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AFDELING ZUIVERE WISKUNDE ZW 187/83 APRIL
(DEPARTMENT OF PURE MATHEMATICS)

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PROBABILISTIC APPROACH FOR COMPARING FIRST EIGENVALUES

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 60J 99-35B99

Probabilistic approach for comparing first eigenvalues

by

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ABSTRACT

We use probabilistic subordination and time change to compare the first eigenvalues for problems of the type $G\psi + \lambda\psi = 0$; $G\psi - q\psi + \lambda\rho\psi = 0$, with zero boundary conditions.

KEY WORDS & PHRASES: *First eigenvalue, subordinate process, time change.*

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Work done during a sabbatical stay at the Mathematical Centre.

1. INTRODUCTION

The aim of this paper is to use probabilistic subordination and time change to compare the first eigenvalues of the problems

(1.1) $G\psi + \lambda\psi = 0$ in D $\psi/\partial D = 0$

(1.2) $G\psi - q\psi + \lambda\psi = 0$ in D $\psi/\partial D = 0$

(1.3) $G\psi - q\psi + \lambda\rho\psi = 0$ in D $\psi/\partial D = 0$

for an adequate class of operators G.

In Section 2 we consider a Hunt process (X_t) with state space $(E,\mathcal{E})$ whose semigroup possesses a nonnegative symmetric density $f(t,x,y)$ with respect to a Radon measure m on $(E,\mathcal{E})$, such that if D is a bounded open subset of E with $m(\partial D) = 0$ then

$$\iint\limits_{DD} f^2(t,x,y)m(dx)m(dy) < \infty$$

and such that, if $T = \inf\{t > 0 : X_t \in D^c\}$ is the first exit time from D, then

$$P^x(T<\infty) = 1 \quad \forall x \in D.$$

By stopping (X_t) at the boundary of D and subjecting it to a local death rate $q(x)$ we are able to compare the first eigenvalues of the problems (1.1) and (1.2), when G is the infinitesimal generator of (X_t).

In Section 3 we consider an adequate time change of the process with generator G-q to compare the first eigenvalues of problems (1.2) and (1.3).

Basic for all the comparison results is the fact that under the stated assumptions on the semigroup one can get eigenvalue expansions for these various processes which allow one to give a simple characterization of the first eigenvalue. This characterization (proposition 1 in Section 2) is well known for diffussions (see chapter 14 of [4]).

Section 4 is devoted to the discussion of examples.

We refer the reader to [1] for the basic notation and definitions throughout the paper.

2. COMPARISON OF (1.1) AND (1.2)

Let (X_t^D) be the process (X_t) killed when it leaves D and (Q_t^D) its transition semigroup, that is

$$Q_t^D f(x) = E^x[f(X_t), t<T].$$

It is proved in [5] that there is a sequence $\{\lambda_j\}$ of eigenvalues of G such that $0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$, $\lambda_n \to \infty$ and a complete orthonormal set of eigenfunctions $\{\phi_j\}$ such that the series $\sum_{j=1}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y)$ converges absolutely and

$$(2.1) \qquad Q_t^D f(x) = \sum_n e^{-\lambda_n t} \phi_n(x) \int_D \phi_n(y) f(y) m(dy)$$

for $f \in L^2(D)$.

A similar expansion can be found for the semigroup $\tilde{Q}_t^D f(x) = E^x[f(X_t)M_t; t<T]$ of the process subordinate to (X_t^D) with respect to the multiplicative functional $M_t = \exp(-\int_0^t q(X_s)ds)$ where q is a continuous positive function on E (see also [5]).

The first eigenvalue of the problem

$$(1.1) \qquad G\psi + \lambda\psi = 0 \quad \text{in} \quad D, \quad \psi/\partial D = 0$$

can now be characterized as follows

PROPOSITION 1. *If* λ_1 *is the first eigenvalue of the problem* (1.1) *then*

$$\lambda_1 = \sup\{\lambda: \sup_{x \in D} E^x[e^{\lambda T}] < \infty\}.$$

PROOF. It follows from (2.1) that

$$P^x(T>t) = \sum_n e^{-\lambda_n t} \phi_n(x) \int_D \phi_n(y) m(dy).$$

Let $K_n = \int_D \phi_n(y)m(dy)$ then one has

$$E^x[e^{\lambda T}] = -\int_0^\infty e^{\lambda t}\, dP^x(T>t) = \sum_n K_n\, \phi_n(x) \int_0^\infty -\lambda_n\, e^{-(\lambda_n-\lambda)t} dt =$$

$$= \sum_n K_n\, \phi_n(x)\left(\frac{\lambda_n}{\lambda_n-\lambda}\right).$$

Clearly, when $\lambda \to \lambda_1$ this series diverges. Also, $E^x[e^{\lambda T}]$ is an increasing function of λ, so it suffices to show that $E^x[e^{\lambda T}] < \infty$ for $\lambda < \lambda_1$, $\forall x \in D$. Now, if $0 < \lambda < \lambda_1$ then

$$\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda} \leq \frac{\lambda_n}{\lambda_n-\lambda}\ ,$$

hence

$$\sum_n K_n\, \phi_n(x)\left(\frac{\lambda_n}{\lambda_n-\lambda}\right) \leq \frac{\lambda_1}{\lambda_1-\lambda}\, Q_0^D\, 1(x) < \infty.$$

REMARK 1. A similar characterization for the first eigenvalue of problem (1.2) clearly holds

REMARK 2. The heuristics of the situation is as follows:

$$E^x[e^{\lambda T}] = -\int_0^\infty e^{\lambda t} dP^x(T>t) = \int_0^\infty \lambda e^{\lambda s} P^x(T>s)\, ds =$$

$$= \int_0^\infty \lambda e^{\lambda s}\, E^x[1(X_s), s<T] = \lambda \int_0^\infty e^{\lambda s}\, Q_s^D 1(x) \text{ when } P^x(T<\infty) = 1$$

So what one does is "look at the poles of the resolvent"

In order to compare the first eigenvalues of problems (1.1) and (1.2) we consider $(\tilde{X}_t)$, the canonical realization of the process subordinated to (X_t) with respect to $M_t = \exp(-\int_0^t q(X_s)ds)$ on the space $\tilde{\Omega} = \Omega\times[0,\infty]$, and denote by $\tilde{P}^x$ its corresponding measures on $\tilde{\Omega}$ (see [1] ch.3 for all the constructions).

Following [1], let $\gamma\colon \tilde{\Omega} \to [0,\infty]$ be the projection $\gamma(\omega,\lambda) = \lambda$, and

$$T = \inf\{t > 0 : X_t \in D^c\}$$

$$\tilde{T} = \inf\{t > 0 : \tilde{X}_t \in D^c\}.$$

LEMMA 1. $\tilde{E}^x[e^{\lambda\tilde{T}}] \le E^x[e^{\lambda T}], \quad \forall x \in D.$

PROOF. It follows from the definition of $(\tilde{X}_t)$ that

$$\{\tilde{T}>t\} = \{\tilde{X}_s \in D,\ s \le t\} = \{X_s \in D,\ s \le t,\ \gamma > t\} =$$
$$= \{T > t;\ \gamma > t\}$$

and therefore, by the definition of $\tilde{P}^x$ we have

$$\tilde{P}^x(\tilde{T}>t) = E^x[M_t;T>t].$$

Now

$$\tilde{E}^x[e^{\lambda\tilde{T}}] = -\int_0^\infty e^{\lambda t} d\tilde{P}^x(\tilde{T}>t) = -\int_0^\infty e^{\lambda t}\, dE^x[M_t;T>t] =$$
$$= 1 + \lambda E^x \int_0^T e^{\lambda t} M_t dt \le 1 + \lambda E^x \int_0^T e^{\lambda t} dt = E^x[e^{\lambda T}]$$

the integration by parts in the third equality is possible since $P^x(T<\infty) = 1\ \forall x \in D$.

The following comparison result now follows easily from Proposition 1 and Lemma 1.

THEOREM 1. *Let λ_1 be the smallest eigenvalue of problem* (1.1) *and $\tilde{\lambda}_1$ the smallest eigenvalue of problem* (1.2) *then*

$$\tilde{\lambda}_1 \ge \lambda_1.$$

REMARK 3. In the same manner, one can compare the first eigenvalues of the problems

$$G\psi - q_1\psi + \lambda\psi = 0 \quad \text{in} \quad D \quad \psi/\partial D = 0$$

$$G\psi - q_2\psi + \mu\psi = 0 \quad \text{in} \quad D \quad \psi/\partial D = 0$$

for $q_2 \geq q_1$. In fact, let (X_t) now denote the process with generator $G-q_1$, and let $(\tilde{X}_t)$ be the process subordinated to (X_t) with respect to $M_t = \exp(-\int_0^t (q_2(X_s)-q_1(X_s))ds)$ then, the same reasoning as before gives

$$\mu_1 \geq \tilde{\lambda}_1.$$

<u>REMARK 4</u>. The comparison results will hold whenever a characterization for the principal eigenvalue, as the one given in proposition 1 holds; therefore, these results will be valid for the elliptic operators considered by FRIEDMAN [4] Vol.2, Chapter 14.

3. COMPARISON OF (1.2) AND (1.3)

We will now compare the first eigenvalues of the problems

(1.2) $$(G-q)\psi + \lambda\psi = 0 \quad \text{in } D \quad \psi/\partial D = 0$$

(1.3) $$(G-q)\psi + \lambda\rho\psi = 0 \quad \text{in } D \quad \psi/\partial D = 0$$

for ρ continuous and $\rho \geq c > 0$.

Let (X_t) now be the process with generator $G-q = \tilde{G}$ (we warn the reader that to be consistent with the notation of section 2 we should work with $(\tilde{X}_t)$, but we feel this would become too confusing).

Suppose (X_t) has a symmetric transition density $p(t,x,y)$ with respect to the Radon measure m which satisfies

(3.1) $$\int_D \int_D p^2(t,x,y)m(dx)m(dy) < \infty$$

(3.2) $$\int_D p(t,x,x)\rho(x)m(dx) < \infty \quad \forall t$$

for bounded sets D.

Let (A_t) be the following additive functional of (X_t)

$$A_t = \int_0^t \rho(X_s)ds$$

and let (τ_t) be the right continuous inverse of (A_t); that is

$$\tau_t = \inf\{u;\ A_u > t\}.$$

One may check that (τ_t) satisfies

$$\tau_t = \int^t \frac{ds}{\rho(X_{\tau_s})}, \qquad (X_{\tau_t}) \text{ being the time changed process.}$$

Call $\hat{X}_t = X_{\tau_t}$, then
$\hat{X} = (\Omega, F, F_{\tau_t}, \hat{X}_t, \theta_{\tau_t}, P^x)$ is a strong Markov process (see [1] Ch.V) and it follows from Dynkin's formula that $(\hat{X}_t)$ has generator $\frac{\underset{\sim}{G}}{\rho}$.

The following proposition allows one to use the same procedure as that of proposition 1 in Section 2 to characterize the first eigenvalue of problem (1.3) as

$$\hat{\lambda}_1 = \sup\{\lambda:\ \sup_{x\in D} E^x[e^{\lambda\hat{T}}] < \infty\}$$

where

$$\hat{T} = \inf\{t > 0:\ \hat{X}_t \in D^c\}.$$

<u>PROPOSITION 1</u>. *Under the assumptions* (3.1) *and* (3.2) *and the symmetry of* $p(t,x,y)$; *the process* $(\hat{X}_t)$ *possesses a symmetric transition density* $\hat{p}(t,x,y)$ *with respect to the measure* $\rho(y)m(dy)$ *and moreover, if* D *is bounded then*

$$\int_D\int_D \hat{p}^2(t,x,y)\rho(x)\rho(y)m(dx)m(dy) < \infty \tag{3.3}$$

<u>PROOF</u>. Observe that

$$\hat{U}^\alpha f(x) = E^x\int_0^\infty e^{-\alpha t} f(X_{\tau_t})dt = E^x\int_0^\infty e^{-\alpha A_t} f(X_t)\rho(X_t)dt =$$

$$= \int_0^\infty E^x[\rho(X_t)f(X_t)e^{-\alpha A_t}]dt = V^0_\alpha(\rho f)(x)$$

where

$$V_\alpha^\beta h(x) = \int_0^\infty e^{-\beta t} Q_t^\alpha h(x)\,dt$$

$$Q_t^\alpha h(x) = E^x[e^{-\lambda A_t} h(X_t)].$$

If (X_t) has a symmetric transition density $p(t,x,y)$ then (Q_t^α) has a symmetric transition density as well (see [5]), call it $q^\alpha(t,x,y)$. It is easy to see that

(3.4) $$q^\alpha(t,x,y) \le e^{-\alpha ct}\, p(t,x,y)$$

(recall c is such that $\rho(x) \ge c > 0$).

Let $v_\alpha^0(x,y) = \int_0^\infty q^\alpha(s,x,y)ds$, then by inverting Laplace transforms we get, for an appropriate contour γ

$$\hat{P}_t f(x) = \frac{1}{2\pi i}\int_\gamma e^{\sigma t}\, \hat{U}^\sigma\, f(x)\,d\sigma =$$

$$= \int \left(\frac{1}{2\pi i}\int_\gamma e^{\sigma t}\, v_\sigma^0(x,y)\,d\sigma\right) f(y)\rho(y)m(dy)$$

which proves the existence of the symmetric transition density $\hat{p}(t,x,y)$.

$$\hat{p}(t,x,y) = \frac{1}{2\pi i}\int_\gamma e^{\sigma t}\, v_\sigma^0(x,y)\,d\sigma.$$

To prove (3.3) it is enough to show that

$$\int_D \hat{p}(2t,x,x)\rho(x)m(dx) < \infty.$$

But, for all $\alpha > 0$

$$\int_0^\infty e^{-\alpha t}\hat{p}(t,x,y)\,dt = v_\alpha^0(x,y) \le \int_0^\infty e^{-\alpha ct}\, p(t,x,y)\,dt =$$

$$= c\int_0^\infty e^{-\alpha t}\, p(\tfrac{t}{c},x,y)\,dt.$$

Hence

$$\hat{p}(t,x,y) \le cp(\tfrac{t}{c},x,y)$$

and the result follows from (3.2).

Let

$$T = \inf\{t > 0: X_t \in D^c\}$$
$$\hat{T} = \inf\{t > 0: \hat{X}_t \in D^c\}$$

then

(3.5) $$\hat{T} = A(T)$$

and so

(3.6) $$\{\hat{T} > t\} = \{A(T) > t\} = \{T > \tau_t\}$$

(see [1] ch.V).

The following analytic lemmas will be needed for the comparison result.

<u>LEMMA 1</u>. *Let* $F(t,\omega)$ *be a measurable function on* $\mathbb{R} \times \Omega$, $((\Omega,\mathcal{F},P)$ *being a probability space) such that* $t \to F(t,\omega)$ *is monotone and right continuous for each* ω. *Then, for* $f \ge 0$

$$\int f(t)\,dE(F(t,\omega)) = E\int f(t)\,dF(t,\omega).$$

(E denotes expectation with respect to P).

<u>PROOF</u>. Easy.

<u>LEMMA 2</u>. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be* 1-1 *and differentiable. If* $\delta(x-a)$ *denotes the delta function at* a *then*

$$\int g(x)\delta(f(x)-a)\,dx = \frac{g(f^{-1}(a))}{f'(f^{-1}(a))}.$$

<u>PROOF</u>. Easy by choosing an approximating δ-sequence (h_n).

THEOREM 1. *If* $\rho < 1$ *then the first eigenvalue of problem* (1.3) *is bigger than the first eigenvalue of problem* (1.2) *and if* $\rho > 1$ *the first eigenvalue of problem* (1.3) *is smaller than the first eigenvalue of problem* (1.2).

PROOF. It suffices to compare

$$E^x[e^{\lambda\hat{T}}] \quad \text{with} \quad E^x[e^{\lambda T}].$$

Now

$$E^x[e^{\lambda\hat{T}}] = -\int_0^\infty e^{\lambda t}\, dP^x(\hat{T}>t) = -\int_0^\infty e^{\lambda t}\, dP^x(T>\tau_t).$$

Using lemmas 1 and 2 we get

$$E^x[e^{\lambda\hat{T}}] = -E^x\int_0^\infty e^{\lambda t}\, d1_{\{T>\tau_t\}} = E^x\int_0^\infty e^{\lambda t}\, \delta(\tau_t - T)\,dt$$

$$= E^x\int_0^\infty e^{\lambda t}\, \rho(X_{A_T})\delta(t-T)\,dt = E^x[e^{\lambda T}\rho(X_{A_T})]$$

the theorem now follows easily.

4. EXAMPLES

In this section we list some situations in which the comparison results hold. In particular, we see that the classical results of examples 1 and 2 (see [3] ch.VI) are valid for more general situations.

EXAMPLE 1. The eigenvalues of the problem $\Delta\phi - q\phi + \lambda\phi = 0$ in D, $\phi = 0$ on ∂D; increase when q increases.

EXAMPLE 2. The eigenvalues of $\Delta\phi - q\phi + \lambda\phi = 0$ in D, $\phi = 0$ on ∂D; increase when the domain D decreases.

EXAMPLE 3. This is a generalization of example 1. Let G be the Laplace-Beltrami operator on a Riemannian manifold or compact Lie group and D an open bounded subset. The same conclusions hold. See [7] or [8] for the appropriate constructions.

EXAMPLE 4. The following construction, going back to PHILLIPS [6] yields a larger class of processes (and eigenvalue problems) for which the comparison results apply. Suppose (X_t) has a transition density $p(t,x,y)$ such that

$$p(t,x,y) = \sum_n e^{-\lambda_n t} \phi_n(x)\phi_n(y)$$

where the (ϕ_n) are an orthonormal, complete set, and (ϕ_n,λ_n) are all the solutions to $G\phi_n + \lambda_n\phi_n = 0$ plus appropriate boundary conditions.

Let $(\tau_t)_{t\geq 0}$ now denote a subordinator (a process with stationary independent increments on $[0,\infty]$) independent of (X_t). Define a new semigroup on the same state space by

$$\begin{aligned}(3.1)\qquad Q_t f(x) &= \int_0^\infty P_u f(x)\tau_t(du) = \sum_n \left(\int_0^\infty e^{-\lambda_n u}\tau_t(du)\right)\phi_n(x)\int \phi_n(y)f(y)\,dy \\ &= \sum_n e^{-\mu_n t}\phi_n(x)\int \phi_n(y)f(y)\,dy\end{aligned}$$

where $\mu_n = a\lambda_n - \int (e^{-\lambda_n u}-1)\nu(du)$, (a,ν) being the characteristics of (τ_t).

In this case, the process $(X_{\tau_t},\Omega,P^x_\tau)$ where P^x_τ is constructed from (Q_t) via the Kolmogorov extension theorem, has transition semigroup (Q_t) and infinitesimal generator

$$(3.2)\qquad G_\tau f(x) = aGf(x) - \int_0^\infty (P_u f(x)-f(x))\nu(du).$$

One has $G_\tau\phi_n + \mu_n\phi_n = 0$ and, from completeness, there are no more solutions to $G_\tau\psi + \lambda\psi = 0$ (with the same boundary conditions as for $G\psi + \lambda\psi = 0$).

EXAMPLE 5. The comparison results can be applied to the symmetric stable process in $\mathbb{R}^n$ of index α, $0 < \alpha \leq 2$; that is, the process with stationary independent increments whose continuous transition density relative to Lebesgue measure in $\mathbb{R}^N$ is

$$p(t,x) = \frac{1}{2\pi N}\int e^{i(x,\xi)}e^{-t|\xi|^\alpha}d\xi \qquad t > 0.$$

See [2] and [5] in connection with this example.

It is interesting to observe the connection with example 4, which is as follows: If (N_t^β) is the one sided stable semigroup in $(0,\infty)$, $\beta \in (0,1)$ and $P_t(x,A)$ the transition semigroup of the Brownian motion process in $\mathbb{R}^N$ then, the semigroup of the symmetric stable process in $\mathbb{R}^N$ satisfies

$$P_t^\alpha(x,A) = \int_0^\infty P_u(x,A)N_t^{\alpha/2}(du)$$

(see [1] ch.I;2.20).

If one now considers the Brownian motion process killed when it first reaches D^c and one performs on it the time change of example (4) with (τ_t) being $(N_t^{\alpha/2})$ one obtains, as might be expected, the symmetric stable process of index α, killed when it first leaves D. This fact follows from the following two lemmas, which might also be interesting for other situations.

Following the notation of example 4 define

$$T_\tau = \inf\{t > 0: X_{\tau_t} \in D^c\}$$

$$T = \inf\{t > 0: X_t \in D^c\}$$

$\ell_t = \inf\{s > 0: \tau_s \geq t\}$ the left continuous inverse of (τ_t).
It is well known that

$$\{\ell_s > t\} \Longleftrightarrow \{\tau_t < s\}$$

and therefore

$$\{\ell(T) > t\} \Longleftrightarrow \{\tau_t < T\}.$$

<u>LEMMA 1</u>. $\ell(T) = T_\tau$.

<u>PROOF</u>. $T_\tau \geq t \Longleftrightarrow X_{\tau_s} \in D$, $\forall s < t \Longleftrightarrow \tau_s < T$, $\forall s < t \Longleftrightarrow s < \ell(T)$ $\forall s < t$ $\Longleftrightarrow t \leq \ell(T)$.

If E[ ] denotes expectation with respect to the distribution of τ_t then one has as an immediate consequence of Lemma 1 and the remarks preceding it, that

LEMMA 2.

$$E[E^x[f(X_{\tau_t}); \tau_t < T]] = E[E^x[f(X_{\tau_t}); t < T_\tau]].$$

EXAMPLE 6. The Ornstein-Uhlenbeck process in $\mathbb{R}$ has a transition density $f(t,x,y)$ with respect to the measure $m(dy) = e^{-y^2}/2\, dy$ which is symmetric see [2].

EXAMPLE 7. The following situation may apply to discretized versions of examples 1 or 2. Consider a symmetric Markov chain on a lattice, with transition among nearest neighbors only. Let D be a subset of the state space and let ∂D denote the subset of D^c consisting of nearest neighbors to points in D. If Q denotes the rate matrix of the process killed on leaving D, then our comparison result applies to the problems

$$\sum_{j\in D} Q(i,j)\phi(j) + \lambda\phi(i) = 0; \ \phi(i) = 0, \ i \in \partial D.$$

$$\sum_{j\in D} Q(i,j)\phi(j) - q(i)\phi(i) + \lambda\phi(i) = 0; \ \phi(i) = 0, \quad i \in \partial D.$$

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