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INTERTWINING FUNCTIONS ON COMPACT LIE GROUPS I,
SUMMARY OF RESULTS

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Intertwining Functions on Compact Lie Groups I, Summary of Results

by

Bob Hoogenboom

ABSTRACT

Let (U,K) and (U,H) be two Riemannian symmetric pairs of the compact type. An intertwining function on U is a left- K -, right- H -invariant function on U which is an eigenfunction of all left- U -, right- H -invariant differential operators on U . Thus an intertwining function is a generalization of a spherical function. We give an outline of the proof that the intertwining functions on U may be considered as orthogonal polynomials with respect to a positive weight function, defined on a region in \mathbb{R}^l .

KEY WORDS & PHRASES: *Intertwining Function, Orthogonal Polynomials in one or more variables, representations of K,H -class 1, generalized Cartan decomposition for a compact Lie group.*

INTRODUCTION

As was proved by Vretare (cf. [Vre]), the spherical functions on a compact Lie group can be considered as orthogonal polynomial in several variables. The present work generalizes this result to intertwining functions. Roughly speaking, an intertwining function on a compact Lie group U is a spherical function, but with the K -biinvariance (here (U, K) is a Riemannian symmetric pair of the compact type) replaced by left- K -, and right- H -invariance (where (U, H) is another Riemannian symmetric pair of the compact type).

Although the line of proof is roughly the same as the original proof for spherical functions (cf. [Vre]), this generalization is far from a routine exercise; many complications of algebraic nature arise. This corresponds to new phenomena, which arise when a complex semisimple Lie algebra is studied with two commuting involutions, instead of one.

This work is planned to appear in a dissertation, University of Leiden, at the end of 1983. Here we present a summary of the most important results.

INTERTWINING FUNCTIONS

Let g be a noncompact real semisimple Lie algebra, with complexification g_c . Let σ be an involution of g , then there exists a Cartan involution θ of g such that $\theta\sigma = \sigma\theta$, cf. [Lo, p. 153]. Let $g = k + p = h + q$ be the decomposition of g in ± 1 eigenspaces of θ and σ , respectively. Put $u := k + ip$, then u is a compact real form of g_c . Let $u = h^0 + q^0$ be the decomposition of u with respect to σ . Let G_c be a simply connected Lie group with Lie algebra g_c . Let G, K, H, H^0 and U be analytic subgroups of G_c with Lie algebras g, k, h, h^0 and u , respectively. Let $a_{pq} \subset p \cap q$ be a maximal abelian subalgebra. Extend a_{pq} to $a_p \subset p$ maximal abelian subalgebra, and also to $a_q \subset q$ maximal abelian subalgebra.

THEOREM 1. *There exists $a \subset g$ maximal abelian subalgebra such that $a_p \subset a, a_q \subset a$.*

The proof is straightforward, see also [Os]. One shows that

$[a_p, a_q] = (0)$, hence $a_p \oplus a_q$ is maximal abelian in $p \cap q \oplus k \cap q \oplus p \cap h$.

DEFINITION 2. An irreducible representation π of U is said to be of K, H^0 -class 1 if there exist nonzero vectors e_K and e_H in H (the representation space of π) such that $\pi(k)e_K = e_K$, $\pi(h)e_H = e_H \forall k \in K, h \in H^0$.

Necessary and sufficient conditions for an irreducible representation of U to be of K, K -class 1 are given in [Wa, Th. 3.3.1.1]. By twice applying [Wa, Th. 3.3.1.1], using Theorem 1, one can give necessary and sufficient conditions for an irreducible representation of U to be of K, H^0 -class 1. This gives Theorem 3. But, before we can state this result, we first need some more notation.

Let ϕ , Σ_p , Σ_q and Σ_{pq} be the sets of roots of the pairs (g_c, a_c) , $(g_c, (a_p)_c)$, $(g_c, (a_q)_c)$, and $(g_c, (a_{pq})_c)$, respectively. These are root systems. Put $a_k := a \cap k$, $a_h := a \cap h$. Let us agree to extend linear forms on $(a_p)_c$, $(a_q)_c$ or $(a_{pq})_c$ to a_c , by making them trivial on $(a_k)_c$, $(a_h)_c$, or both, respectively. Then roots in ϕ , Σ_p , Σ_q or Σ_{pq} are real-valued on $ia_k + a_p$. The Killing form of g_c induces an inner product (\cdot, \cdot) on the real dual of $ia_k + a_p$. Choose compatible positive systems Σ_{pq}^+ , Σ_p^+ , Σ_q^+ , ϕ^+ (this is possible, cf. [Os]). Let Λ be the set of weights corresponding to ϕ , $\Lambda^+ \subset \Lambda$ the set of dominant weights. For $\lambda \in \Lambda^+$, let π_λ denote the irreducible finite dimensional representation of G with highest weight λ .

THEOREM 3. Let $\lambda \in \Lambda^+$. Then π_λ is a representation of K, H -class 1 iff

- (1) $\lambda|_{a_h \cup a_k} = 0$
- (2) $\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \quad \forall \alpha \in \Sigma_p \cup \Sigma_q$.

Let a_{pq} have dimension ℓ . As a corollary to Theorem 3 we obtain that the representations of U of K, H^0 -class 1 are parametrized by a lattice \mathbb{Z}_+^ℓ of real forms on a_{pq} , analogous to the spherical case, cf. [Vre]. Let the generators of this lattice be denoted by μ_1, \dots, μ_ℓ .

For $\beta \in \phi$, let $\tilde{\beta}$, $\hat{\beta}$, $\hat{\beta}$ denote the restriction of β to $(a_p)_c$, $(a_q)_c$, $(a_{pq})_c$, respectively.

Put, for $\alpha \in \Sigma_{pq}$:

$$c(\alpha) := \max_{\substack{\tilde{\beta} \in \Phi \\ \tilde{\beta} = \alpha}} \left\{ \frac{(\tilde{\beta}, \tilde{\beta})}{(\alpha, \alpha)}, \frac{(\tilde{\beta}, \tilde{\beta})}{(\alpha, \alpha)} \right\}$$

LEMMA 4. Condition (2) in Theorem 3 can be replaced by

$$(2') \quad \frac{(\lambda, \alpha)}{(\alpha, \alpha)} \in c(\alpha)\mathbb{Z} \quad \forall \alpha \in \Sigma_{pq}.$$

Let Σ_{pq}^c be defined by

$$\Sigma_{pq}^c := \{c(\alpha)\alpha \mid \alpha \in \Sigma_{pq}\}.$$

LEMMA 5. Σ_{pq}^c is a root system.

The proofs of Lemmas 4 and 5 are straightforward. Let $(\Sigma_{pq}^c)'$ be the root system consisting of roots $c(\alpha)\alpha$ in Σ_{pq}^c such that $2c(\alpha)\alpha \notin \Sigma_{pq}^c$. Let Λ_c be the weight lattice corresponding to $(\Sigma_{pq}^c)'$. Then Lemmas 4 and 5 imply:

THEOREM 6. $\mathbb{Z}^\ell = 2\Lambda_c$.

Let W_{pq} be the Weyl group of Σ_{pq} . Then Theorem 6 implies:

THEOREM 7. \mathbb{Z}^ℓ is W_{pq} -invariant.

DEFINITION 8. Let π be an irreducible representation of U of K, H^0 -class 1. Then the function $\phi : x \mapsto (e_K | \pi(x) e_H)$ ($x \in U$) is called an *intertwining function*. (Here $(\cdot | \cdot)$ denotes an inner product in H according to which π is unitary). If $\pi = \pi_\lambda$ with $\lambda = \lambda_1 \mu_1 + \dots + \lambda_\ell \mu_\ell$ then denote ϕ by ϕ_λ .

Equivalent conditions for a function on U to be an intertwining function can be given, just as for spherical functions (cf. [He, ch.X], see also [Du]). Therefore, let $\mathbb{D}_0(U)$ be the algebra of differential operators on U which are left- U -, and right- H^0 -invariant.

THEOREM 9. Let ϕ be a function on U . The following conditions are equivalent:

1. ϕ is an intertwining function.
2. ϕ is continuous, not identically zero, and there exists $c \neq 0$ such that:

$$\phi(x)\overline{\phi(z)}\phi(y) = c \int_K \int_{H^0} \phi(xhz^{-1}ky) dhdk \text{ for all } x,y,z \in U.$$

3. ϕ is C^∞ , left- K -, and right- H^0 -invariant, $\phi \neq 0$, and there exists $\lambda : \mathbb{D}_0(U) \rightarrow \mathbb{C}$ such that:

$$D\phi = \lambda(D)\phi \text{ for all } D \in \mathbb{D}_0(U).$$

$$\text{Put } A_{pq} := \exp ia_{pq}.$$

THEOREM 10. $U = KA_{pq}H^0$.

For the proof, see [Ho,Th. 3.6]. Theorem 10 will be referred to as the "generalized Cartan decomposition". For the concompact analogue of this decomposition [FJ, Th. 2.6] gives an integral formula. We shall now give the compact version of Flensted-Jensen's formula. Therefore, let $g^{\pm\sigma\theta}$ be the ± 1 eigenspace of $\sigma\theta$ in g . Put $p_\alpha := \dim(g_\alpha \cap g^{+\sigma\theta})$, $q_\alpha := \dim(g_\alpha \cap g^{-\sigma\theta})$.

DEFINITION 11. For $X \in ia_{pq}$ let the function δ be defined by:

$$\delta(\exp X) := \left| \prod_{\alpha \in \Sigma_{pq}^+} \sin^{p_\alpha} ia(X) \cos^{q_\alpha} ia(X) \right|.$$

THEOREM 12.

$$\int_{A_{pq}} \delta(a) da \int_U f(u) du = \int_K \int_{A_{pq}} \int_{H^0} f(kah) \delta(a) dk da dh \quad \forall f \in C(U).$$

For the proof, see [Ho, Th. 4.7]. Let Σ_0 be the set of roots in Σ_{pq} such that $p_\alpha > 0$. Then Σ_0 is a root system. Let W_0 be the Weyl group of Σ_0 . We know that the spherical functions (i.e. K,K -intertwining functions) are invariant under the Weyl group of Σ_p . For intertwining functions this is no longer true, but we have the following theorem, which implies that intertwining functions are invariant under W_0 .

THEOREM 13. Let $\alpha \in \Sigma_{pq}$, and let s_α be the reflection corresponding to α .

Let ϕ be an intertwining function. Put $H_\alpha := [X_\alpha, \theta X_\alpha]$ with $X_\alpha \in g_\alpha$ normalized such that $B(X_\alpha, \theta X_\alpha) = -2(\alpha, \alpha)^{-1}$

1. If $\mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0)$ then $\phi(\exp_\alpha X) = \phi(\exp X)$ for all $X \in \mathfrak{ia}_{pq}$.
2. If $\mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta} \neq (0)$ then $\phi(\exp_\alpha X) = \phi(\exp(X + \frac{1}{2}\pi i H_\alpha))$ for all $X \in \mathfrak{ia}_{pq}$.

The proof of Theorem 10 is quite straightforward, using Definition 8. If $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}_+^\ell$, denote the monomial $x_1^{\lambda_1} \dots x_\ell^{\lambda_\ell}$ by x^λ . A polynomial $p(x) = \sum_{\nu \prec \lambda} q(\nu) x^\nu$ ($q(\nu) \in \mathbb{C}$, $q(\lambda) \neq 0$) is said to be of *degree* λ , where \prec denotes the partial ordering on the root lattice as defined in [Vre].

THEOREM 14. Let $\lambda \in \mathbb{Z}^\ell$. Then ϕ_λ is a polynomial of degree λ in the variables $\phi_{\mu_1}, \dots, \phi_{\mu_\ell}$.

The proof proceeds in exactly the same way as the corresponding proof for spherical functions, cf. [Vre, Th. 3.1], after having made some preliminaries which we'll omit here. For $i = 1, \dots, \ell$, put $\phi_i := \phi_{\mu_i}$.

DEFINITION 15. Define a function $F: \mathfrak{ia}_{pq} \rightarrow \mathbb{C}^\ell$ by

$$F(X) := (\phi_1(\exp X), \dots, \phi_\ell(\exp X)) \quad (X \in \mathfrak{ia}_{pq}).$$

Put

$$\Omega_0 := F(\mathfrak{ia}_{pq}) \subset \mathbb{C}^\ell.$$

Denote the function on \mathfrak{A}_{pq} , defined by $\exp X \mapsto F(X)$ ($X \in \mathfrak{ia}_{pq}$) also by F . There exists a function ψ which embeds Ω_0 in \mathbb{R}^ℓ (just as in the case of spherical functions, cf. [Vre]). Put $\Omega := \psi(\Omega_0)$. For $\alpha \in \Sigma_{pq}$ put

$$k(\alpha) := \min_{\substack{\mu \in \mathbb{Z}^\ell \\ (\mu, \alpha) \neq 0}} \left| \frac{(\mu, \alpha)}{(\alpha, \alpha)} \right|.$$

Put $\Sigma'_{pq} := \{\alpha \in \Sigma_{pq}^+ : \frac{1}{2}\alpha \notin \Sigma_{pq}^+\}$, and for $j = 0, 1$ put $\Sigma_j := \Sigma_j \cap \Sigma'_{pq}$.

THEOREM 16. $\det dF(X) = c \prod_{\alpha \in \Sigma_0'} \sin(k(\alpha) \text{ia}_\alpha(X)) \prod_{\alpha \in \Sigma_1'} \sin(k(\alpha) \text{ia}_\alpha(X + \frac{1}{2}\pi i H_\alpha))$

$$X \in \mathfrak{ia}_{pq}.$$

The proof of Theorem 16 again follows the idea of the corresponding proof for spherical functions, but of course is much more complicated. Put

$$D := \{X \in ia_{pq} \mid k(\alpha)\alpha(X) \in \pi i \mathbb{Z} \text{ for some } \alpha \in \Sigma'_0, \text{ or} \\ k(\alpha)(\alpha(X) + \frac{1}{2}\pi i) \in \pi i \mathbb{Z} \text{ for some } \alpha \in \Sigma'_1\},$$

$$A'_{pq} := A \setminus \exp D.$$

Let F' denote the restriction of F to A'_{pq} .

DEFINITION 17. Let J be the set of all pairs (s, mh) , with $m \in N_K(ia_{pq})$, $h \in H^0$, and $s = \text{Adm}|ia_{pq}$.

Then J is a finite set. Let $j := |J|$. Let $w := |W_{pq}|$. Let $M := C_K(ia_{pq})$, and put $k := |MH^0 \cap A_{pq}|$.

THEOREM 18. $j = wk$.

THEOREM 19. F' is a regular wk -to-one mapping of A' onto an open dense subset Ω'_0 of Ω_0 .

Regularity follows from Theorem 16, and A'_{pq} is open dense in A_{pq} , hence Ω'_0 is open dense in Ω_0 . The fact that F' is wk -to-one is proved first for a dense subset A''_{pq} of A_{pq} , and the extension to A'_{pq} then follows by a topological argument.

DEFINITION 20. Let the positive function w on Ω be defined by:

$$w(\psi(F(X))) := \left| \prod_{\alpha \in \Sigma'_+} \sin^{p_\alpha} ia(X) \cos^{q_\alpha} ia(X) \cdot \prod_{\alpha \in \Sigma'_0} \sin^{-1}(k(\alpha) ia(X)) \cdot \prod_{\alpha \in \Sigma'_1} \sin(k(\alpha)(ia(X) - \frac{1}{2}\pi)) \right|, \quad X \in ia_{pq}.$$

THEOREM 21. The mapping $P \mapsto P \circ \psi \circ F$ is an isomorphism of the algebra of polynomials on Ω onto the algebra of functions on A_{pq} spanned by the intertwining functions such that the orthogonal polynomial $P \circ \psi$ of degree $\lambda \in \mathbb{Z}_+^l$ with respect to the weight function w is mapped onto the intertwining function ϕ_λ .

Theorem 21 follows from Theorem 12, Theorem 16 and Theorem 19, by using the orthogonality relation of Schur.

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