations of χ_n and ϕ_n.

LEMMA 4. *In terms of the expansion* (3.8) *the functions* χ_n, ϕ_n *have the following representations*:

(3.29) $$\chi_n(\mu) = C_n\lambda_n^{-1} \sum_{k=2}^{\infty} k(k+1)a_{k,n}P_k(\mu),$$

(3.30) $$\phi_n(\mu) = C_n\lambda_n^{-1} \sum_{k=2}^{\infty} a_{k,n}P_k(\mu).$$

The normalization condition $(P1_J, \phi_n) = 1$ *takes the form*

(3.31) $$a_{2,n} = -\frac{5}{4}\lambda_n C_n^{-1}.$$

The inner product (χ_n, ϕ_n) *becomes*

$$(3.32)\qquad (\chi_n,\phi_n) = \|\chi_n\|_K^2 = C_n^2\,\lambda_n^{-2}\sum_{k=2}^{\infty}\frac{2k(k+1)}{(2k+1)}\,a_{k,n}^2$$

$$= C_n^2\,\lambda_n^{-1}\sum_{k=2}^{\infty}\frac{4(k+1)}{(2k+1)(2k+3)}\,a_{k,n}a_{k+1,n}.$$

PROOF. The representation (3.29) follows from (2.44) if one uses (3.10) and (3.12). The representation (3.30) follows from the identities $\phi_n = K\chi_n$ and $KP_n = (n(n+1))^{-1}\,P_n$, $n = 1,2,\ldots$ Since $P1_J(\mu) = -2P_2(\mu)$, the condition $(P1_J,\phi_n) = 1$ is equivalent with $(-2P_2, C_n\lambda_n^{-1}a_{2,n}P_2) = 1$. Relation (3.31) follows then by the property $(P_k,P_k) = 2/(2k+1)$, $k = 0,1,\ldots$ The first few coefficients become by (3.31)

$$(3.33)\qquad a_{0,n} = C_n^{-1}\,\lambda_n/2,$$

$$(3.34)\qquad a_{1,n} = 0,$$

$$(3.35)\qquad a_{2,n} = -(5/4)C_n^{-1}\lambda_n,$$

$$(3.36)\qquad a_{3,n} = -(35/2)C_n^{-1}.$$

The identity (3.32) is found by taking the inner product, using (3.29), (3.30) and the property $(P_k,P_k) = 2/(2k+1)$, $k = 0,1,\ldots$ The second identity in (3.32) follows from (3.12). □

REMARK. From (3.33) it follows that

$$(3.37)\qquad C_n = \lambda_n\left(\int_{-1}^{1} u_n(x)dx\right)^{-1},$$

since $a_{0,n} = (P_0,u_n)/2$.

4. ASYMPTOTIC EXPANSIONS OF EIGENFUNCTIONS AND EIGENVALUES

In this section we construct asymptotic expansions for the eigenfunctions of (3.1), (3.2). Since we want to use the expansion theorems of OLVER [13], we write (3.1) into a form without first derivative.

LEMMA 5. *The following boundary value problems are equivalent:*

$$(4.1)\qquad u_n'' - \frac{2x}{1-x^2}\,u_n' + \frac{\lambda_n x}{1-x^2}\,u_n = 0, \qquad -1 < x < 1,$$

$$u_n(1) = 1, \quad n > 0, \quad u_n(-1) = 1, \quad n < 0,$$

$$(4.2)\qquad w''_n + \left[\frac{\lambda_n x}{1-x^2} + \frac{1}{(1-x^2)^2}\right] w_n = 0, \; -1 < x < 1,$$

$$\lim_{x\uparrow 1} (1-x^2)^{-\frac{1}{2}} w_n(x) = 1, \quad n > 0,$$

$$\lim_{x\downarrow -1} (1-x^2)^{-\frac{1}{2}} w_n(x) = 1, \quad n < 0,$$

$$(4.3)\qquad g''_n + \lambda_n(\mathrm{tgh}\, z/\cosh^2 z) g_n = 0, \quad -\infty < z < \infty,$$

$$\lim_{z\to\infty} g_n(z) = 1, \quad n > 0, \; \lim_{z\to-\infty} g_n(z) = 1, \quad n < 0,$$

$$(4.4)\qquad k''_n + \mathrm{cotg}\,\theta\, k'_n + \lambda_n \cos\theta\, k_n = 0, \quad 0 < \theta < \pi,$$

$$k_n(0) = 1, \quad n > 0, \quad k_n(\pi) = 1, \quad n < 0,$$

$$(4.5)\qquad \ell''_n + \left[\lambda_n \cos\theta + \frac{1+\sin^2\theta}{4\sin^2\theta}\right]\ell_n = 0, \quad 0 < \theta < \pi.$$

$$\lim_{\theta\downarrow 0} (\sin\theta)^{-\frac{1}{2}} \ell_n(\theta) = 1, \quad n > 0,$$

$$\lim_{\theta\uparrow\pi} (\sin\theta)^{-\frac{1}{2}} \ell_n(\theta) = 1, \quad n > 0.$$

The solutions u,w,g,k *and* ℓ *are related by the identities*

$$(4.6)\qquad w(x) = (1-x^2)^{\frac{1}{2}} u(x),$$

$$(4.7)\qquad g(z) = \cosh z\; w(\mathrm{tgh}\, z) = u(\mathrm{tgh}\, z).$$

$$(4.8)\qquad k(\theta) = u(\cos\theta),$$

$$(4.9)\qquad \ell(\theta) = (\sin\theta)^{\frac{1}{2}} u(\cos\theta).$$

PROOF. Straight-forward calculation. □

REMARK. In the original formulation of the electron scattering problem equation (4.4) was derived, see BOTHE [6].

As we mentioned in section 2, two independent solutions of the Legendre differential equation N[f] = 0 (see (2.2)) are $f_1(x) = 1$ and $f_2(x) = \ln((1+x)/(1-x))$. In general, the equation $N[f] = \lambda f$ admits a solution f_1 which is bounded near x = 1 and another solution which is unbounded near x = 1. If one continues these solutions to the other singular endpoint x = -1, f_1 remains bounded for $\lambda = n(n+1)$, $n \in \mathbb{N}$, only. For these values of λ, $f_1(x) = P_n(x)$. In the case under consideration the situation near x = 1 is qualitatively the same. However, when crossing the turning point $x = 0$, the character of the solutions f_1 and f_2 changes drastically. The solutions u_n have the symmetry property:

(4.10) $$u_{-n}(x) = u_n(-x), \quad \lambda_{-n} = -\lambda_n, \quad n \in \mathbb{N}.$$

Thus it is sufficient to treat only positive eigenvalues λ_n. We assume that $\lambda_n > 0$ if $n > 0$.

We handle the eigenvalue problem in the form (4.2). This is the form for which OLVER ([13], Ch.10,11 & 12) summarized the so-called Liouville-Green approximation technique for Sturm-Liouville equations on a domain J in the complex plane:

(4.11) $$\frac{d^2w}{dz^2} - (u^2 f(z)+g(z))w = 0, \quad \text{for } u^2 \to \infty.$$

In his notation we have

(4.12) $$u^2 = \lambda, \quad f(z) = -z/(1-z^2), \; g(z) = -1/(1-z^2)^2.$$

Transition points are those points where f vanishes or where either f or g becomes singular. We distinguish three cases:

case I: J is free from transition points,

case II: J has one transition point z_0 where f vanishes and g is analytic,

case III: J has one transition point z_0 where f has a simple pole and $(z-z_0)^2 g$ is analytic.

Restricting ourselves to real values of the independent variable, we consider (4.2) on $\bar{J} = [-1,1]$. If we split $\bar{J}$ into three parts: $J_1 = [q,1]$, $J_2 = [p,q]$, $J_3 = [-1,p]$ where p and q are arbitrary points with $-1 < p < 0$, $0 < q < 1$, then we are dealing with case III on J_1 and J_3 and with case II on J_2. The Liouville-Green approximation consists of two transformations on w and z:

(4.13) $$W(\xi) = \left(\frac{dz}{d\xi}\right)^{-\frac{1}{2}} w(z), \quad \xi = \xi(z).$$

Then (4.11) becomes

(4.14) $$\frac{d^2W}{d\xi^2} - \{u^2\left(\frac{dz}{d\xi}\right)^2 f(z) + \phi(\xi)\}W = 0, \quad z = z(\xi),$$

with

(4.15) $$\phi(\xi) = \left(\frac{dz}{d\xi}\right)^2 g(z) + \left(\frac{dz}{d\xi}\right)^{\frac{1}{2}} \frac{d^2}{d\xi^2}\left[\left(\frac{dz}{d\xi}\right)^{-\frac{1}{2}}\right], \quad z = z(\xi).$$

The transformation $\xi = \xi(z)$ is chosen in such a way that

(i) ξ and z are analytic functions of each other, and

(ii) the solutions of the differential equation (4.14) are approximated by the solutions of the same equation with $\phi(\xi) = 0$ (or part of it). The choices of ξ are:

$$\text{case I} \ : (\tfrac{dz}{d\xi})^2 f(z) = 1, \qquad \xi = \int f^{\frac{1}{2}}(z)dz,$$

$$(4.16) \qquad \text{case II} \ : (\tfrac{dz}{d\xi})^2 f(z) = \xi, \qquad \tfrac{2}{3}\xi^{3/2} = \int_{z_0}^{z} f^{\frac{1}{2}}(z)dz,$$

$$\text{case III} : (\tfrac{dz}{d\xi})^2 f(z) = \xi^{-1}, \qquad 2\xi^{\frac{1}{2}} = \int_{z_0}^{z} f^{\frac{1}{2}}(z)dz.$$

Thus, (4.14) reduces to the standard form

$$(4.17) \qquad \frac{d^2W}{d\xi^2} - \{u^2\xi^m + \phi(\xi)\}W = 0,$$

with $m = 0$ (case I), $m = 1$ (case II), $m = -1$ (case III). In cases I and II ϕ becomes a holomorphic function; in case III, ϕ has a single or double pole at $\xi = 0$ if g does. The approximating equation is

$$(4.18) \qquad \frac{d^2W}{d\xi^2} - \{u^2\xi^m - c\xi^{-2}\}W = 0,$$

with m as above; $c = 0$ for the cases I and II, and for the case III if ϕ has no double pole; $c \neq 0$ for case III if ϕ has a double pole. The theory given in OLVER [13] also supplies bounds for the error terms.

LEMMA 6. *The asymptotic expansion of the solution of* (3.1), (3.2), *on* $J_1 = [q,1]$ *is given by*

$$(4.19) \qquad u(x) = (1-x^2)^{-\frac{1}{2}}(\tfrac{d\zeta}{dx}(x))^{-\frac{1}{2}}W_1(\lambda^{\frac{1}{2}},(-\zeta(x))^{\frac{1}{2}}),$$

where, with J_0, J_1 *the Bessel functions of order zero and one,*

$$(4.20) \qquad (-\zeta(x))^{\frac{1}{2}} = \int_x^1 \sqrt{\frac{t}{1-t^2}}\,dt, \qquad 0 < x \le 1, \qquad \zeta \le 0,$$

$$(4.21) \qquad W_1(\lambda^{\frac{1}{2}},(-\zeta)^{\frac{1}{2}}) = (-\zeta)^{\frac{1}{2}}J_0(\lambda^{\frac{1}{2}}(-\zeta)^{\frac{1}{2}})[2^{\frac{1}{2}}+A_1(\zeta)\lambda^{-1}] + -(-\zeta)\lambda^{-\frac{1}{2}}J_1(\lambda^{\frac{1}{2}}(-\zeta)^{\frac{1}{2}})B_0(\zeta) + O(\lambda^{-3/2}),$$

uniformly for $\zeta \in J_1' = [\zeta(q),0]$, $\lambda \to \infty$,

$$(4.22)\qquad B_0(\zeta) = 2^{\frac{1}{2}}(-\zeta)^{\frac{1}{2}} \int_\zeta^0 \psi(v)(-v)^{-\frac{1}{2}}dv,$$

$$(4.23)\qquad A_1(\zeta) = 2^{\frac{1}{2}}[-\psi(\zeta) + \int_0^\zeta \psi(\zeta')(-\zeta')^{-\frac{1}{2}}\{\int_{\zeta'}^0 \psi(v)(-v)^{-\frac{1}{2}}dv\}d\zeta'] + \tfrac{1}{2}B_0(\zeta),$$

$$(4.24)\qquad \psi(\zeta(x)) = \frac{1}{16\zeta(x)} + \frac{(3x^2-5)(x^2+1)}{64x^3(x^2-1)}.$$

The derivative $\frac{du}{dx}$ *is given by the derivative of* (4.19) *with respect to* x. *The error term becomes* $O(\lambda^{-1})$, $\lambda \to \infty$.

PROOF. For the proof we refer to OLVER ([13], Ch.12 Theorem 4.1 and section 5.2). □

Observe that $\zeta = 4\xi$, ζ defined in (4.20), ξ defined in (4.16), case III. By this transformation the interval [q,1] is mapped onto $[\zeta(q),0]$, $\zeta(q) < 0$. In terms of the original functions we have

$$(4.25)\qquad \psi(\zeta(x)) = \frac{1}{16\zeta(x)} + \frac{g(x)}{4f(x)} + \frac{4f(x)f''(x)-5(f'(x))^2}{64f^3(x)},$$

and

$$(4.26)\qquad \left(\frac{d\zeta}{dx}(x)\right)^{-\frac{1}{2}} = 2^{-\frac{1}{2}}(-\zeta(x))^{-\frac{1}{4}}x^{-\frac{1}{4}}(1-x^2)^{\frac{1}{4}},\quad 0 < x \le 1.$$

It is possible to give an infinite asymptotic series with the coefficients A_n, B_n defined recursively. However, the information given by (4.19), (4.20), (4.21), (4.22) and (4.23) is sufficient. For the actual calculation of $B_0(\zeta)$ and $A_1(\zeta)$ one needs to transform the variable of integration to x, since it is not possible to give an explicit expression for $x = x(\zeta)$ other than in the form of the inverse of an incomplete elliptic integral. For $x \uparrow 1$, the behaviour of $(-\zeta(x))$ is given by

$$(4.27)\qquad (-\zeta(x)) \sim 2(1-x),\quad x \uparrow 1.$$

The approximating equation (see (4.18)) becomes

$$(4.28)\qquad \frac{d^2W}{d\zeta^2} - \{\lambda(4\zeta)^{-1} - (4\zeta^2)^{-1}\}W = 0.$$

Independent solutions of (4.28) are $(-\zeta)^{\frac{1}{2}}J_0(\lambda^{\frac{1}{2}}(-\zeta)^{\frac{1}{2}})$ and $(-\zeta)^{\frac{1}{2}}Y_0(\lambda^{\frac{1}{2}}(-\zeta)^{\frac{1}{2}})$; Y_0 the other Bessel function of zero order. The solution (4.19) uses only the former, because Y_0 does not have the right boundary behaviour. The constant $A_0 = 2^{\frac{1}{2}}$ has been chosen to satisfy the requirement $u_n(1) = 1$.

LEMMA 7. *The asymptotic expansion of the solution of* (3.1), (3.2) *on* $J_3 = [-1,p]$ *is given by*

(4.29) $$u(x) = (1-x^2)^{-\frac{1}{2}}\left(\frac{d\hat{\zeta}}{dx}(x)\right)^{-\frac{1}{2}} W_3(\lambda^{\frac{1}{2}},(\hat{\zeta}(x))^{\frac{1}{2}}),$$

where, with I_0, I_1 *the modified Bessel functions of order zero and one,*

(4.30) $$(\hat{\zeta}(x))^{\frac{1}{2}} = \int_{-1}^{x} \sqrt{\frac{-t}{1-t^2}}\, dt, \quad -1 \le x < 0, \quad \hat{\zeta} \ge 0,$$

(4.31) $$W_3(\lambda^{\frac{1}{2}},\hat{\zeta}^{\frac{1}{2}}) = \hat{\zeta}^{\frac{1}{2}} I_0(\lambda^{\frac{1}{2}}\hat{\zeta}^{\frac{1}{2}})[\hat{A}_0+\hat{A}_1(\zeta)\lambda^{-1}] + \hat{\zeta}\lambda^{-\frac{1}{2}} I_1(\lambda^{\frac{1}{2}}\hat{\zeta}^{\frac{1}{2}})\hat{B}_0(\hat{\zeta}) + O(\lambda^{-3/2}).$$

uniformly for $\hat{\zeta} \in J_3' = [0,\hat{\zeta}(p)]$, $\lambda \to \infty$,

(4.32) $$\hat{B}_0(\hat{\zeta}) = \hat{A}_0\hat{\zeta}^{-\frac{1}{2}} \int_0^{\hat{\zeta}} \hat{\psi}(v)v^{-\frac{1}{2}}dv,$$

(4.33) $$\hat{A}_1(\hat{\zeta}) = \hat{A}_0[-\hat{\psi}(\hat{\zeta})+\int_0^{\hat{\zeta}} \hat{\psi}(\hat{\zeta}')\hat{\zeta}'^{-\frac{1}{2}}\{\int_0^{\hat{\zeta}'} \hat{\psi}(v)v^{-\frac{1}{2}}dv\}d\hat{\zeta}'+\tfrac{1}{2}\hat{B}_0(\hat{\zeta}),$$

(4.34) $$\hat{\psi}(\hat{\zeta}(x)) = \frac{1}{16\hat{\psi}(x)} + \frac{(3x^2-5)(x^2+1)}{64x^3(x^2-1)}.$$

The derivative $\frac{du}{dx}$ *is given by the derivative of* (4.29) *with respect to* x. *The error term becomes* $O(\lambda^{-1})$, $\lambda \to \infty$.

PROOF. For the proof we refer to OLVER ([13], Ch.12, Theorem 3.1 and section 5.2). □

Observe that, again, $\hat{\zeta} = 4\xi$, $\hat{\zeta}$ defined in (4.30), ξ defined in (4.16), case III. The factor in front of W_3 is

(4.35) $$\left(\frac{d\hat{\zeta}}{dx}(x)\right)^{-\frac{1}{2}} = 2^{-\frac{1}{2}}(\hat{\zeta}(x))^{-\frac{1}{4}}(-x)^{-\frac{1}{4}}(1-x^2)^{\frac{1}{4}}, \quad -1 \le x < 0,$$

and the approximating equation (see (4.18)) is

(4.36) $$\frac{d^2W}{d\hat{\zeta}^2} - \{\lambda(4\hat{\zeta})^{-1}-(4\hat{\zeta}^2)^{-1}\}W = 0.$$

Independent solutions of (4.36) are $\hat{\zeta}^{\frac{1}{2}}I_0(\lambda^{\frac{1}{2}}\hat{\zeta}^{\frac{1}{2}})$ and $\hat{\zeta}^{\frac{1}{2}}K_0(\lambda^{\frac{1}{2}}\hat{\zeta}^{\frac{1}{2}})$; K_0 the other modified Bessel function of zero order. The solution (4.29) uses only the former, because K_0 does not have the right boundary behaviour. The constant A_0 has to be determined by matching the solution in J_1 to J_3 across J_2. For $x \downarrow -1$, the behaviour of $\hat{\zeta}(x)$ is given by

(4.37) $$\hat{\zeta}(x) \sim 2(1+x), \quad x \downarrow -1.$$

LEMMA 8. *The asymptotic expansions of two independent solutions of* (3.1) *on* $J_2 = [p,q]$ is given by

(4.38) $$u_i(x) = (1-x^2)^{-\frac{1}{2}}\left(\frac{d\xi}{dx}(x)\right)^{\frac{1}{2}} W_{2,i}(\lambda^{\frac{1}{2}},\xi), \quad i = 1,2,$$

where, with Ai, Bi *the Airy functions,*

(4.39) $$\frac{2}{3}(\xi(x))^{3/2} = \int_0^x \sqrt{\frac{t}{1-t^2}}\, dt, \quad 0 \le x < 1, \quad \xi \ge 0,$$

$$\frac{2}{3}(-\xi(x))^{3/2} = \int_x^0 \sqrt{\frac{-t}{1-t^2}}\, dt, \quad -1 < x \le 0, \quad \xi \le 0,$$

(4.40) $$W_{2,1}(\lambda^{\frac{1}{2}},\xi) = \mathrm{Ai}(\lambda^{1/3}(-\xi))[\bar{A}_0+\bar{A}_1(-\xi)\lambda^{-1}] + \lambda^{-2/3}\mathrm{Ai}'(\lambda^{1/3}(-\xi))\bar{B}_1(-\xi) + O(\lambda^{-3/2}),$$

uniformly for $\xi \in J_2' = [\xi(p),\xi(q)]$, $\lambda \to \infty$,

(4.41) $$W_{2,2}(\lambda^{\frac{1}{2}},\xi) = \mathrm{Bi}(\lambda^{1/3}(-\xi))[\bar{A}_0+\bar{A}_1(-\xi)\lambda^{-1}] + \lambda^{-2/3}\mathrm{Bi}'(\lambda^{1/3}(-\xi))\bar{B}_1(-\xi) + O(\lambda^{-3/2}),$$

uniformly for $\xi \in J_2' = [\xi(p),\xi(q)]$, $\lambda \to \infty$,

(4.42) $$\bar{B}_0(\xi) = \bar{A}_0 2^{-1}\xi^{-\frac{1}{2}} \int_0^\xi \bar{\psi}(v)v^{-\frac{1}{2}}dv, \quad \xi > 0,$$

$$\bar{B}_0(\xi) = \bar{A}_0 2^{-1}(-\xi)^{-\frac{1}{2}} \int_\xi^0 \bar{\psi}(v)(-v)^{-\frac{1}{2}}dv, \quad \xi < 0,$$

(4.43) $$\bar{A}_1(\xi) = \bar{A}_0[-\tfrac{1}{4}\bar{\psi}(\xi)\xi^{-1}+\tfrac{1}{8}\xi^{-3/2} \int_0^\xi \bar{\psi}(v)v^{-\frac{1}{2}}dv + \tfrac{1}{4} \int_0^\xi \bar{\psi}(\xi')\xi'^{-\frac{1}{2}}\{ \int_{\xi'}^0 \bar{\psi}(v)v^{-\frac{1}{2}}dv\}d\xi'], \quad \xi > 0,$$

$$\bar{A}_1(\xi) = \bar{A}_0[-\tfrac{1}{4}\bar{\psi}(\xi)(-\xi)^{-1}-\tfrac{1}{8}(-\xi)^{-3/2} \int_\xi^0 \bar{\psi}(v)(-v)^{-\frac{1}{2}}dv + \tfrac{1}{4} \int_0^\xi \bar{\psi}(\xi')(-\xi')^{-\frac{1}{2}}\{\int_{\xi'}^0 \bar{\psi}(v)(-v)^{-\frac{1}{2}}dv\}d\xi'], \xi < 0,$$

(4.44) $$\bar{\psi}(\xi(x)) = \frac{5}{16\xi^2(x)} + \xi(x)\,\frac{(3x^2-5)(x^2+1)}{16x^3(x^2-1)}.$$

The derivative $\frac{du}{dx}$ *is given by the derivative of* (4.38) *with respect to* x. *The error term becomes* $O(\lambda^{-7/6})$, $\lambda \to \infty$.

PROOF. For the proof we refer to OLVER ([13], Ch.12, Theorem 7.1 and section 7.4). □

The transformation $x \to \xi$ maps the interval $[p,q]$ onto $[\xi(p),\xi(q)]$, with $\xi(p) < 0$, $\xi(q) > 0$. In terms of the original functions we have

(4.45) $$\left(\frac{d\xi}{dx}(x)\right)^{\frac{1}{2}} = (\xi(x))^{\frac{1}{4}}(1-x^2)^{-\frac{1}{4}}, \quad x > 0,$$

$$\left(\frac{d\xi}{dx}(x)\right)^{\frac{1}{2}} = (-\xi(x))^{\frac{1}{4}}(1-x^2)^{-\frac{1}{4}}, \quad x < 0.$$

The approximating equation (see (4.18)) is

(4.46) $$\frac{d^2W}{d\xi^2} + \lambda\xi W = 0.$$

Independent solutions of (4.46) are $Ai(\lambda^{1/3}(-\xi))$ and $Bi(\lambda^{1/3}(-\xi))$. The function $W_{2,2}$ can be deleted, because of the matching condition between J_2 and J_3 (see Lemma 13). Both functions are oscillatory for $\xi > 0$, while $Ai(\lambda^{1/3}(-\xi))$ is exponentially decreasing and $Bi(\lambda^{1/3}(-\xi))$ exponentially increasing for $\xi < 0$. The constant $\bar{A}_0$ has to be determined by matching the solutions in J_1 and J_2. For $x \to 0$, the behaviour of $\xi(x)$ is given by

(4.47) $$\xi(x) \sim x, \quad x \to 0.$$

The following relations exist between the transformations $\zeta, \hat{\zeta}, \xi$ defined in (4.20), (4.30) and (4.39):

(4.48) $$(-\zeta(x))^{\frac{1}{2}} = L - \tfrac{2}{3}(\xi(x))^{3/2}, \qquad x > 0,$$

(4.49) $$(\hat{\zeta}(x))^{\frac{1}{2}} = L - \tfrac{2}{3}(-\xi(x))^{3/2}, \qquad x < 0,$$

where

(4.50) $$L = \int_0^1 \sqrt{\frac{t}{1-t^2}}\, dt.$$

In the sequel we shall need the values of some integrals; we list them in the lemma below. We recall the definition of the complete elliptic integrals E and F:

(4.51) $$E = E(\tfrac{\pi}{2},\tfrac{1}{2}\sqrt{2}) = \int_0^1 \sqrt{\frac{1-\frac{1}{2}x^2}{1-x^2}}\, dx,$$

(4.52) $$K = F(\tfrac{\pi}{2},\tfrac{1}{2}\sqrt{2}) = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}}\, dx.$$

LEMMA 9. *The following identies hold:*

(4.53) $$L = L_1 = \int_0^1 \sqrt{\frac{x}{1-x^2}}\, dx = 2^{3/2}(E-\tfrac{1}{2}K) = 2^{-\frac{1}{2}}\pi K^{-1},$$

(4.54) $$L_2 = \int_0^1 \sqrt{\frac{1}{x(1-x^2)}}\, dx = 2^{1/2}K,$$

(4.55) $$L_3 = \int_0^1 \sqrt{\frac{1+x}{x(1-x)}}\, dx = 2^{3/2}E,$$

(4.56) $$L_4 = \int_0^1 \sqrt{\frac{1-x}{x(1+x)}}\, dx = 2^{3/2}(K-E).$$

PROOF. The identies follow from BYRD & FRIEDMAN ([7], 235.06 & 318.02; 235.00; 235.05 & 315.02; 235.07 & 320.02). The second identity in (4.53) follows from the Legendre relation $2EK-K^2 = \pi/2$ (see [7], 110.10). □

LEMMA 10. *The integrals in Lemma 9 can also be expressed in terms of the constant* $c = \Gamma(\frac{1}{4})$:

$$L_1 = 2^{3/2}\pi^{3/2}c^{-2}, \tag{4.57}$$

$$L_2 = 2^{-3/2}\pi^{-1/2}c^2, \tag{4.58}$$

$$L_3 = 2^{3/2}\pi^{3/2}(c^{-2}-2^{-3}\pi^{-2}c^2), \tag{4.59}$$

$$L_4 = 2^{3/2}(2^{-3}\pi^{-1/2}c^2-\pi^{3/2}c^{-2}). \tag{4.60}$$

PROOF. From ABRAMOWITZ & STEGUN ([1], 17.3.9,17.3.10) or BYRD & FRIEDMAN ([13],118.02) we conclude that

$$E = \frac{\pi}{2}\ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1}{2}), \tag{4.61}$$

$$K = \frac{\pi}{2}\ {}_2F_1(-\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1}{2}). \tag{4.62}$$

If we evaluate E and K by means of [1] (15.1.24,15.1.25) we find

$$E = \pi^{3/2}(c^{-2}+2^{-2}\Gamma^{-2}(\tfrac{3}{4})) = \pi^{3/2}(c^{-2}+2^{-3}\pi^{-2}c^2), \tag{4.63}$$

$$K = 2^{-1}\pi^{3/2}\Gamma^{-2}(\tfrac{3}{4}) = 2^{-2}\pi^{-1/2}c^2. \tag{4.64}$$

Using the relation $\Gamma(\frac{3}{4}) = 2^{1/2}\pi c^{-1}$, i.e. the Reflection Formula for Γ-functions ([1], 6.1.17), we find (4.57),...,(4.60) and the second identities in (4.63), (4.64). □

In the sequel we shall also need the functions $B_0(\zeta)$, (4.22), and $\bar{B}_0(\xi)$, (4.42), for $\xi > 0$. Define

$$Q(r,s) = \int_r^s \frac{(3x^2-5)(x^2+1)}{32x^3(x^2-1)} \sqrt{\frac{x}{1-x^2}}\, dx, \quad 0 < r,\ s < 1. \tag{4.65}$$

Observe that the integrand becomes singular for $r \downarrow 0$, $s \uparrow 1$.

LEMMA 11. *The following relations hold:*

$$B_0(\zeta(x)) = \sqrt{2}[\tfrac{1}{8}(-\zeta(x))^{-1} + (-\zeta(x))^{-\frac{1}{2}}\lim_{\varepsilon_2\downarrow 0}\{Q(x,1-\varepsilon_2)-\frac{1}{8\sqrt{2}}\varepsilon_2^{-\frac{1}{2}}\}], \tag{4.66}$$

$$\bar{B}_0(\xi(x)) = \bar{A}_0[-\tfrac{5}{48}(\xi(x))^{-2}-(\xi(x))^{-\frac{1}{2}}\lim_{\varepsilon_1\downarrow 0}\{Q(\varepsilon_1,x) - \tfrac{5}{48}\varepsilon_1^{-3/2}\}]. \tag{4.67}$$

PROOF. Since $\lim_{\varepsilon_1 \downarrow 0} \zeta(1-\varepsilon_1)/(-2\varepsilon_1) = 1$ by (4.27), $B_0(\zeta)$ can be written as

$$B_0(\zeta(x)) = \lim_{\varepsilon_1 \downarrow 0} 2^{\frac{1}{2}}(-\zeta(x))^{\frac{1}{2}} \int_{\zeta(x)}^{\zeta(1-\varepsilon_1)/2} \psi(v)(-v)^{-\frac{1}{2}}dv. \tag{4.68}$$

Using (4.24) and performing the integration, we find

$$B_0(\zeta(x)) = \lim_{\varepsilon_1 \downarrow 0} [\tfrac{1}{8}(-\zeta(x))^{-1} - \tfrac{1}{8}(-\zeta(x))^{-\frac{1}{2}}(-\zeta(1-\varepsilon_1)/2)^{-\frac{1}{2}} + (-\zeta(x))^{-\frac{1}{2}}Q(x,1-\varepsilon_1)], \tag{4.69}$$

from which (4.66) follows. The proof for (4.67) proceeds along the same lines. □

LEMMA 12. *The following identity holds:*

$$\lim_{\varepsilon_1,\varepsilon_2 \downarrow 0} \{Q(\varepsilon_1,1-\varepsilon_2) - \frac{1}{8\sqrt{2}}\varepsilon_2^{-\frac{1}{2}} - \frac{5}{48}\varepsilon_1^{-3/2}\} = \frac{1}{32}[\tfrac{14}{3}L_2+L_3+L_4] = \frac{5}{24}\sqrt{2}K = \frac{5}{96}\sqrt{2}\pi^{-\frac{1}{2}}c^2, \tag{4.70}$$

where the L_i, $i = 2,3,4$, *are defined in Lemma* 9 *and* K *is defined in* (4.52).

PROOF. $Q(\varepsilon_1,1-\varepsilon_2)$ can be written as

$$Q(\varepsilon_1,1-\varepsilon_2) = \frac{1}{32}\int_{\varepsilon_1}^{1-\varepsilon_2}[3+\frac{5}{x^2}+\frac{2}{1+x}+\frac{2}{1-x}]\frac{1}{\sqrt{x(1-x)(1+x)}}dx. \tag{4.71}$$

Using [13] (230.03), we evaluate (4.71) in terms of the L_i, $i = 2,3,4$:

$$Q(\varepsilon_1,1-\varepsilon_2) = \frac{1}{32}[3L_2+\tfrac{5}{3}L_2+L_3+L_4+\tfrac{10}{3}\varepsilon_1^{-3/2}+2\sqrt{2}\varepsilon_2^{-\frac{1}{2}} + O(\varepsilon_1^{\frac{1}{2}})+O(\varepsilon_2^{\frac{1}{2}})], \quad \varepsilon_1,\varepsilon_2 \downarrow 0. \tag{4.72}$$

Inserting (4.72) into the left-hand side of (4.70) and taking the limits, we obtain the first identity. The second identity follows from (4.54), (4.55) and (4.56); the third identity from (4.64).□

The next step in the procedure for finding the asymptotic representations of the eigenfunctions consists of a matching of the three representations obtained above. We take arbitrary points in the intervals (-1,0) and (0,1) to match (4.29) with (4.40) and (4.41), and (4.19) with (4.40) and (4.41) respectively. The matching is performed by putting the Wronskian $\{u,v\} = uv'-u'v$ equal to zero and by using the asymptotic expansions of the Bessel and

Airy functions. Let $\kappa = x - \frac{\pi}{4}$, $\eta = \frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}$, $\mu = \frac{2}{3}x^{3/2}$, $\nu = \frac{2}{3}(-x)^{3/2}$. Then,

(4.73) $$J_0(x) = \sqrt{\frac{2}{\pi x}}\{\cos\kappa + \frac{1}{8x}\sin\kappa + O(x^{-2})\}, \quad x \to \infty,$$

(4.74) $$J_1(x) = -J_0'(x) = \sqrt{\frac{2}{\pi x}}\{\sin\kappa + \frac{3}{8x}\cos\kappa + O(x^{-1})\}, x \to \infty,$$

(4.75) $$J_1'(x) = \sqrt{\frac{2}{\pi x}}\{\cos\kappa - \frac{7}{8x}\sin\kappa + O(x^{-2})\}, \quad x \to \infty,$$

(4.76) $$I_0(x) = \frac{e^x}{\sqrt{2\pi x}}\{1 + \frac{1}{8x} + O(x^{-2})\}, \quad x \to \infty,$$

(4.77) $$I_1(x) = I_0'(x) = \frac{e^x}{\sqrt{2\pi x}}\{1 - \frac{3}{8x} + O(x^{-2})\}, \quad x \to \infty,$$

(4.78) $$I_1'(x) = \frac{e^x}{\sqrt{2\pi x}}\{1 - \frac{7}{8x} + O(x^{-2})\}, \quad x \to \infty,$$

(4.79) $$Ai(x) = 2^{-1}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}e^{-\mu}\{1 - \frac{5}{72}\mu^{-1} + O(\mu^{-2})\}, \quad x \to \infty,$$

(4.80) $$Ai(x) = \pi^{-\frac{1}{2}}(-x)^{-\frac{1}{4}}\{\sin\eta - \frac{5}{72}\nu^{-1}\cos\eta + O(\nu^{-2})\}, x \to -\infty,$$

(4.81) $$Ai'(x) = -2^{-1}\pi^{-\frac{1}{2}}x^{\frac{1}{4}}e^{-\mu}\{1 + \frac{7}{72}\mu^{-1} + O(\mu^{-2})\}, \quad x \to \infty,$$

(4.82) $$Ai'(x) = -\pi^{-\frac{1}{2}}(-x)^{\frac{1}{4}}\{\cos\eta - \frac{7}{72}\nu^{-1}\sin\eta + O(\nu^{-2})\}, x \to -\infty,$$

(4.83) $$Bi(x) = \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}e^{\mu}\{1 + \frac{5}{72}\mu^{-1} + O(\mu^{-2})\}, \quad x \to \infty$$

(4.84) $$Bi(x) = \pi^{-\frac{1}{2}}(-x)^{-\frac{1}{4}}\{\cos\eta - \frac{5}{72}\nu^{-1}\sin\eta + O(\nu^{-2})\}, x \to -\infty,$$

(4.85) $$Bi'(x) = \pi^{-\frac{1}{2}}x^{\frac{1}{4}}e^{\mu}\{1 - \frac{7}{72}\mu^{-1} + O(\mu^{-2})\}, \quad x \to \infty,$$

(4.86) $$Bi'(x) = \pi^{-\frac{1}{2}}(-x)^{\frac{1}{4}}\{\sin\eta - \frac{7}{72}\nu^{-1}\cos\eta + O(\nu^{-2})\}, x \to -\infty.$$

See ABRAMOWITZ & STEGUN ([1], Ch.9 & 10). Since the asymptotic expressions for $u(x)$ share the common factor $(x(1-x^2))^{-\frac{1}{4}}$, $x > 0$, or $((-x)(1-x^2))^{-\frac{1}{4}}$, $x < 0$, we omit this factor in the calculation of the Wronskian. It is also possible to differentiate all formulas with respect to ξ, using the relations (4.48) and (4.49), because the common factor $\frac{d\xi}{dx}$ does not influence the equation $\{u,v\} = 0$. The relevant representations are:

on J_1: $(-\zeta)^{-\frac{1}{4}}W_1$, see (4.21),
on J_2: $|\xi|^{\frac{1}{4}}\{\alpha W_{2,1}+\beta W_{2,2}\}$, see (4.40), (4.41),
on J_3: $\hat{\zeta}^{-\frac{1}{4}}W_3$, see (4.31).

LEMMA 13. *The matching of the representation* (4.29) *for* u *on* J_3 *with the representation* (4.38) *for* $\alpha u_1+\beta u_2$ *on* J_2, $x < 0$, *implies that* $\beta = 0$.

PROOF. Performing the calculations with the asymptotic expansions for the Bessel and Airy functions and omitting the common factors $2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}z^{-1}e^{z}|_{z=\lambda^{\frac{1}{2}}\hat{\zeta}^{\frac{1}{2}}}$ and $2^{-1}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}|_{z=\lambda^{1/3}(-\xi)}$ we obtain the leading term $-2\beta\zeta^{\frac{1}{4}}(-\xi)^{\frac{3}{4}}e^{\nu}\lambda^{\frac{1}{4}}$. This implies that $\beta = 0$. The remaining terms cancel out, which implies that the representation given for u on J_3 matches with that given for u_1 on J_2, $x < 0$. □

THEOREM 7. *The eigenvalue* λ_n *is asymptotically given by*

(4.87) $$\lambda_n = A(n+\tfrac{1}{2})^2+B+O(n^{-1}), \quad n \to \infty,$$

where

(4.88) $$A = L^{-2}\pi^2 = 2^{-3}\pi^{-1}c^4 (= 6.87518581),$$

(4.89) $$B = 2\delta L^{-1} = -5(96)^{-1}\pi^{-2}c^4 (= -0.91184984),$$

with $\delta = -\frac{5}{24}\sqrt{2}\,K, c = \Gamma(\frac{1}{4}) (= 3.62560991)$.

PROOF. Performing the calculations with the asymptotic expansions for the Bessel and Airy functions and omitting the common factors $2^{\frac{1}{2}}\pi^{-\frac{1}{2}}z|_{z=\lambda^{\frac{1}{2}}(-\zeta)^{\frac{1}{2}}}$ and $\pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}|_{z=\lambda^{1/3}(-\xi)}$, we obtain the leading term $(-\zeta)^{\frac{1}{4}}\xi^{\frac{3}{4}}\cos(\lambda^{\frac{1}{2}}L)\lambda^{\frac{1}{2}}$; the next term is $O(1)$. The first approximation for λ_n is therefore equal to $\lambda_n \sim L^{-2}\pi^2(n+\frac{1}{2})^2$, $n \to \infty$. Taking $\lambda^{\frac{1}{2}}L = (n+\frac{1}{2})\pi + \delta\lambda^{-\frac{1}{2}}$ as the second approximation, which implies $\cos(\lambda^{\frac{1}{2}}L) = (-1)^n \sin(-\delta\lambda^{-\frac{1}{2}}) = (-1)^{n+1}\delta\lambda^{-\frac{1}{2}}+O(\lambda^{-3/2})$, $\lambda \to \infty$, we calculate the second order term ($O(1)$) of the Wronskian. After some tedious calculations, using Lemmas 11 and 12, we find that this term equals

(4.90) $$(-1)^{n+1}(-\zeta)^{\frac{1}{2}}\xi^{\frac{3}{4}}[\delta+\tfrac{5}{24}\sqrt{2}\,K],$$

from which we conclude that $\delta = -\frac{5}{24}\sqrt{2}K$. After some manipulation of these results we finally find (4.87). □

REMARK. BETHE, ROSE & SMITH [3] gave the result $\lambda_{\pm n} \sim \pm\, 6.88(n+\frac{1}{2})^2$ without a further specification of the constant.

REMARK. The result (4.87) refines a statement in BIRMAN & SOLOMYAK ([4], formula (16)), from which a general formula for only the coefficient A for this type of eigenvalue problems can be calculated.

REMARK. The eigenvalue problem (3.1), (3.2) is the most simple example of an equation with one turning point and two regular signularities. The full asymptotic behaviour of the eigenvalues for such cases is still an area of research, see [4].

Having found the asymptotic expression for the eigenvalues λ_n, we can now give the full representation of the eigenfunctions u_n by determining the constants $\hat{A}_0$ and $\bar{A}_0$.

THEOREM 8. *The asymptotic representation of the eigenfunction* $u_n(x)$ *of* (3.1),(3.2) *is given by* (4.19) *for* $x \in J_1 = [q,1]$, *by* (4.38), $i = 1$ *for* $x \in J_2 = [p,q]$ *and by* (4.29) *for* $x \in J_3 = [-1,p]$, *with* p,q *arbitrary*, $-1 < p < 0$, $0 < q < 1$, *and where* $\lambda = \lambda_n$ *is given by* (4.87) *and*

$$\bar{A}_{0,n} = (-1)^n 2^{\frac{1}{2}} \lambda_n^{-1/6}(1+o(n^{-1/3})), \quad n \to \infty, \tag{4.91}$$

$$\hat{A}_{0,n} = (-1)^n 2^{\frac{1}{2}} e^{-\lambda_n^{\frac{1}{2}} L}(1+o(e^{-nL})), \quad n \to \infty. \tag{4.92}$$

Asymptotic representations of the eigenfunctions u_n *and eigenvalues* λ_n *with* $n < 0$ *follow from the symmetry relations* $u_{-n}(x) = u_n(-x)$, $\lambda_{-n} = -\lambda_n$.

PROOF. The leading term of (4.19) equals

$$u(x) = x^{-\frac{1}{4}}(1-x^2)^{-\frac{1}{4}} 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \lambda_n^{-\frac{1}{4}} \cos(\lambda_n^{\frac{1}{2}}(-\zeta(x))^{\frac{1}{2}} - \tfrac{\pi}{4})(1+o(\lambda_n^{-\frac{1}{4}})), \quad n \to \infty, \tag{4.93}$$

by virtue of (4.73), while the leading term of (4.38), $i = 1$, equals

$$u_1(x) = \bar{A}_0 x^{-\frac{1}{4}}(1-x^2)^{-\frac{1}{4}} \pi^{-\frac{1}{2}} \lambda_n^{-\frac{1}{4}} \sin(\tfrac{2}{3}\lambda_n^{\frac{1}{2}}(\xi(x))^{3/2} + \tfrac{\pi}{4}) \cdot (1+o(\lambda_n^{-1/12})), \quad n \to \infty, \tag{4.94}$$

by virtue of (4.80). The identity (4.48) and the relation $\lambda_n^{\frac{1}{2}} L = (n+\frac{1}{2})\pi + o(\lambda_n^{-\frac{1}{2}})$, $n \to \infty$, imply the result (4.91). The relation (4.92) is found in a similar way. □

5. ASYMPTOTIC EXPRESSION FOR σ_n

In this section we give asymptotic representations of the inner products $(1,u_n)$ and (xu_n,u_n), the norm $\|\chi_n\|_K^2$ and the weight $\sigma_n = \|\chi_n\|_K^{-2}$.

We shall confine ourselves to first-order approximations, just as we have confined ourselves to first-order approximations for the constants $\bar{A}_0$ and $\bar{A}_0$ (see (4.91) and (4.92)). We therefore consider only the first term of each of the expansions for u_n in J_i, $i = 1,2,3$. The first inner product is evaluated by the theory summarized in BLEISTEIN & HANDELSMAN [5]. These authors treat integrals of the form

$$(5.1) \qquad I(\lambda) = \int_a^b f(x)h(\lambda\phi(x))dx, \qquad \lambda \to \infty.$$

Under some restrictions on ϕ and the kernel h, asymptotic expansions are constructed which use the Mellin transform of h. The treatment depends on whether h is oscillatory or monotone (exponentially increasing or decreasing). We shall encounter both cases. Since the character of the eigenfunction u_n is different on each domain, it is not possible to handle the three integrals in a uniform manner. Therefore it is necessary to treat each domain in its own specific way.

LEMMA 14. *The following relation holds:*

$$(5.2) \qquad (1,u_n) = \int_{-1}^{1} u_n(x)dx = (-1)^n 2^{\frac{1}{2}}|\lambda_n|^{-\frac{1}{2}}(1+o(|\lambda_n|^{-\frac{1}{2}})), \qquad |\lambda_n| \to \infty.$$

PROOF. Denote the contributions of the three domains J_i by I_i, $i = 1,2,3$. The endpoints of the domain J_2: p,q, $-1 < p < 0$, $0 < q < 1$, are arbitrary. According to the theory in [5], it is necessary to treat the neighbourhood of the upper endpoint of the domain of integration separately from that of the lower endpoint by the technique of neutralization. We denote the contributions from these neighbourhoods by $I_i^{\pm}$, $i = 1,2,3$, where the plus-sign refers to the upper, the minus-sign to the lower endpoint. Since we restrict ourselves to the first order, we have for $n > 0$

$$(5.3) \qquad I_1 = \int_q^1 x^{-\frac{1}{4}}(1-x^2)^{-\frac{1}{4}}(-\zeta(x))^{\frac{1}{4}}J_0(\lambda_n^{\frac{1}{2}}(-\zeta(x))^{\frac{1}{2}})dx.$$

In the notation of [5], $f(x) = x^{-\frac{1}{4}}(1-x^2)^{-\frac{1}{4}}(-\zeta(x))^{\frac{1}{4}}$, $h = J_0$, $\phi = (-\zeta)^{\frac{1}{2}}$, $\lambda = \lambda_n^{\frac{1}{2}}$. Performing the calculations, we find

$$(5.4) \qquad I_1^{+} = O(\lambda_n^{-1}), \quad \lambda_n \to \infty, \quad ([5], 6.3.34),$$

(5.5) $$I_1^- = 2^{\frac{1}{2}}\pi^{\frac{1}{2}}q^{-\frac{3}{4}}(1-q^2)^{-\frac{1}{4}}\cos(\lambda_n^{\frac{1}{2}}(-\zeta(q))^{\frac{1}{2}} - \frac{3\pi}{4})\lambda_n^{-\frac{3}{4}}(1+o(\lambda_n^{-\frac{3}{4}})),$$
$$\lambda_n \to \infty, \quad ([5],\ 6.3.28).$$

On J_2 the integral becomes for $n > 0$

(5.6) $$I_2 = \bar{A}_0 \int_p^q x^{-\frac{1}{4}}(1-x^2)^{-\frac{1}{4}}|\xi(x)|^{\frac{1}{4}}\ Ai(\lambda_n^{1/3}(-\xi(x)))dx.$$

Since $\phi(x) = \xi(x)$, and ϕ becomes zero for $x = 0$, it is necessary to treat the contribution from the integrand around $x = 0$ separately. Therefore we split I_2 into integrals over $(p,0)$ and $(0,q)$, denoting these integrals by $I_{2,-}$ and $I_{2,+}$ respectively. Performing the calculations, we find

(5.7) $$I_{2,+}^+ = 2^{\frac{1}{2}}\pi^{\frac{1}{2}}q^{-\frac{3}{4}}(1-q^2)^{-\frac{1}{4}}\sin(\lambda_n^{\frac{1}{2}}\ \frac{2}{3}(\xi(q))^{3/2} - \frac{\pi}{4})\lambda_n^{-\frac{3}{4}}(1+o(\lambda_n^{-\frac{3}{4}})),$$
$$\lambda_n \to \infty, \quad ([5],\ 6.3.14),$$

(5.8) $$I_{2,+}^- = (-1)^n 2^{3/2}3^{-1}\lambda_n^{-\frac{1}{2}}(1+o(\lambda_n^{-\frac{1}{2}})),\ \lambda_n \to \infty\ ([5],\ 6.3.11),$$

(5.9) $$I_{2,-}^+ = (-1)^n 2^{\frac{1}{2}}3^{-1}\lambda_n^{-\frac{1}{2}}(1+o(\lambda_n^{-\frac{1}{2}})), \quad \lambda_n \to \infty, ([5],\ 5.3.5),$$

(5.10) $$I_{2,-}^- = o(\lambda_n^{-s}),\ \lambda_n \to \infty, \text{ for every } s > 0,\ ([5],\ 5.2.11).$$

Finally, it is easily seen that $I_3 = O(e^{-\lambda_n^{\frac{1}{2}}L})$, $\lambda_n \to \infty$, so summing up all contributions

(5.11) $$I = (-1)^n 2^{\frac{1}{2}}\lambda_n^{-\frac{1}{2}}(1+o(\lambda_n^{-\frac{1}{2}})), \quad \lambda_n \to \infty.$$

Note that the contributions (5.5) and (5.7) cancel, because of (4.48). In view of the symmetry relation $u_{-n}(x) = u_n(-x)$, the result (5.2) follows. □

LEMMA 15. *The following relation holds:*

(5.12) $$(xu_n,u_n) = \int_{-1}^1 xu_n^2(x)dx = L\pi^{-1}|\lambda_n|^{-\frac{1}{2}}(1+o(|\lambda_n|^{-\frac{1}{2}})), |\lambda_n| \to \infty.$$

PROOF. Denote the contributions of the three domains J_i by M_i, $i = 1,2,3$. The remaining notation is the same as in Lemma 14. Since we restrict ourselves to the first order, we have for $n > 0$ by the transformation $t(x) = \lambda_n^{\frac{1}{2}}(-\zeta(x))^{\frac{1}{2}}$

(5.13) $$M_1 = \int_q^1 x^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}(-\zeta(x))^{\frac{1}{2}}J_0^2(\lambda_n^{\frac{1}{2}}(-\zeta(x))^{\frac{1}{2}})dx$$
$$= \lambda_n^{-1} \int_0^{t(q)} tJ_0^2(t)dt$$
$$= \lambda_n^{-1}[\tfrac{1}{2}t^2\{J_0^2(t)+J_1^2(t)\}]\Big|_{t=0}^{t=t(q)}$$

$$= \pi^{-1}(-\zeta(q))^{\frac{1}{2}}\lambda_n^{-\frac{1}{2}}(1+o(\lambda_n^{-\frac{1}{2}})), \quad \lambda_n \to \infty,$$

by the asymptotic relations (4.73), (4.74) for J_0, J_1. Further, for $n > 0$, by (4.91) and the transformation $s(x) = \lambda_n^{1/3}\xi(x)$

(5.14) $$M_{2,+} = \bar{A}_0^2 \int_0^q x^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}(\xi(x))^{\frac{1}{2}}Ai^2(\lambda_n^{1/3}(-\xi(x)))dx$$

$$= 2\lambda_n^{-1} \int_0^{s(q)} s\ Ai^2(-s)ds.$$

Now we use the relation $Ai(-s) = \frac{1}{3}\ s^{\frac{1}{2}}\{J_{1/3}(w)+J_{-1/3}(w)\}$, $w = \frac{2}{3}\ s^{3/2}$([1], 10.4.15). Then relation (5.14) becomes

(5.15) $$M_{2,+} = 3^{-1}\lambda_n^{-1} \int_0^{w(q)} w\{J_{1/3}^2(w)+2J_{1/3}(\bar{w})J_{-1/3}(w)+J_{-1/3}^2(w)\}dw.$$

An explicit expression for this integral follows from [1] (11.4.2, 11.3.31):

(5.16) $$M_{2,+} = 3^{-1}\lambda_n^{-1}\{\frac{w^2}{2}[J_{1/3}^2(w)-J_{-2/3}(w)J_{4/3}(w)+$$

$$+\ 2J_{1/3}(w)J_{-1/3}(w)+2J_{4/3}(w)J_{2/3}(w)+J_{-1/3}^2(w)-$$

$$J_{-1/3}^2(w)-J_{-4/3}(w)J_{2/3}(w)]\}\Big|_{w=0}^{w=w(q)}.$$

Finally, we use the asymptotic relation ([1], 9.2.1)

(5.17) $$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\ \cos(x-\tfrac{1}{2}\nu\pi-\ \tfrac{\pi}{4})(1+o(x^{-1})), \quad x \to \infty,$$

to obtain the expression

(5.18) $$M_{2,+} = \pi^{-1}\ \tfrac{2}{3}(\xi(q))^{3/2}\lambda_n^{-\frac{1}{2}}(1+o(\lambda_n^{-\frac{1}{2}})), \quad \lambda_n \to \infty.$$

Furthermore, for $n > 0$, by (4.91), and the transformations $r(x) = \lambda_n^{1/3}(-\xi(x))$, $v = \frac{2}{3}\ r^{3/2}$

(5.19) $$M_{2,-} = \bar{A}_0^2 \int_p^0 x^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}(-\xi(x))^{\frac{1}{2}}Ai^2(\lambda_n^{1/3}(-\xi(x)))dx$$

$$= -2\lambda_n^{-1} \int_0^{r(p)} r\ Ai^2(r)dr$$

$$= -\lambda_n^{-1}\pi^{-2} \int_0^{v(p)} vK_{1/3}^2(v)dv$$

$$= O(\lambda_n^{-1}), \quad \lambda_n \to \infty,$$

by the relation $Ai(r) = \pi^{-1}3^{-\frac{1}{2}}r^{\frac{1}{2}}K_{1/3}(\frac{2}{3}r^{3/2})$, $r > 0$ ([1], 10.4.14). Finally

(5.20) $$M_3 = O(e^{-\lambda_n^{\frac{1}{2}}L}), \quad \lambda_n \to \infty.$$

The last relation is proved with the same type of transformations as for (5.13), working with the Bessel functions I_0, I_1 instead of J_0, J_1. Summing up all contributions we find

(5.21) $$M = L\pi^{-1}\lambda_n^{-\frac{1}{2}}(1+o(\lambda_n^{-\frac{1}{2}})), \quad \lambda_n \to \infty.$$

Finally, (5.12) follows from the usual symmetry relation. □

THEOREM 9. *The following asymptotic relation holds for the weights* $\sigma_n = \|\chi_n\|_K^{-2}$:

(5.22) $$\sigma_n = 2L^{-1}\pi|\lambda_n|^{-3/2}(1+o(|\lambda_n|^{-3/2})), \quad |n| \to \infty,$$
$$= 2L^2\pi^{-2}(|n|+\tfrac{1}{2})^{-3}(1+o(|n|^{-3})), \quad |n| \to \infty,$$

where L is defined in (4.53).

PROOF. Since $\sigma_n = \|\chi_n\|_K^{-2}$, the first relation follows from (3.5), (3.37), and the Lemmas 14 and 15, and the second one from (4.87). □

6. COMPARISON OF ASYMPTOTIC AND NUMERICAL RESULTS

In this section we compare the asymptotic formulas (4.87), (5.2) and (5.13) with the results of numerical calculations. Using the procedures F01AEF and F01AFF of the NAG-library (Numerical Algorithms Group, Oxford) for generalized eigenvalue problems of the form $Ax = \mu Bx$, where A is a real matrix and B a real symmetrix positive - definite matrix, we calculated the coefficients $a_{k,n}$ (see (3.8)) and the eigenvalues $\lambda_n = \mu_n^{-1}$. Table 1 gives the eigenvalues calculated from the asymptotic expression (4.87),

(6.1) $$\lambda_n^{asym} = A(n+\tfrac{1}{2})^2 + B,$$

(6.2) $$A = 6.87518581, \quad B = -0.91184984,$$

and the calculated eigenvalues λ_n^{num}, for n = 1,2,...,33. The numerical calculations were based on a 100-dimensional matrix approximation, which yields λ_n and λ_{-n} for n = 1,2,...,50. Because the error in the calculated eigenvalues and coefficients grows with increasing index n for this matrix approximation, we compare λ_n only for n = 1,2,...,33. In Table 2 we compare the asymptotic expression for $(1,u_n)$,

(6.3) $$(1,u_n)^{asym} = (-1)^n\, 2^{\frac{1}{2}}|\lambda_n^{asym}|^{-\frac{1}{2}},$$

with the calculated expression $(1,u_n)^{num} = -\frac{4}{5} a_{2,n}$, see (3.35) and (3.37), for n = 1,...,15. In Table 3 we compare the asymptotic expression for (xu_n,u_n),

(6.4) $$(xu_n,u_n)^{asym} = L\pi^{-1}|\lambda_n^{asym}|^{-\frac{1}{2}},$$

(6.5) $$L\pi^{-1} = 2^{3/2}\pi^{1/2}c^{-2} (= 0.38137988),$$

with the expression

(6.6) $$(xu_n,u_n)^{num} = \sum_{k=2}^{50} \frac{4(k+1)}{(2k+1)(2k+3)} a_{k,n}\, a_{k+1,n},$$

which is an approximation of (xu_n,u_n), see (3.5) and (3.32), for n = 1,...,15. The calculations for Table 2 and 3 were based on a 50-dimensional matrix approximation; we list only the first 15 entries.

Figure 1 gives the graphs of the eigenfunctions u_n, n = 1,2,...,5, based on the results of the calculations for Table 2 and 3. Notice that the oscillatory behaviour is in agreement with a theorem of KWONG (see [12], Theorem 5.3): u_n (n>0) has precisely n zeros, and all zeros lie in the interval (0,1).

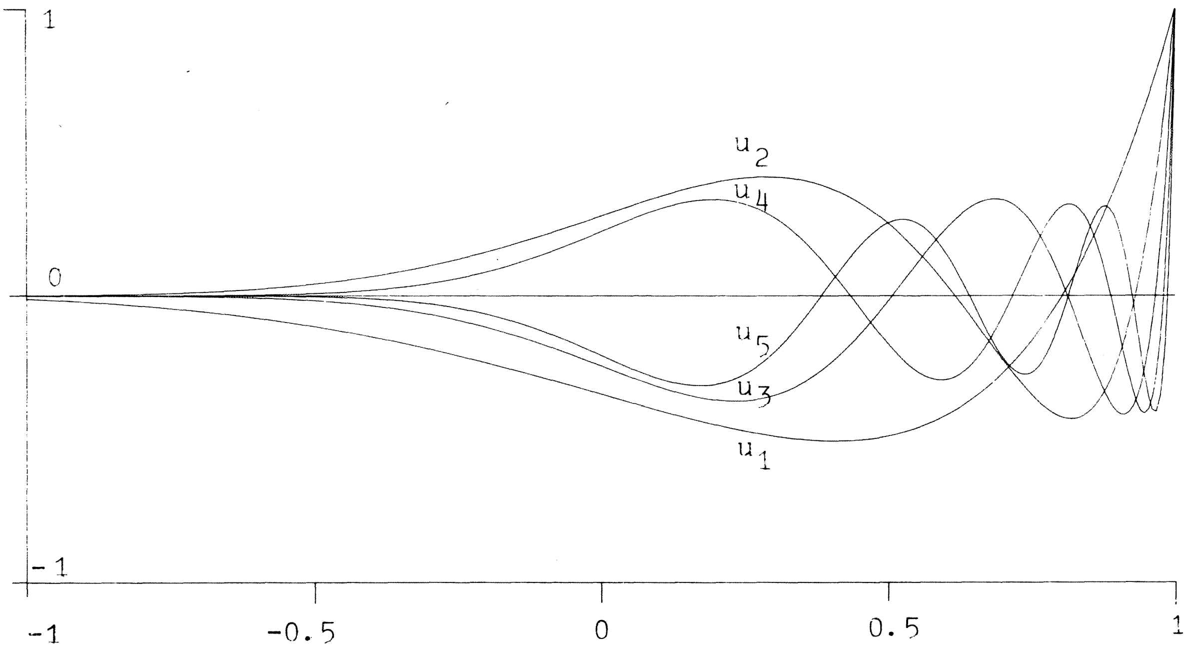

Figure 1. The eigenfunctions $u_n(x)$, n = 1,2,...,5

n	λ_n^{asym}	λ_n^{num}	n	λ_n^{asym}	λ_n^{num}
1	14.55732	14.52800	18	2352.12049	2352.12033
2	42.05806	42.04855	19	2613.37756	2613.37741
3	83.30918	83.30444	20	2888.38499	2888.38486
4	138.31066	138.30782	21	3177.14279	3177.14267
5	207.06252	207.06063	22	3479.65097	3479.65086
6	289.56475	289.56334	23	3795.90951	3795.90942
7	385.81735	385.81634	24	4125.91843	4125.91834
8	495.82033	495.81954	25	4469.67772	4469.67764
9	619.57367	619.57304	26	4827.18739	4827.18731
10	757.07739	757.07687	27	5198.44742	5198.44735
11	908.33147	908.33105	28	5583.45783	5583.45776
12	1073.33593	1073.33557	29	5982.21860	5982.21854
13	1252.09076	1252.09045	30	6394.72975	6394.72970
14	1444.59597	1444.59570	31	6820.99127	6820.99122
15	1650.85154	1650.85131	32	7261.00316	7261.00312
16	1870.85749	1870.85728	33	7714.76543	7714.76539
17	2104.61381	2104.61362			

Table 1.

n	$(1,u_n)^{asym}$	$(1,u_n)^{num}$	n	$(1,u_n)^{asym}$	$(1,u_n)^{num}$
1	-0.3707	-0.3710	9	-0.05682	-0.05682
2	0.2181	0.2181	10	0.05140	0.05140
3	-0.1549	-0.1550	11	-0.04692	-0.04692
4	0.1203	0.1202	12	0.04317	0.04317
5	-0.09828	-0.09832	13	-0.03997	-0.03997
6	0.08311	0.08312	14	0.03721	0.03721
7	-0.07200	-0.07200	15	-0.03481	-0.03481
8	0.06351	0.06351			

Table 2.

n	$(xu_n,u_n)^{asym}$	$(xu_n,u_n)^{num}$	n	$(xu_n,u_n)^{asym}$	$(xu_n,u_n)^{num}$
1	0.09996	0.09675	9	0.01532	0.01531
2	0.05881	0.05817	10	0.013861	0.013852
3	0.04178	0.04155	11	0.012654	0.012649
4	0.03243	0.03232	12	0.011641	0.011636
5	0.02650	0.02645	13	0.010778	0.010774
6	0.02241	0.02238	14	0.010034	0.010031
7	0.01942	0.01939	15	0.009387	0.009385
8	0.01713	0.01711			

Table 3.

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