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STABILITY IN LINEAR MULTISTEP METHODS FOR PURE DELAY EQUATIONS

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Stability in linear multistep methods for pure delay equations *)

by

P.J. van der Houwen & B.P. Sommeijer

ABSTRACT

The stability regions of linear multistep methods for pure delay equations are compared with the stability region of the delay equation itself. A criterion is derived stating when the numerical stability region contains the analytical stability region. This criterion yields an upper bound for the integration step (conditional Q-stability). These bounds are computed for the Adams-Bashforth, Adams-Moulton and backward differentiation methods of orders ≤ 8. Furthermore, symmetric Adams methods are considered which are shown to be unconditionally Q-stable. Finally, the extended backward differentiation methods of Cash are analysed.

KEY WORDS & PHRASES: *Numerical analysis, delay equations, linear multistep methods,* Q-*stability*

*) This report will be submitted for publication.

1. INTRODUCTION

Consider the retarded differential equation

$$(1.1) \qquad y'(t) = f(t,y(t-\omega)), \quad \omega = \omega(t,y(t)) \geq 0.$$

Following CRYER [4], WIEDERHOLT [8] and BARWELL [2] we adopt the scalar equation

$$(1.2) \qquad y'(t) = \lambda\, y(t-\omega), \quad \lambda \text{ and } \omega \text{ constant}$$

as the stability test equation. A linear multistep (LM) method $\{\rho,\sigma\}$ with

$$(1.3) \qquad \rho(\zeta) = \sum_{j=0}^{k} a_j\, \zeta^{k-j}, \quad \sigma(\zeta) = \sum_{j=0}^{k} b_j\, \zeta^{k-j},$$

when applied to (1.1) (cf. TAVERNINI [7]), reduces for (1.2) to the relation

$$(1.4) \qquad \rho(E)y_n - \lambda\Delta t\, \sigma(E)y_{n-\nu} = 0, \quad \nu := \frac{\omega}{\Delta t},$$

where E is the forward shift operator and where we will assume that ν is a (positive) integer. To (1.4) we can associate the characteristic equation

$$(1.5) \qquad \rho(\zeta) - z\,\zeta^{-\nu}\sigma(\zeta) = 0, \quad z := \lambda\Delta t.$$

In analogy with the stability theory for ODEs, one may define (in the complex z-plane) the stability region of the LM method $\{\rho,\sigma\}$ by the boundary locus curve

$$(1.6) \qquad z = e^{i\nu\psi}\, \frac{\rho}{\sigma}\,(e^{i\psi}), \quad -\pi \leq \psi \leq \pi.$$

Following Barwell we compare this region with the stability region of (1.2) which can be defined by the analytical boundary locus curve

$$(1.7) \qquad z = \frac{1}{\nu}\, e^{i\phi}.\begin{cases} \phi - \pi/2 & \text{for } \pi/2 \leq \phi \leq \pi \\ 3\pi/2 - \phi & \text{for } \pi \leq \phi \leq 3\pi/2. \end{cases}$$

BARWELL [2] called the LM method $\{\rho,\sigma\}$ *Q-stable* if the numerical stability region contains the analytical stability region for all $\nu \geq 1$ (see figure 1.1).

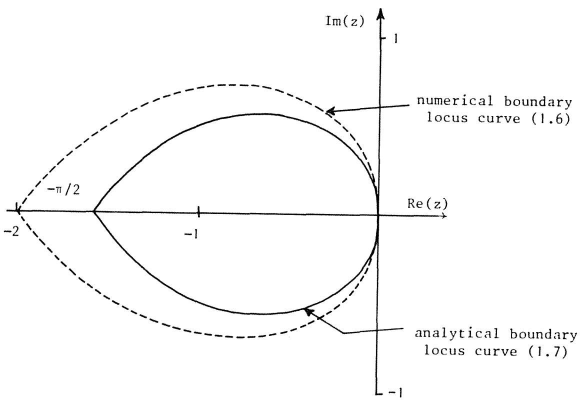

Figure 1.1. Analytical and numerical stability region

Q-stability generalizes the concept of DA_0-*stability* introduced by CRYER [4], which requires that the interval $[-\pi/2\nu,0]$ is contained into the numerical stability region for all $\nu \geq 1$. Evidently, Q-stability implies DA_0-stability.

Cryer showed that the backward Euler rule and the trapezoidal rule are DA_0-stable, Barwell proved Q-stability for backward Euler, and ALLEN [1] proved Q-stability for the trapezoidal rule and the 4-th order centred difference rule defined by

$$(1.8) \qquad \rho(\zeta) = \zeta^2 - \zeta\ , \quad \sigma(\zeta) = \frac{1}{24}[-\zeta^3 + 13\,\zeta^2 + 13\,\zeta - 1].$$

In this paper we investigate a form of stability which is a slight modification of DA_0 and Q-stability.

DEFINITION 1.1 The LM method will be called $Q_0(r)$-*stable* if there exists an $r \in \mathbb{Z}_+$ such that the interval $(-\pi/2\nu,0)$ is contained in the numerical stability region for all $\nu \geq r$. It is called $Q(r)$-*stable* if the numerical stability region contains the analytical stability region for all $\nu \geq r$.

Evidently, $Q_0(1)$ and $Q(1)$-stability are equivalent with DA_0 and Q-stability, respectively. These modified definitions are justified by the fact that many methods are *not* DA_0 or Q-stable but can be proved to be $Q_0(r)$-or $Q(r)$-stable for some $r > 1$. Thus, for $\nu \geq r$, i.e. $\Delta t \leq \frac{\omega}{r}$ it is guaranteed that the numerical stability region encloses the analytical stability region.

2. A NECESSARY CONDITION

We start with the derivation of necessary conditions for $Q_0(r)$- and $Q(r)$-stability by considering the boundary locus curve in the neighbourhood of the origin. In this derivation it is convenient to introduce the functions

$$R(\psi) := \begin{cases} \dfrac{m(\psi)}{\psi} - 1 & \text{for } \psi > 0 \\[2ex] -\dfrac{m(\psi)}{\psi} - 1 & \text{for } \psi < 0 \end{cases}$$

(2.1a)

$$\alpha(\psi) := \begin{cases} \theta(\psi) - \dfrac{\pi}{2} & \text{for } \psi > 0 \\[2ex] \theta(\psi) + \dfrac{\pi}{2} & \text{for } \psi < 0 \end{cases}$$

where $m(\psi)$ and $\theta(\psi)$ are the modulus and argument of the complex function $\rho(e^{i\psi})/\sigma(e^{i\psi})$, respectively. It will be convenient to write this function in the form

$$\text{(2.1b)} \qquad \frac{\rho}{\sigma}(e^{i\psi}) = \psi\,(1+R(\psi))e^{i(\frac{\pi}{2}+\alpha(\psi))}, \quad |\psi| \leq \pi.$$

It will turn out that $R(\psi)$ and $\alpha(\psi)$ vanish as $|\psi| \to 0$. Furthermore, we notice that $R(\psi) \geq -1$.

By deriving the first terms in the Taylor expansions of $R(\psi)$ and $\alpha(\psi)$ about $\psi = 0$ and by eliminating ψ, a relation between the argument and the modulus of the points on the boundary locus is obtained which can be compared with (1.7). In this relation the error constants occurring in the truncation error of the LM method play an important role. These constants are defined by the equation

$$\text{(2.2)} \qquad [\rho(E) - \sigma(E)\,\Delta t \frac{d}{dt}]\,y(t) = \left[C_{p+1}\left(\Delta t \frac{d}{dt}\right)^{p+1} + C_{p+2}\left(\Delta t \frac{d}{dt}\right)^{p+2} + \ldots\right] y(t)$$

where p is the order of the LM method and $y(t)$ denotes a sufficiently differentiable function. We always assume $p \geq 1$ and $\sigma(1) \neq 0$.

THEOREM. 2.1 (a) *A* $Q_0(r)$-*stable* LM *method has a* ρ-*polynomial which satisfies the strong root condition (i.e., all zeros are within the unit circle except for a simple root* $\zeta = 1$).
(b) *A* $Q(r)$-*stable* LM *method satisfies, in addition, the inequality*

$$(2.3)\qquad (-1)^{\lfloor \frac{p+3}{2} \rfloor} \sigma(1)\, C_{p+1} < 0, \quad \lfloor x \rfloor \textit{ denotes the integer part of } x.$$

(c) *A* $Q(r)$-*stable* LM *method of even order satisfies*

$$(2.4)\qquad r > \frac{C_{p+2}}{C_{p+1}} - \frac{\sigma'(1)}{\sigma(1)}.$$

PROOF. (a) If the LM method is $Q_0(r)$-stable the characteristic equation (1.5) necessarily has a non-empty negative interval of stability for $\nu \geq r$. For $\nu = 0$ we know from the stability theory for ODEs that then ρ should satisfy the strong root condition. For $\nu \geq r$ the proof can be given along the same lines.
(b) Evidently, a $Q(r)$-stable method is also $Q_0(r)$-stable. In addition however, if $(\phi_1, |z|)$ and $(\phi_2, |z|)$ are two points in the upper halfplane on the numerical and analytical boundary curves, respectively, we necessarily have that the arguments satisfy $\phi_1 < \phi_2$ as the modulus $|z| \to 0$. From (1.6), (1.7) and (2.1) it follows that this condition can be written as

$$(2.5)\qquad \left.\begin{array}{l} 0 < \nu\, \psi_1 + \frac{\pi}{2} + \alpha(\psi_1) < \frac{\pi}{2} + \nu\, |z| \\ \\ |\psi_1\, (1+R(\psi_1))\, | = |\, z\, | \end{array}\right. \qquad \text{as } |z| \to 0.$$

In order to find the behaviour of $\alpha(\psi)$ and $R(\psi)$ as $\psi \to 0$, we use the relation (cf.(2.2))

$$\rho(e^z) - z\sigma(e^z) = C_{p+1}\, z^{p+1} + C_{p+2}\, z^{p+2} + \ldots \quad \text{as } |z| \to 0$$

from which we derive the expansion

$$\frac{\rho}{\sigma}(e^z) = z + \frac{C_{p+1}}{\sigma(1)} z^{p+1} + \left(\frac{C_{p+2}}{\sigma(1)} - \frac{\sigma'(1)\, C_{p+1}}{\sigma^2(1)}\right) z^{p+2} + \ldots\, .$$

Hence,

$$\frac{\rho}{\sigma}(e^{i\psi}) = i\psi + i^{p+1}\frac{C_{p+1}}{\sigma(1)}\psi^{p+1} + i^{p+2}\left(\frac{C_{p+2}}{\sigma(1)} - \frac{\sigma'(1)C_{p+1}}{\sigma^2(1)}\right)\psi^{p+2} + \dots .$$

If p is *odd* we write

$$\frac{\rho}{\sigma}(e^{i\psi}) = (-1)^{\frac{p+1}{2}} c_1\psi^{p+1} + i\psi\left[1 + (-1)^{\frac{p+1}{2}} c_2\psi^{p+1}\right] + O(\psi^{p+3}),$$

where

$$c_1 := \frac{C_{p+1}}{\sigma(1)}, \quad c_2 := \frac{C_{p+2}}{\sigma(1)} - \frac{\sigma'(1)C_{p+1}}{\sigma^2(1)} .$$

Comparison with (2.1) yields

(2.6a) $\quad R(\psi) \approx (-1)^{\frac{p+1}{2}} c_2\,\psi^{p+1} + \tfrac{1}{2}c_1^2\,\psi^{2p}, \quad \alpha(\psi) \approx (-1)^{\frac{p-1}{2}} c_1\psi^{p}$ as $\psi \to 0$

and substitution into (2.5) yields the necessary condition for Q(r)-stability (2.3) for odd values of p.

If p is *even* we write

$$\frac{\rho}{\sigma}(e^{i\psi}) = (-1)^{\frac{p+2}{2}} c_2\psi^{p+2} + i\psi\left[1 + (-1)^{\frac{p}{2}} c_1\psi^{p}\right] + O(\psi^{p+3})$$

from which we derive

(2.6b) $\quad R(\psi) \approx (-1)^{p/2} c_1\,\psi^{p}, \quad \alpha(\psi) \approx (-1)^{p/2} c_2\,\psi^{p+1}$ as $\psi \to 0$.

Substitution into (2.5) leads to the condition

(2.7) $\quad -(-1)^{p/2}\sigma(1)\,C_{p+1}\,\nu < (-1)^{p/2}\left[\sigma'(1)\,C_{p+1} - \sigma(1)C_{p+2}\right].$

Thus, Q(r)-stability is only possible if the coefficient of ν is negative. This results into the necessary condition (2.3).

(c) Furthermore, it follows from (2.7) that r satisfies the inequality (2.4) □

The condition (2.3) will be called the *$Q(r)$-stability condition* . In the next section it will be shown that (2.3) is also necessary for $Q_0(r)$-stability. By means of this necessary condition a large number of LM methods can already be dropped as possible $Q(r)$-stable methods. We also note that the strong root condition to be imposed on $\rho(\zeta)$ excludes methods such as the (explicit) Nyström methods and (implicit) Milne-Simpson methods which have $\rho(\zeta) = \zeta^k - \zeta^{k-2}$ with roots at $\zeta = \pm 1$.

2.1. Adams - Bashforth methods

These methods have a *positive* error constant C_{p+1} and since in all Adams methods $\rho(\zeta) = \zeta^k - \zeta^{k-1}$, they satisfy the strong root condition and have

$$\sigma(1) = \rho'(1) = 1, \quad \sigma'(1) = \frac{\rho'(1) + \rho''(1)}{2} = k - \tfrac{1}{2}, \qquad k \geq 2.$$

It follows from theorem 2.1 that the Adams-Bashforth methods of orders $p = 1, 2; 5, 6; 9, 10, \ldots$ cannot be $Q(r)$-stable. The methods of orders $p = 3, 4; 7, 8; \ldots$ will be proved in section 4 to be $Q(r)$-stable for some finite value of r. For $p = 4, 8, 12, \ldots$ we already conclude from (2.4) that r should satisfy

$$(2.8) \qquad r > \frac{C_{p+2}}{C_{p+1}} - \frac{\sigma'(1)}{\sigma(1)} = \frac{C_{p+2}}{C_{p+1}} - k + \tfrac{1}{2}.$$

Although Adams-Bashforth methods also have a positive error constant C_{p+2}, this lower bound for r was checked to be negative for $k = p = 4,8$.

2.2. Adams-Moulton methods

Adams-Moulton methods have $C_{p+1} < 0$, hence by the $Q(r)$-stability condition (2.3), the methods of orders $p = 3,4; 7,8; 11, 12, \ldots$ cannot be $Q(r)$-stable. The other ones will be proved to be $Q(r)$-stable in section 4. For $p = 2, 6, 10, \ldots$ the lower bound (2.8) applies which was checked to be non-positive for $p = k - 1 = 2,6$.

2.3. Backward differentiation methods

These methods also have *negative* principal error constants, but satisfy only for $p \leq 6$ the strong root condition. Since $\sigma(1) > 0$, only for $p = 1,2\ ;\ 5,6$ $Q(r)$-stability can be expected (see section 4). Furthermore, the right-hand side of (2.4) turns out to be negative for $k = p = 2,6$.

2.4. Symmetric Adams methods

A particularly interesting family of LM methods are the *symmetric Adams methods* defined by

$$\rho(\zeta) = \zeta^{\frac{k-1}{2}}(\zeta-1)\ , \quad \sigma(\zeta) = \sum_{j=0}^{(k-1)/2} b_j\ (\zeta^{k-j} + \zeta^{j})\ , \tag{2.9}$$

where k is assumed to be odd. We notice that $\rho(\zeta)$ satisfies the strong root condition, necessary to obtain $Q_0(r)$-or $Q(r)$-stability. By an appropriate choice of the coefficients b_j the maximal order of accuracy can be shown to be $p = k + 1$ which is, at the same time, the maximal attainable order for all convergent LM methods with odd step number k(cf. e.g. LAMBERT [1973, p. 38]). Examples of symmetric Adams methods are the trapezoidal rule, the fourth order method defined in (1.8) and the methods of order 6 and 8 respectively defined by

$$\rho(\zeta) = \zeta^2(\zeta-1)\ , \ \sigma(\zeta) = \frac{1}{1440}\,[11\,(\zeta^5+1) - 93(\zeta^4+\zeta) + 802(\zeta^3+\zeta^2)] \tag{2.10}$$

and

$$\begin{aligned} \rho(\zeta) &= \zeta^3(\zeta-1) \\ \sigma(\zeta) &= \frac{1}{120960}[-191(\zeta^7+1) + 1879\,(\zeta^6+\zeta) - 9531(\zeta^5+\zeta^2) + \\ &\qquad + 68323\,(\zeta^4+\zeta^3)]. \end{aligned} \tag{2.11}$$

The error constants C_{p+1} of these methods can be proved to *alternate in sign*; the first four members of the symmetric Adams family have the principal error constants

$$C_3 = -\frac{1}{12}, \quad C_5 = \frac{11}{720}, \quad C_7 = -\frac{191}{60480}, \quad C_9 = \frac{2497}{3628800}.$$

Since $\sigma(1) = 1$, the Q(r)-stability condition (2.3) is always satisfied.

Furthermore, it is easily verified that

$$\sigma(1) = \rho'(1) = 1, \quad \sigma'(1) = \frac{\rho''(1) + \rho'(1)}{2} = \tfrac{1}{2}k,$$

and since symmetric methods satisfy

$$C_{p+2} = \tfrac{1}{2}k \, C_{p+1}$$

if follows that (2.14) reduces to $r > 0$. Therefore, all members of the symmetric Adams methods are candidates to be $Q_0(1)$ - or Q(1)-stable, and will be shown in section 4 to be Q-stable indeed.

2.5. Extended backward differentiation methods

Finally, we consider the family of *extended backward differentiation methods* proposed by CASH [3], which have characteristic polynomials of the form

$$\rho(\zeta) = \sum_{j=0}^{k} a_j \zeta^{k-j}, \quad \sigma(\zeta) = b_0 \zeta^k + b_{-1} \zeta^{k+1}.$$

These methods are of particular interest in case of retarded differential equations. The error constants C_{p+1} turn out to be *positive*, as in the case of the Adams-Bashforth methods, and the strong root condition on ρ is satisfied. Again $\sigma(1) > 0$, so that the methods of orders $p = 3, 4; 7,8; \ldots$ are possibly Q(r)-stable, the others are not.

3. Q_0(r)-STABILITY

In this section we give a sufficient condition for $Q_0(r)$ - stability with $r < \infty$ and a relatively simple formula for computing numerically the minimal value of r.

THEOREM 3.1. (a) *Let the* LM *method be such that:*

(i) $\rho(\zeta)$ *satisfies the strong root condition*

(ii) C_{p+1} *satisfies the* $Q(r)$-*stability condition* (2.3)

Then there exists a finite r for which the method is $Q_0(r)$-*stable.*

(b) *The minimal value of r is given by*

$$(3.1)\qquad r_{min} = 1 + \left\lfloor \max\left\{0, \max_{\ell=0,1,\dots}\left(\sup_{\Psi_\ell^+} f_\ell(\psi)\right)\right\}\right\rfloor ,$$

$$\Psi_\ell^+ := \left\{\psi \,\big|\, |\psi| \le \pi;\ F_\ell(\psi) \ge 0\right\} ,$$

where

$$(3.2)\qquad f_\ell(\psi) := \frac{\pm(2\ell+1)\pi - \theta(\psi)}{\psi} \quad \textit{for } \psi \gtrless 0 ,$$

$$(3.3)\qquad F_\ell(\psi) := \frac{\pi}{2m(\psi)} - f_\ell(\psi) ,$$

with $m(\psi)$ *and* $\theta(\psi)$ *defined by* $\frac{\rho}{\sigma}(e^{i\psi}) = m(\psi)\, e^{i\theta(\psi)}$. □

PROOF. Since $\rho(\zeta)$ satisfies the strong root condition we are sure that there is a non-empty negative interval of stability with 0 as its right end-point. Hence, we can find the length of this interval by looking for the points where the numerical boundary locus curve intersects the negative axis. If we find such a point in the interval $(-\pi/2\nu, 0)$ for $\nu \ge r$, then there is no $Q_0(r)$-stability. Thus, the situation where there are values of $\nu \ge r$ such that (cf. (2.1))

$$(3.4)\qquad 0 < |\psi\,(1 + R(\psi))| < \frac{\pi}{2\nu} ,\quad \nu\psi + \frac{\pi}{2} + \alpha(\psi) =$$

$$= \begin{cases} -2\ell\pi & \text{for } \psi < 0,\ \ell = 0, 1, 2, \dots \\ \pi + 2\ell\pi & \text{for } \psi > 0,\ \ell = 0, 1, 2, \dots \end{cases}$$

simultaneously hold, should be avoided. Geometrically, this means that in the (ψ,ν)-plane the curves $\nu = f_\ell(\psi)$, with $f_\ell(\psi)$ defined by

$$(3.2') \qquad f_\ell(\psi) := \frac{\pm(2\ell+\frac{1}{2})\,\pi-\alpha(\psi)}{\psi} \quad \text{for } \psi \gtrless 0,\ \ell = 0, 1, \ldots$$

should not enter the domain defined by (see the shaded region in figure 3.1)

$$(3.5) \qquad \nu \le \frac{\pi}{2|\psi(1+R(\psi))|}, \quad |\psi| \le \pi, \quad \nu \ge r.$$

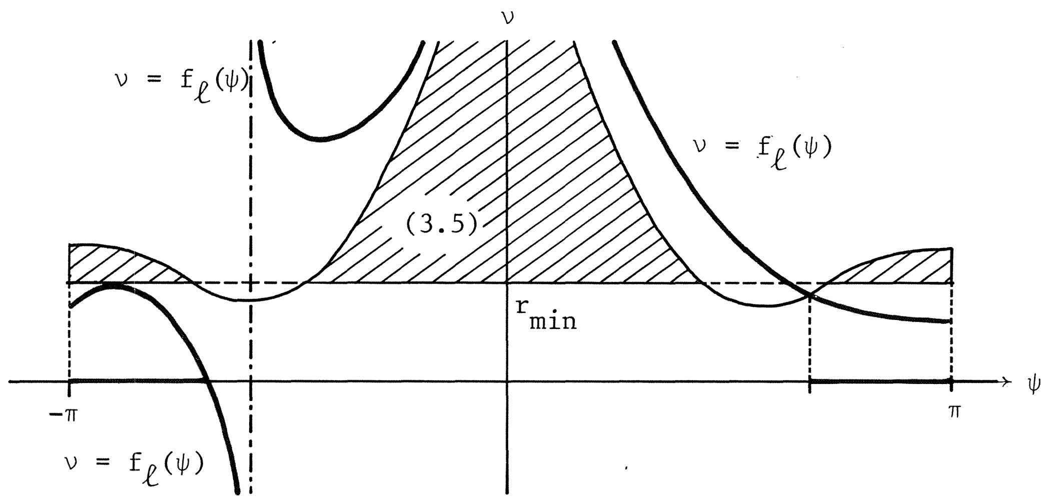

Figure 3.1. Determination of the minimal value of r for which $Q_0(r)$-stability is obtained.

We observe that the function $\pi/2|\psi(1+R(\psi))|$ is an even function that is bounded in $[-\pi,\pi]$ except for the point $\psi = 0$ (by virtue of the strong root condition on ρ). The curves $\nu = f_\ell(\psi)$ also have an asymptote at $\psi = 0$ and $\nu \to +\infty$ as $\psi \to 0$. Thus, we can always find a finite value of r such that $\nu = f_\ell(\psi)$ does not enter (3.5) if the difference functions

$$(3.3') \qquad F_\ell(\psi) := \frac{\pi}{2|\psi(1+R(\psi))|} - f_\ell(\psi), \quad \ell = 0, 1, 2, \ldots$$

satisfy the condition

$$F_\ell(\psi) < 0 \quad \text{as } |\psi| \to 0,\ \ell = 0, 1, 2, \ldots .$$

Since

$$F_{\ell}(\psi) \approx \frac{\pm(4\ell+R(\psi))\,\pi + 2\,\alpha(\psi)}{2\psi} \quad \text{as} \quad \psi \to \mp 0\,,\ \ell = 0,\ 1,\ 2,\ \ldots\,,$$

we derive from (2.6) that $F_{\ell}(\psi)$ does assume negative values in the neighbourhood of the origin if the Q(r)-stability condition (2.3) is satisfied. A simple geometrical consideration reveals that the minimal value of r is given by (3.1), where we have expressed $f_{\ell}(\psi)$ and $F_{\ell}(\psi)$ in terms of $\theta(\psi)$ and $m(\psi)$ instead of $\alpha(\psi)$ and $R(\psi)$ (see 2.1a)). □

It should be remarked that in (3.1) only a *finite number* of functions $f_{\ell}(\psi)$ are involved. In other words, the set Ψ_{ℓ}^{+} is empty for sufficiently large values of ℓ. This follows from the relation

$$F_{\ell+1}(\psi) = F_{\ell}(\psi) - \frac{2\pi}{|\psi|}$$

which shows that eventually $F_{\ell}(\psi)$ becomes negative in the interval $[-\pi,\pi]$ as ℓ increases, with the possible exception of points ψ in the neighbourhood of the origin where $F_{\ell}(\psi)$ tends to infinity. However, as shown in the proof of the theorem, $F_{\ell}(\psi) \to -\infty$ as $|\psi| \to 0$ provided that the conditions of the theorem are satisfied. Thus, we may conclude that there exists a finite ℓ_0 such that Ψ_{ℓ}^{+} is empty if $\ell \geq \ell_0$.

In table 3.1 we have listed the values of r obtained from (3.1) for the families of LM methods of orders $p \leq 8$ discussed in the preceding sections ($r = \infty$ indicates $Q_0(r)$- instability).

Table 3.1 $Q_0(r)$ - stable LM methods

method	p = 1	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7	p = 8
Adams-Bashforth	∞	∞	4	5	∞	∞	34	65
Adams-Moulton	-	1	∞	∞	4	2	∞	∞
Backward differentiation	1	1	∞	∞	6	2	∞	∞
Symmetric Adams	-	1	-	1	-	1	-	1
Extended backw. diff.	-	∞	1	1	∞	∞	6	2

4. Q(r) - STABILITY

The analogue of theorem 3.1 reads:

THEOREM 4.1. *Let the* LM *method be such that*

(i) $\rho(\zeta)$ *satisfies the strong root condition*

(ii) C_{p+1} *satisfies the* Q(r)-*stability condition* (2.3)

Then there exists a finite r *for which the method is* Q(r)-*stable*

(b) *Let*

(4.1) $$\nu_0 := \frac{C_{p+2}}{C_{p+1}} - \frac{\sigma'(1)}{\sigma(1)} \text{ for p even }, \quad \nu_0 := 0 \text{ for p odd.}$$

Then the minimal value of r *is given by*

(4.2) $$r_{min} = 1 + \left\lfloor \max\left\{0, \nu_0, \max_{\ell=0,1,\ldots} \left(\sup_{\Psi_\ell^+} g_\ell(\psi)\right)\right\}\right\rfloor,$$

$$\Psi_\ell^+ := \{\psi \,|\, |\psi| \leq \pi;\ G_\ell(\psi) \geq 0\},$$

where

(4.3) $$g_\ell(\psi) := \frac{2\theta(\psi) + (4\ell+3)\pi}{2(m(\psi) - \psi)} \text{ for } \psi < 0, \quad g_\ell(\psi) := \frac{2\theta(\psi) - (4\ell+1)\pi}{2(m(\psi) - \psi)}$$

$$\text{for } \psi > 0$$

(4.4) $$G_\ell(\psi) := \frac{\pi}{2m(\psi)} - g_\ell(\psi). \quad \square$$

PROOF. (a) First of all we remark that the conditions of this theorem imply that the necessary conditions stated in theorem 2.1 are satisfied if $r > \nu_0$. Restricting our considerations to the upper half plane, we conclude from theorem 2.1 that we have Q(r)-stability iff for $\nu \geq r$ the analytical and numerical boundary locus curves do not intersect except for the trivial point of intersection at $z = 0$ and possibly $z = -\pi/2\nu$. Thus, we should avoid the situation where there are values of $\nu \geq r$ such that

$$(4.5) \qquad |\psi(1+R(\psi))| = \frac{\phi-\pi/2}{\nu} \text{ for } \frac{\pi}{2} < \phi < \pi,$$

$$\nu\psi + \frac{\pi}{2} + \alpha(\psi) = \begin{cases} \phi - (2\ell+1)\pi & \text{for } \psi < 0,\ \ell = 0, 1, 2,\dots \\ \phi + 2\ell\pi & \text{for } \psi > 0,\ \ell = 0, 1, 2,\dots \end{cases}$$

simultaneously hold. Elimination of ϕ yields (cf.(3.4))

$$(4.5') \quad 0 < |\psi(1+R(\psi))| < \frac{\pi}{2\nu},$$

$$\nu\psi + \alpha(\psi) = \begin{cases} -(2\ell+1)\pi - \nu\psi(1+R(\psi)) & \text{for } \psi < 0,\ \ell = 0,1,\dots \\ 2\ell\pi + \nu\psi(1+R(\psi)) & \text{for } \psi > 0,\ \ell = 0,1,\dots \end{cases}.$$

From this we conclude that the curves $\nu = g_\ell(\psi)$ with

$$(4.3') \qquad \begin{aligned} g_\ell(\psi) &:= -\frac{\alpha(\psi) + (2\ell+1)\pi}{2\psi + \psi R(\psi)} \text{ for } \psi < 0, \\ g_\ell(\psi) &:= \frac{\alpha(\psi) - 2\ell\pi}{\psi R(\psi)} \text{ for } \psi > 0; \end{aligned} \qquad \ell = 0, 1, \dots$$

should not enter the region defined by (3.5). This is true for a finite value of r if the difference functions

$$(4.4') \qquad G_\ell(\psi) = \frac{\pi}{2|\psi(1+R(\psi))|} - g_\ell(\psi), \quad \ell = 0, 1, 2,\dots,$$

either satisfy

$$(4.6a) \qquad G_\ell(\psi) < 0 \text{ as } |\psi| \to 0, \quad \ell = 0, 1, 2, \dots$$

or simultaneously satisfy

$$(4.6b) \qquad G_\ell(\psi) > 0, \ g_\ell(\psi) < \infty \text{ as } |\psi| << 1; \ \ell = 0, 1, 2,\dots.$$

For $|\psi| \to 0$ we may write

$$G_\ell(\psi) \approx \frac{\pi R(\psi) - 2\alpha(\psi) + 4\ell\pi}{2\psi R(\psi)}, \quad g_\ell(\psi) \approx \frac{\alpha(\psi) - 2\ell\pi}{\psi R(\psi)} \quad \text{as } \psi \downarrow 0$$

and

$$G_\ell(\psi) \approx \frac{(1-2\ell)\pi R(\psi) + 2\alpha(\psi) + 4\ell\pi}{4\psi}, \quad g_\ell(\psi) \approx -\frac{2\ell+1}{2\psi}\pi \text{ as } \psi \uparrow 0.$$

If $\psi \uparrow 0$ we have that $g_\ell(\psi) \to +\infty$, hence we should require $G_\ell(\psi) < 0$ as $\psi \uparrow 0$. For $\ell \geq 1$ this is obvious. For $\ell = 0$ we find by virtue of (2.6) that

$$G_0(\psi) \approx \begin{cases} \frac{1}{2}(-1)^{\frac{p-1}{2}} \frac{C_{p+1}}{\sigma(1)} \psi^{p-1}, & p \text{ odd} \\ \frac{1}{4}\pi(-1)^{p/2} \frac{C_{p+1}}{\sigma(1)} \psi^{p-1}, & p \text{ even} \end{cases} \quad \text{as } \psi \uparrow 0.$$

It is easily verified that $G_0(\psi) < 0$ as $\psi \uparrow 0$ iff condition (2.3) is satisfied. If $\psi \downarrow 0$ we find for $\ell \geq 1$ that $g_\ell(\psi)$ and $G_\ell(\psi)$ have opposite sign, so that either (4.6a) or (4.6b) is trivially satisfied. For $\ell = 0$ we derive

$$G_0(\psi) \approx \begin{cases} \dfrac{-\sigma(1)C_2}{\frac{1}{2}C_2^2 + \sigma'(1)C_2 - \sigma(1)C_3} \dfrac{1}{\psi^2}, & p = 1 \\ \dfrac{-\sigma(1)C_{p+1}}{\sigma'(1)C_{p+1} - \sigma(1)C_{p+2}} \dfrac{1}{\psi^2}, & p \text{ odd},\ p \neq 1 \\ \dfrac{\pi}{2\psi}, & p \text{ even} \end{cases} \quad \text{as } \psi \downarrow 0$$

and

$$g_0(\psi) \approx \begin{cases} -G_0(\psi), & p \text{ odd}, \\ \dfrac{C_{p+2}}{C_{p+1}} - \dfrac{\sigma'(1)}{\sigma(1)}, & p \text{ even} \end{cases} \quad \text{as } \psi \downarrow 0.$$

It is easily verified that the condition (4.6a) or (4.6b) is satisfied by $g_0(\psi)$ and $G_0(\psi)$ if the Q(r)-stability condition (2.3) holds. This completes the proof of part (a) of the theorem.

(b) The minimal value of r given by (4.2) follows from a geometrical argument and takes into account the necessary condition (2.4) □

In this case of Q(r)-stability it is slightly more complicated to show that only a finite number of $g_\ell(\psi)$ functions are involved in the expression (4.2). From the relation

$$G_{\ell+1}(\psi) = G_\ell(\psi) \pm \frac{2\pi}{m(\psi)-\psi} \quad \text{for } \psi \gtrless 0$$

it follows that $G_\ell(\psi)$ decreases as ℓ increases for all $\psi < 0$ where $G_\ell(\psi)$ has no singularities, and therefore in these points it will become negative for sufficiently large ℓ. In the singular points of $G_\ell(\psi)$ we have $g_\ell(\psi) \to \pm \infty$. Since $\pi/2m(\psi)$ is only singular at $\psi = 0$ this implies that the singular points of $G_\ell(\psi)$ do not affect the value of r_{min}. For $\psi > 0$ we distinguish the cases $\psi < m(\psi)$ and $\psi > m(\psi)$. If $\psi < m(\psi)$ then $g_\ell(\psi) < 0$ so that the value of r_{min} will not increase in (4.2). If $\psi > m(\psi)$ the values of $G_\ell(\psi)$ again decreases as ℓ increases and will eventually become negative. Thus, the set Ψ_ℓ^+ is empty for ℓ sufficiently large.

We have computed the values of r defined by (4.2) for all methods listed in table 3.1 and found identical values. However, we did not succeed in proving our conjecture: *any $Q_0(r)$-stable method is also $Q(r)$-stable for the same value of r.*

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