stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 178/82

B.N. COOPERSTEIN

A CHARACTERIZATION OF A GEOMETRY RELATED TO $\Omega^{+}_{2n}(K)$

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 51B25

A characterization of a Geometry Related to $\Omega^{+}_{2n}(K)$ *)

by

Bruce N. Cooperstein **)

ABSTRACT

The halved dual polar spaces related to $\Omega^{+}_{2n}(K)$ are characterized as incidence structures in terms of a short list of axioms on points and lines.

KEY WORDS & PHRASES: *graphs, incidence structures, (dual) polar spaces, buildings of type* D_n.

*) Supported in part by the National Science Foundation. I would also like to thank the Mathematisch Centrum in Amsterdam for their support and hospitality during which part of this paper was written.

**) University of California, Santa Cruz, CA 95064

0. INTRODUCTION

Let V be a vector space of dimension $2n \geq 8$ over a field K equipped with a non-degenerate quadratic form Q with maximal Witt index (so totally singular subspaces of dimension n exist). Let M denote the collection of maximal totally singular subspaces of V. If we define the relation $x \approx y$, for $x,y \in M$, if and only if $\dim_K x/x \cap y$ is even, then it is well-known that $\approx$ is an equivalence relation with two equivalence classes. Let P denote one of these classes. Let L be the collection of totally singular subspaces of V with linear dimension n-2. Then $(P,L, \subseteq \cup \supseteq)$ is an incidence structure known as $D_{n,max}(K)$ or $D_{n,n}(K)$. The purpose of this paper is to characterize these incidence structures. This extends part of Theorem B of [4]. As an application of our results, in sections 5 and 6 we obtain another proof of Cameron's characterization of the dual polar spaces of type D_n.

1. DEFINITION AND NOTATION

(1.1) DEFINITION. By *an incidence structure* here we will mean a pair of disjoint sets P and L whose members we call *points* and *lines* respectively, together with a symetric relation between them, such that each line is incident with at least two points. If every line is incident with at least three points then we say (P,L) is *thick*.

(1.2) DEFINITION. An incidence structure (P,L;I) is a *partial linear space* (pls) if two points lie on at most one line.

When (P,L;I) is a pls then no two lines are incident with the same points. Then we may identify a line with the points it is incident to and replace I with symmetrized inclusion. We will do this throughout this paper, and drop the relation I.

(1.3) DEFINITION. The *point-graph* of (P,L) is the graph (P,Γ) with vertex set P and edge set consisting of pairs of points which are collinear.

(1.4) NOTATION. If (P,Γ) is the point-graph of (P,L), then $x^{\perp} = \{x\} \cup \{y : \{x,y\} \in \Gamma\}$.

If $X \subseteq P$, $X^{\perp} = \bigcap_{x \in X} x^{\perp}$, and $\mathrm{Rad}(X) = X \cap X^{\perp}$.

(1.5) DEFINITION. If (P,Γ) is a graph and $x,y \in P$, then a *path of of length* n from x to y is a sequence $x = x_0, x_1, \ldots, x_n = y$ with $\{x_i, x_{i+1}\} \in \Gamma$ for $i = 0,1,\ldots,n-1$. If such a path exists, then the *distance* from x to y, denoted $d(x,y)$, is the length of the shortest path from x to y (such a path is called a *geodesic* or *g* - path). If no path connects x and y, then we write $d(x,y) = +\infty$.

(P,L) is *connected* if for each pair $x,y \in P$, $d(x,y) < \infty$, and in this case diam $(P,\Gamma) = \sup\{d(x,y): x,y \in P\}$. If $X,Y \subseteq P$, then $d(X,Y) = \min\{d(x,y): x \in X,\ y \in Y\}$.

(1.6) NOTATION. If (P,Γ) is a graph, $x \in P$, then $\Gamma_k(x) = \{y \in P: d(x,y) = k\}$. In [10] D.G. Higman introduced the notion of a *gamma space*. This notion is generalized in [3] to

(1.7) DEFINITION. An incidence structure (P,L) with point graph (P,Γ) is a *strong gamma space* if whenever $x \in P$, $\ell \in L$ with $d(x,\ell) = k$, then either $\ell \subseteq \Gamma_k(x)$ or $|\ell \cap \Gamma_k(x)| = 1$.

(1.8) DEFINITION. (P,L) an incidence structure with point-graph (P,Γ). A subset X of P is a *subspace* if whenever a line ℓ m-ets X in a least two points, then ℓ is contained in X. X is a *singular subspace* if X is a clique. The *rank* of a singular subspace X, denote rk(x), is defined to be the length of a maximal chain of properly ascending subspaces. For example the rank of a point os 0, of a line 1. We will call singular subspaces of rank two *planes*. By convention the empty set has rank -1.

(1.9) NOTATION. If (P,L) is an incidence structure, and $\mathcal{K}$ some collection of subspaces, and $X \subseteq P$, then $\mathcal{K}_X = \{K \in \mathcal{K} : X \subseteq K\}$ and $\mathcal{K}(X) = \{K \in \mathcal{K}: K \subseteq X\}$. We denote the collection of all subspaces of by Sub, planes by V, and singular subspaces by Sing.

(1.10) DEFINITION. For $X \subseteq P$, $\langle X\rangle$ will denote the *subspace spanned by* X, $\langle X\rangle = \cup_{S \in \mathrm{Sub}_X} S$.

(1.11) DEFINITION. A *polar space* is an incidence structure (P,L) such that for any point-line pair x,ℓ, either X is collinear with one or all points of ℓ [alternatively a (strong) gamma space in which $d(x,\ell) \leq 1$]. The polar space is non-degenerate if Rad(P) = $\emptyset$. The theorems of BUEKENHOUT and SHULT, [1], TITS [5] and VELDKAMP [7] classify the non degenerate polar spaces all of whose singular subspaces have finite rank. Then rk(P,L) = = max{rk M : M $\in$ Sing} + 1.

It is our goal in this paper to characterize incidence structures (P,L) with point graph (P,Γ) which satisfy the following axioms

(D1) (P,L) is thick and connected, (P,Γ) is not complete;
(D2) For $d(x,y) = 2, (\{x,y\}^{\perp}, L(\{x,y\}^{\perp}))$ is a thick non-degenerate polar space of rank three. If x,ℓ is a point line pair with $\ell \subseteq \Gamma_2(x)$, then $x^{\perp} \cap \ell^{\perp}$ is a singular subspace maximal in $\{x,y\}^{\perp}$ for each $y \in \ell$.
(D3) (P,L) is a strong gamma space. If $\ell \subseteq \Gamma_k(x)$ with $k \geq 3$, then $\emptyset \neq \ell^{\perp} \cap \Gamma_{k-1}(x) \in$ Sing.

We now describe the typical example:

Let V be a vector space of dimension $2n \geq 8$ ever a field K and Q a non-degenerate quadratic form on V with maximal with index (i.e. so that there exists subspaces U of dimension n with $Q(U) = \{0\}$). Let M be the collection of such subspaces. Define $U_1 \approx U_2$, for $U_1,U_2 \in M$ if $\dim U_i/U_1 \cap U_2$ is even. Then it is well known that $\approx$ is an equivalence relation with two equivalence classes. Let P be either of these classes. We will define a set of lines on P : for $U_1,U_2 \in$ P we define U_1 and U_2 to be collinear if $\dim U_i/U_1 \cap U_2 = 2$ and then $\ell(U_1,U_2) = \{U \in P : U \supseteq U_1 \cap U_2\}$. Define $L = \{\ell(U_1,U_2) : U_1,U_2$ collinear}. Then we denote (P,L) by $D_{n,n}(K)$.

In [4] it is remarked that $D_{n,n}(K)$ arises as a Lie incidence structure and satisfies (D1) and (D2). By [3] it follows that $D_{n,n}(K)$ is a strong gamma space, we next prove

(1.13) PROPOSITION. (P,L) *satisfies* (D3).

PROOF. Let $\ell \in L$, $X \in P$ with $\ell \subseteq \Gamma_k(x), k \geq 3$. We must show $\ell^{\perp} \cap \Gamma_{k-1}(x) \neq \emptyset$ a singular subspace. Let $y \in \ell$, and $z \in y^{\perp} \cap \Gamma_{k-1}(x)$. We assert that $z \geq y \cap x$. If not, then there is linear three-subspace, N, contained in $z \cap x$, with $y \cap N = \emptyset$. Then $z \cap y \subseteq N' \cap y$ (here N' is the collection of all vectors of V orthogonal to N), but $\dim z \cap y = n-2$, $\dim N' \cap y = n-3$, so we have a contraction. Thus our assertion follows.

Now set $\mathcal{U} = \bigcap_{y \in \ell} y$, so $\mathcal{U}$ is a totally singular n − 2 subspace of V. Since $\ell \subseteq \Gamma_k(x)$, dim ${}^x/x \cap y$ = 2k for each $y \in \ell$. Then we must have dim $\mathcal{U} \cap x$ n −1 − 2k and dim $\mathcal{U} \cap x$ = n + 1 − 2k, so that there is a subspace A of dimension two in $\mathcal{U}' \cap x$ complimenting $\mathcal{U} \cap x$. Set $M = \mathcal{U} \oplus A$, $N = M \cap x$. Note that $M \in \mathcal{M} \setminus P$. Let

$$\Delta = \{z = (M \cap W') + W : W \subseteq x, W \supseteq M \cap x, \dim {}^W/M \cap x = 1\} .$$

Then clearly Δ is a singular subspace of (P,L) with rank 2k − 2, and $\Delta \subseteq \ell^\perp \cap \Gamma_{k-1}(x)$. Thus to prove the proposition it suffices to prove $\ell^\perp \cap \Gamma_{k-1}(x) \subseteq \Delta$.

Let $z \in \ell^\perp \cap \Gamma_{k-1}(x)$. Then from the very beginning of the proof $z \supseteq \langle y \cap x : y \in \ell \rangle = \mathcal{U}' \cap x = M \cap x$. Now since dim $z \cap x = n+2-2k$, if $W = z \cap x$, then W contains $M \cap x$ as a hyperplane. Now z must equal $(W' \cap y) + W$, for each $y \in \ell$. But $(W' \cap y) + W = (M \cap W') + W$ and $z \in \Delta$ as desired.

The main result of this paper is

(1.14) <u>THEOREM</u>. *Let* (P,L) *be an incidence structure whose maximal singular subspaces all have finite rank, and satisfies* (D1)-(D3). *Then either* (P,L) *is a thick, non-degenerate polar space of rank* 4 *or for some* k ≥ 5 *and field* K, (P,L) *is isomorphic to* $D_{n,n}(K)$.

2. PRELIMINARY LEMMAS

(2.1) <u>LEMMA</u>. *Let* $y \in \Gamma_2(x)$. *Then* $S(x,y) = \langle x, y, \{x,y\}^\perp \rangle$ *is a polar spce of rank four. Moreover, if* $x', y' \in S(x,y)$ *with* $y' \notin (x')^\perp$, *then* $S(x',y') = S(x,y)$.

<u>PROOF</u>. See (3.9) and the corollary to (3.11) in [4].

(2.2) <u>NOTATION</u>. The subspaces $S(x,y) = \langle x,y\}^\perp \rangle$, where $d(x,y) = 2$, will be called *Symplectons* or *Symps*. We denote the collection of all symps by <u>Symp</u>.

(2.3) LEMMA. *If* $x \in P$, $\ell \in L$ *with* $\ell \subseteq x^\perp \setminus \{x\}$, then there is an $S \in$ <u>Symp</u>, PROOF. See (3.12) of [4].

(2.4) COROLLARY. *If* M $\in$ Sing, *then* (M,L(M)) *is a Desarguesion projective space.*

PROOF. By VEBLEN and YOUNG [6], we need only prove the result if $M = \langle \ell, x \rangle$ with $x \in P$, $\ell \in L$, $\ell \subseteq x^{\perp} \setminus \{x\}$. However, this case follows from (2.3) and Tits' classification of polar spaces [5] .

(2.5) NOTATION. $\underline{\underline{V}}$ is the subset of Sing of singular subspaces which contain lines as maximal subspaces. We call elements of $\underline{\underline{V}}$ *planes*.

(2.6) LEMMA. *If there exists a pair* x,w $\in$ P *with* d(x,w) = 2 *and for each* $\ell \in L_x$, $\ell \cap \Gamma(w) \neq \emptyset$, *then* (P,L) *is a thick, nondegenerate polar space of rank* 4.

PROOF. See (3.13) of [4] .

3. INCIDENCE STRUCTURES INDUCED AT A POINT

In this section we induce an incidence structure at a point, called the residue of the point and identify its structure. Thus, let $x \in P$. The points of the residue are the lines on x,L_x, the lines are the planes on x, $\underline{\underline{V}}_x$, with ordinary inclusion as incidence. Thus, if ℓ, $m \in L_x$, ℓ,m will be collinear in the residue if and only if $m \subseteq \ell^{\perp}$, and then the line on ℓ and m is $L_x(\langle \ell, m \rangle)$. For $\ell \in L_x$, $\Gamma_x(\ell) = \{m \in L_x(\ell^{\perp}) - \{\ell\}\}$. We first prove

(3.1) LEMMA. $(L_x, \underline{\underline{V}}_x)$ *is a thick, gamma space whose point graph* (L_x, Γ_x) *has diameter two and satisfies*

(A1) *It* ℓ, $m \in L_x$ *and* $m \notin \Gamma_x(\ell)$, *then* $\Gamma_x(\ell) \cap \Gamma_x(m)$, *together with its lines, is a non-degenerate generalized quadrangle and*

(A2) *If* $V \in \underline{\underline{V}}_x$, $\ell \in L_x$ *such that* $L_x(V) \cap \Gamma_x(\ell) = \emptyset$, *and* $C_x(V,\ell) = \langle m \in L_x : \ell , L_x(V) \subseteq \Gamma_x(m) \rangle \in \underline{\underline{V}}_x$.

PROOF. Clearly $(L_x, \underline{\underline{V}}_x)$ is thick. We first show (L_x, N_x) is a gamma space. Let $\ell \in L_x, V \in \underline{\underline{V}}_x$ and suppose $|\Gamma_x(\ell) \cap L_x(V)| \geq 2$. Then there are $m_1, m_2 \in L_x, (V)$ such that $m_1, m_2 \subseteq \ell^{\perp}$. Then $V = \langle m_1, m_2 \rangle \subseteq \ell^{\perp}$, and hence

$L_x(V) \subseteq \Gamma_x(\ell)$.

Next suppose $\ell = xa$, $m = xb \in L_x$, $m \notin \Gamma_x(\ell)$. Then $d(a,b) \geq 2$. Since $x \in \{a,b\}^\perp$, $d(a,b) = 2$. Then $\{a,b\}^\perp$ is a polar-space of rank 3, in particular $\{a,b\}^\perp \cap x^\perp \neq \emptyset$. If $c \in \{a,b\}^\perp \cap x^\perp$, then $xc \in \Gamma_x(\ell) \cap \Gamma_x(m)$, so diam $\{L_x,\Gamma_x\} = 2$. Also see that $\Gamma_x(\ell) \cap \Gamma_x(m) = L_x(\{a,b\}^\perp)$, and so is a non-degenerate generalized quadrangle. Therefore (A1) is satisfied.

Finally, suppose $V \in \underline{\underline{V}}_x$, $\ell \in L_x$, $\Gamma_x(\ell) \cap L_x(V) = \emptyset$. Let $k \in L(V)\backslash L_x$, $a \in \ell\backslash\{x\}$. Then $a^\perp \cap m = \emptyset$. However, $a^\perp \cap m^\perp \neq \emptyset$, since $x \in a^\perp \cap m^\perp$. Therefore $a^\perp \cap m^\perp \in \underline{\underline{V}}_x$. It is clear to see that $C_x(V,\ell) =$ $= a^\perp \cap m^\perp$, and the lemma is completed.

(3.2) <u>COROLLARY</u>. *For each* x, *there is an integer* $N_x \geq 3$, *and division ring* K_x *such that* $(L_x,\underline{\underline{V}}_x)$ *is isomorphic to* $A_{n_x,2}(K_x)$.

<u>PROOF</u>. Here $A_{n,2}(K)$ is the gamma space whose points are the projective lines in PG(n+1,K), and the lines are in one-one corresponse with incident pairs (π_0,π_2) where $_0$ is a projective point and π_2 a projective plane, and the line is the pencil determined by (π_0,π_2). The corollary follows from (3.1) and Theorem A of [2] and [4].

(3.3) <u>LEMMA</u>. *The graph* (P,Γ_2) *is connected.*

<u>PROOF</u>. Since (P,Γ) is connected it suffices to prove if $y \in \Gamma(x)$, then $\Gamma_2(x) \cap \Gamma_2(y) \neq \emptyset$. By (2.3), if $\ell = xy$, then $\underline{\underline{\text{Symp}}}_\ell \neq \emptyset$. If $S \in \underline{\underline{\text{Symp}}}_\ell$, then $\Gamma_2(x) \cap \Gamma_2(y) \cap S \neq \emptyset$.

(3.4) <u>LEMMA</u>. *For each* $x \in P$, K_x *is a field. Moreover all the* K_x *are isomorphic.*

<u>PROOF</u>. Let $x \in P$, $S \in \underline{\underline{\text{Symp}}}_x$, $L_x(S)$ is a Symp of $(L_x,\underline{\underline{V}}_x)$, and so $L_x(S) \cong A_{3,2}(K_x)$. From Tits' classification of polar spaces (see section 8 of [5]), it follows that K_x is a field and $S \cong D_4(K_x)$. To prove the latter part of the lemma it suffices to prove for $d(x,y) = 2$, then $K_x \cong K_y$. Thus if $d(x,y) = 2$, let $S = S(x,y)$. Then $S \cong D_4(K_x)$ and $S \cong D_4(K_y)$. By (6.13) of [5] it follows that $K_x \cong K_y$.

For the sequel we let K be the underlying field. Note that now all singular subspaces are projective spaces over K. Those of rank t we denote by ${}_tP$.

(3.5) <u>LEMMA</u>. *Let* $x,y \in P$. *Then* $n_x = n_x$.

<u>PROOF</u>. By connectedness of (P,Γ) suffices to prove $n_x = n_y$ for $y \in \Gamma(x)$. Set $\ell = xy$. Then $\ell \in L_x$, and $(L_x,\underline{\underline{V}}_x) = A_{n_x,2}(K)$. Then if $M \in \underline{\underline{\mathrm{Sing}}}_\ell$ is choosen so that rk (M) is maximal, then as a singular subspace of $(L_x,\underline{\underline{V}}_x)$, $(L_x(M)) = n_x-1$. It therefore follows that rk (M) = n_x. By similarly considering $(L_y,\underline{\underline{V}}_y)$, we see rk (M) = n_y and so $n_x = n_y$ as claimed.

4. PROOF OF THE MAIN THEOREM

We now have that there is an integer $n \geq 3$, and field K such that for each point x in P, $(L_x,\underline{\underline{V}}_x) \cong A_{n,2}(K)$. We will prove by induction on n that $(P,L) \cong D_{n+1,n+1}(K)$.

(4.1) <u>LEMMA</u>. *If* $n = 3$, *then* $(P,L) \cong D_4(K) \cong D_{4,4}(K)$.

<u>PROOF</u>. Let $d(x,w) = 2$, $S = S(x,w)$. Then in section three we saw $S \cong D_4(K)$. However, it follows that $x^\perp \subseteq S$, and so by (2.5) that $P = S$.

(4.2) <u>NOTATION</u>. π_x will denote a projective space of rank n over K which underlies $(L_x,\underline{\underline{V}}_x)$.
$R_t = \{x,X\}$ $x \in X \subseteq x^\perp$, $X \in \underline{\underline{\mathrm{Sub}}}$, $L_x(X) \cong A_{t,2}(K)$. For $(x,X) \in R_t$, $y \in X - \{x\}$, we set X_y equal to

$$\bigcup_{z \in X - y^\perp} [S(y,z) \cap y^\perp].$$

Finally let $P^+ = {}_n\mathcal{P}$ and $\mathcal{P}^- = \{M \in {}_3P : M^\perp = M\}$.

(4.3) <u>LEMMA</u>. *Let* $S \in \underline{\underline{\mathrm{Symp}}}$, $x \in P\backslash S$. *If* $L(S \cap x^\perp) \neq \emptyset$, *then* $S \cap x^\perp \in {}_3P\backslash P^-$.

<u>PROOF</u>. Clearly $S \cap x^\perp \in \underline{\underline{\mathrm{Sing}}}$ by (2.1), let $\ell \in L(S\cap x^\perp)$ and $y \in \ell$. Set $m = x_y$. Consider L_y. There is a subspace $\pi_y(S)$ of π_y of rank three such that $L_y(S)$ consists of all lines of $\pi_y(S)$. Now $\ell \in \Gamma_y(m) \cap L_y(S)$, and, therefore, the line of π_y which m is identified with meets $\pi_y(S)$. Then $\Gamma_y(m) \cap L_y(S)$

is a singular plane of L_y. Now it follows that $S \cap x^{\perp} \in {}_3P$. As $x \notin S \cap x^{\perp}$, $S \cap x^{T} \in {}_3\mathcal{P}\backslash\mathcal{P}^{-}$.

(4.4) <u>LEMMA</u>. *Assume* $S_1, S_2 \in \underline{\underline{\mathrm{Symp}}}$ *and* $\underline{\underline{V}}(S_1 \cap S_2) \neq \emptyset$. *Then* $S_1 \cup S_2 \in \mathcal{P}^{-}$.

<u>PROOF</u>. By (2.1), $S_1 \cap S_2 \in \underline{\underline{\mathrm{Sing}}}$. Let $x \in S_1 \cap S_2$. $L_x(S_i)$ are symps of L_x, and since $\underline{\underline{V}}(S_1 \cap S_2) \neq \emptyset$, $L_x(S_1) \cap L_x(S_2) = L_x(S_1 \cap S_2)$ contains a line of $(L_x, \underline{\underline{V}}_x)$. It then follows that $L_x(S_1 \cap S_2)$ is a maximal singular subspace of rank two, hence, $S_1 \cap S_2 \in \mathcal{P}^{-}$.

(4.5) <u>LEMMA</u>. *Let* $(x,X) \in \mathcal{R}_t$, $y \in X - \{x\}$. *Then* $(y, X_y) \in \mathcal{R}_t$.

<u>PROOF</u>. If $t = 3$, then the result is immediate: for any $z \in X - y^{\perp}$, $X = S(y,z) \cap x^{\perp}$. Then $X_y = S(y,z) \cap y^{\perp}$ and $(y, X_y) \in \mathcal{R}_3$, we proceed to prove the lemma in a sequence of short steps. We first introduce some notation. $\underline{\underline{\mathrm{Symp}}}_x(X) = \{S \in \underline{\underline{\mathrm{Symp}}}_x : S \cap x^{\perp} \subseteq X\}$.

I. $X_y \in \underline{\underline{\mathrm{Sub}}}$: Let $u_1, u_2 \in X_y$ with $u_2 \in u_1^{\perp}$. If $u_2 \in yu_1$ then result is clear. Let $S_i \in \underline{\underline{\mathrm{Symp}}}_x(X)$ with $yu_i \subseteq S_i$, $i = 1,2$. If $S_1 = S_2$, then the result is obvious, so we may assume $S_1 \neq S_2$. In particular we may assume $u_1, u_2 \in \Gamma_2(x)$, so $S_i = S(x, u_i)$. Now since $S_1 \cap u_2^{\perp} \supseteq yu_1$, by (4.3), $S_1 \cap u_2^{\perp} \in {}_3\mathcal{P}\backslash\mathcal{P}^{-}$. Then $S_1 \cap u_2^{\perp} \cap x^{\perp} \in {}_2\mathcal{P}$, and hence by (4.4), $\langle x, S_1 \cap u_2^{\perp} \cap x^{\perp} \rangle = S_1 \cap S_2 \in \mathcal{P}^{-}$. Set $M = S_1 \cap S_2$.
Note that $u_1^{\perp} \cap M = u_2^{\perp} \cap M$. Let $N \in {}_2\mathcal{P}_x(M)$, i.e. a hyperplane of M containing x, with $y \notin N$. Let $\{M_i\} \in {}_3\mathcal{P}_N(S_i)$, $i = 1,2, M_i \neq M$ (there are unique such choices). Then by consideration of L_x we see that $M_2 \subseteq M_1^{\perp}$ and $\langle M_1, M_2 \rangle \in {}_4\mathcal{P}$. Let $v_i \in M_i \cap u_i^{\perp}\backslash M$, $i = 1,2$. Now $v_1 \notin u_2^{\perp}$, for if $v_1 \in u_2^{\perp}$, then $v_1 \in \{u_2, x\}^{\perp} \cap S_1 \subseteq S_1 \cap S_2 = M$, a contradiction. However, $u_1, u_2, v_1, v_2 \in S(u_2, v_1)$, a symp, and so $u^{\perp} \cap v_1v_2$ is a point, say v. Now $v \notin y^{\perp}$, for if $v \in y^{\perp}$, then $v \in \{v_1, v_2\}^{\perp} \cap y^{\perp} \subseteq S_1 \cap S_2 = M$. But then $v_2 \in \langle M, v_1 \rangle \subseteq S_1$, a contradiction. Thus $S(u,x) = S(y,s)$. Since $v_1, v_2 \in X$ and X is a subspace, $v \in X$. Hence $S(u,x) \in \underline{\underline{\mathrm{Symp}}}_x(X)$ and $u \in X_y$.

II. If $u_1, u_2 \in X_y$, $d(u_1, u_2) = 2$, then $S(u_1, u_2) \cap y^{\perp} \subseteq X_y$.
Pf: Let $S_i \in \underline{\underline{\mathrm{Symp}}}_x(X) \cap \underline{\underline{\mathrm{Symp}}}_{x_i}$, $i = 1,2$. If $S_1 = S_2$, then the result is clear, so assume $S_1 \neq S_2$. Then we may also assume $u_1, u_2 \in \Gamma_2(x)$.
Let $v \in \{u_1, u_2\}^{\perp} \cap y^{\perp}$. If $v \in x^{\perp}$, then $v \in \{x,u\}^{\perp} \subseteq S_1$, so $v \in X_y$ in this

case. Thus assume $v \in \Gamma_2(x)$. Now consider L_y. The three subspace $\pi_y(S_i)$ of π_i meet in a plane $\mathcal{U}$, and this plane contains the line which xy is identified with. The lines which $u_i y$ are identified with meet $\mathcal{U}$ in projective points ρ_i moreover, since $(u_1,u_2),(u_i,x) \in \Gamma_2, \rho_i$ are not on xy and, $\rho_1 \neq \rho_2$. Now vy "meets" both u_1y and u_2y. If ρ_i is on vy for some i, then vy is contained in $\pi_y(S_j)$, where $\{i,j\} = \{1,2\}$, that is $vy \in L_y(S_j)$ and $v \in S_j$, in which case $v \in X_y$. Thus $u_i y$ "meets" vy in a point $\delta_i \neq \rho_i$, $i = 1,2$. From this it follows that there are lines $m_i = w_i y \in L_y(S_i) \cap \Gamma_y(xy) \cap \Gamma_y(vy)$ with $m_1 \in \Gamma_y(m_2)$ (choose lines m_i to contain δ_i and meet xy in points q_i with $q_1 \neq q_2$). Now $w_i \in S_i \cap v^\perp \cap x^\perp \cap y^\perp$ and so $w_i \in X$, also $d(w_1,w_2) = 2$. Since $y,v \in \{w_1,w_2\}^\perp$, $S(w_1,w_2) \in \underline{\underline{\mathrm{Symp}}}_x(X) \cap \underline{\underline{\mathrm{Symp}}}_y$. As $v \in S(w_1,w_2) \cap y^\perp$ it follows that $v \in X_y$.

III. $X_y \cap x^\perp = X \cap y^\perp$

<u>Pf</u>: Let $z \in X \cap y^\perp$. Then clearly $X \cap z^\perp \backslash y^\perp \neq \emptyset$. Let $w \in X \cap z^\perp \backslash y^\perp$. Then $z \in S(y,w) \cap y^\perp \subseteq X_y$. Thus $z \in X_y \cap x^\perp$ and we have shown $X \cap y^\perp \subseteq X_y \cap x^\perp$. Conversely, suppose $z \in X_y \cap x^\perp$. Let $S \in \underline{\underline{\mathrm{Symp}}}_x(X) \cap \underline{\underline{\mathrm{Symp}}}_{yz}$. Then $z \in S \cap x^\perp \cap y^\perp \subseteq X \cap y^\perp$, and we have equality.

IV. $(y,X_y) \in \mathcal{R}_t$.

<u>Pf</u>: From I. and II., $L_y(X_y)$ is a subspace of L_y, is connected, has diameter two, and is 2-closed (i.e. if $m_1,m_2 \in L_y(X_y)$ with $m_1 \notin \Gamma_y(m_2)$, then $\Gamma_y(m_1) \cap \Gamma_y(m_2) \subseteq L_y(X_y)$. From this it follows that $L_y(X_y) \cong A_{t',2}(K)$ for some t'. Now let $M \in {}_t\mathcal{P}_{xy}(X)$. Then $M \subseteq X \cap y^\perp = xy \cap x^\perp$. Hence $M \in {}_t\mathcal{P}_{xy}(X_y)$ and so $t \leq t'$. On the other hand, by choosing $M' \in {}_{t'}\mathcal{P}_{xy}(X_y)$ we get $M' \in {}_{t'}\mathcal{P}_{xy}(X)$, and so $t' \leq t$. Thus $t = t'$ and the lemma is proved.

(4.6) <u>DEFINITION</u>. For $(x,X), (y,Y) \in \mathcal{R}_t$, write $(x,X) \sim (y,Y)$ if $(x,X) = (y,Y)$ or if there exists a sequence $\{(x_i,X_i)\}_{i=0}^s \subseteq \mathcal{R}_t$ with $(x_0,X_0) = (x,X)$, $(x_s,X_s) = (y,Y)$ and such that for each i, $x_{i+1} \in X_i$ and $(X_i)_{x_{i+1}} = X_{i+1}$.

Suppose $(x,X) \in \mathcal{R}_t, y \in \mathcal{P}$, and $\pi = (x_0,x_1,\ldots,x_s)$ a path from x to y. We shall say X_π is *defined* if there exists a sequence $\{(x_i,X_i)\}_{i=0}^s$ in $\mathcal{R}_t$ such that $x_i \in X_i$ and $(X_i)_{x_{i+1}} = X_{i+1}$. When X_π is defined each X_i is uniquely determined and we set $X_\pi = X_s$.

(4.7) LEMMA. *Let* $(x,X) \in \mathcal{R}_t, y \in x^\perp \backslash X$. *Set* $Y = \cup_{z \in X \backslash y^\perp} [S(y,z) \cap y^\perp]$.
(i) *If* $X \cap y^\perp = \{x\}$, *then* $(y,Y) \in \mathcal{R}_{t+2}$.
(ii) *If* $X \cap y^\perp \supsetneq \{x\}$, *then* $(y,Y) \in \mathcal{R}_{t+1}$.

PROOF. In either case $Y = (\bar{X})_y$ where $\bar{X} = \langle X,y \rangle$. In (i) clearly $(x,\bar{X}) \in \mathcal{R}_{t+2}$ and in (ii) $(x,\bar{X}) \in \mathcal{R}_{t+1}$. The result follows from (4.5).

(4.8) LEMMA. *Let* $(x,X) \in \mathcal{R}_t, y \in X - \{x\}$. *Then* $X = (X_y)_x$.

PROOF. Since X, X_y are isomorphic it suffices to prove $X \subseteq (X_y)_x$. Let $u \in X$. If $u \in y^\perp$, then $u \in X_y$. Then since $u \in X_y \cap x^\perp$, $u \in (X_y)_x$. Thus assume $u \in \Gamma_2(y)$. Then $S(u,y) \cap y^\perp \subseteq X_y$, $x \in S(u,y) \cap y^\perp$, but $S(y,y) \cap y^\perp \subseteq x^\perp$. Choose $v \in S(u,y) \cap y^\perp$, $v \in \Gamma_2(x)$. Then $v \in X_y$ and $S(x,v) \cap x^\perp \subseteq (X_y)_x$. But $S(x,v) = S(u,y)$ and hence $u \in S(x,v) \cap x^\perp \subseteq (X_y)_x$.

(4.9) LEMMA. *Let* $(x,X) \in \mathcal{R}_t$, $a,b \in X - \{x\}$ *with* $b \in a^\perp$. *Then* $(X_a)_b = X_b$.

PROOF. Since $(X_a)_b$, $X_b \in (\mathcal{R}_t)_b$, it suffices to prove $X_b \subseteq (X_a)_b$. Let $d \in X - b^\perp$, $c \in S(b,d) \cap b^\perp$. Suppose first that $d \in a^\perp$. Then $d \in X_a$ and then $S(b,d) \cap b^\perp \subseteq (X_a)_b$. Thus we may assume $d \in \Gamma_2(a)$.

Since $(X_a)_b$ is a subspace it suffices to show $bc \cap (X_a)_b \neq \{b\}$. Since $b \in \Gamma_2(d)$ and $d^\perp \cap bc \neq \emptyset$, we may assume $c \in d^\perp$. Suppose $cd \cap a^\perp \neq \emptyset$. If $a \in c^\perp$, then $c \in X_a$ and then $c \in X_a \cap b^\perp \subseteq (X_a)_b$. Thus we may assume $c \in \Gamma_2(a)$. Let $c' = cd \cap a^\perp$. Then $c' \in S(a,d) \cap a^\perp \subseteq X_a$ and $c' \in \Gamma_2(b)$. Then $S(b,c') \cap b^\perp \subseteq (X_a)_b$ and this implies $c \in (X_a)_b$. Thus we may assume $cd \subseteq \Gamma_2(a)$.

Suppose now that $x \in c^\perp$. Then $x \in (cd)^\perp \cap a^\perp$. Therefore $a^\perp \cap (cd)^\perp \in {}_2\mathcal{P}(\{a,c\}^\perp)$, and so $a^\perp \cap (cd)^\perp$ is maximal in $\{a,c\}^\perp$. Therefore there is an $e \in a^\perp \cap (cd)^\perp \cap \Gamma_2(b)$. Note $e \in x^\perp$ since $a^\perp \cap (cd)^\perp$ contains x. Since $e \in X \cap a^\perp$, $e \in X_a$. Thus $S(b,e) \cap b^\perp \subseteq (X_a)_b$. However, $c \in b^\perp \cap e^\perp$, so $c \in (X_a)_b$.

Therefore we may assume $x \notin (cd)^\perp$. In particular $c \in \Gamma_2(x)$. Now note that $S(b,d) \supseteq dc$ and $S(b,d) \cap a^\perp \supseteq bx$. Then by (4.3) $S(b,d) \cap a^\perp \in {}_3\mathcal{P}$. If $M = S(b,d) \cap a^\perp$, then $M \cap (cd)^\perp \neq \emptyset$, and hence $(cd)^\perp \cap a^\perp \neq \emptyset$, and hence by (D_2), $a^\perp \cap (cd)^\perp \in {}_2\mathcal{P}$. Set $a^\perp \cap (cd)^\perp = N$. $x,b \notin N$. However, N is maximal in $\{a,c\}^\perp$ and $b \in \{a,c\}^\perp \backslash N$. Therefore, there is an $e \in N \backslash b^\perp$. Now $e \in S(a,d) \cap a^\perp$, so $e \in X_a$. $c \in S(b,e) \cap b^\perp$, so $c \in (X_a)_b$ and we have shown $X_b \subseteq (X_a)_b$.

(4.10) LEMMA. (i) *Suppose* $d(x,y) = k \geq 1$. *Then* $y^{\perp} \cap \Gamma_{k-1}(x)$, *together with its lines is isomorphic to* $A_{2k-1,2}(K)$.
(ii) *If* $\ell \subseteq \Gamma_k(x)$, *then* $\ell^{\perp} \cap \Gamma_{k-1}(x) \in {}_{2k-2}\mathcal{P}$.

PROOF. We first show that (ii) is a consequence of (i). Choose $y \in \ell$. By (i) $y^{\perp} \cap \Gamma_{k-1}(x) \cong A_{2k-1,2}(K)$. Set $Y = \langle y^{\perp} \cap \Gamma_{k-1}(x), y\rangle$ so $(y,Y) \in \mathcal{R}_{2k-1}$ and consider L_y, $L_y(Y)$ and ℓ. Now either $\Gamma_y(\ell) \cap L_y(Y) = \emptyset$ or $\Gamma_y(\ell) \cap L_y(Y)$ is a maximal singular subspace of $L_y(Y)$. Thus, either $\ell^{\perp} \cap \Gamma_{k-1}(x) = \emptyset$ or $\ell^{\perp} \cap \Gamma_{k-1}(x) \in {}_{2k-2}\mathcal{P}$. Since $\ell^{\perp} \cap \Gamma_{k-1}(x) \neq \emptyset$ by (D2) and (D3), (ii) now follows.

We prove (i) by induction on $k \geq 1$. (i) is obvious for $k = 1$ and 2. Thus assume (i) is true for $k = t \geq 2$ and suppose $k = t + 1$. Now let $a \in y^{\perp} \cap \Gamma_t(x)$. By induction $a^{\perp} \cap \Gamma_{t-1}(x) \cong A_{2t-1,2}(K)$. Set $A = \langle a, a^{\perp} \cap \Gamma_{t-1}(x)\rangle$, so $(a,A) \in \mathcal{R}_{2t-1}$. Note that for $\ell \in L_a(A)$, $\ell \cap \Gamma_{t-1}(x)$ is a point. Since $y \in \Gamma_{t+1}(x)$, $A \cap y^{\perp} = \{a\}$. Now let $b \in A - \{a\}$, so $b \in \Gamma_2(y)$. Let $c \in S(b,y) \cap y^{\perp}$. Then $yc \cap b^{\perp} \neq \emptyset$, and if $c' \in yc \cap b^{\perp}$, then $c' \in \Gamma_t(x) \cap y^{\perp}$. Thus, if $\ell \in L_y(S(b,y))$, then ℓ contains a unique point in $y^{\perp} \cap \Gamma_t(x)$. Now by (4.7), if $Y = \cup[S(b,y) \cap y^{\perp}]$, $b \in A - \{a\}$ then $(y,Y) \in \mathcal{R}_{2t+1}$. Since each $\ell \in L_y(Y)$ contains a unique point in $y^{\perp} \cap \Gamma_t(x)$, if we set $Z = Y \cap \Gamma_t(x)$, then $Z \cong A_{2t+1,2}(K)$.

We next show that $W = \Gamma_t(x) \cap y^{\perp}$ is a subspace. Suppose $u,v \in \ell \cap W$. Then either $\ell \subseteq W$ or there is a unique point $w \in \ell \cap \Gamma_{t-1}(x)$. But then $d(x,y) \leq d(x,w) + d(w,y) = t - 1 + 1 = t$, a contradiction. Now suppose $a,b \in W$, $d(a,b) = 2$, $c \in \{a,b\}^{\perp} \cap y^{\perp}$. Claim $yc \cap W \neq \emptyset$. If $yc \cap W = \emptyset$, then $yc \subseteq \Gamma_{t+1}(x)$. Then by (D3), $(yc)^{\perp} \cap \Gamma_t(x) \in \underline{\underline{\mathrm{Sing}}}$. However, $a,b \in (yc)^{\perp} \cap \Gamma_t(x)$ and $b \notin a^{\perp}$, a contradiction. Thus $yc \cap W \neq \emptyset$. It follows that $\{a,b\}^{\perp} \cap W$ is a non-degenerate generalized quadrangle, and therefore that $W \cong A_{s,2}(K)$ for some $s \geq 2t+1$ (since $W \supseteq Z$). Thus to complete the proof it suffices to prove $s = 2t+1$.

Now let $a \in W$ and $m \in L_a(W)$. Then $m \subseteq \Gamma_t(x)$ and therefore by induction $m^{\perp} \cap \Gamma_{t-1}(x) \in {}_{2t-2}\mathcal{P}(\mathcal{U})$, $U = \Gamma_{t-1}(x) \cap a^{\perp}$. Suppose that $m_1, \in L_a(W)$, but $m_1 \subseteq m_2^{\perp}$. Then $m_1^{\perp} \cap \Gamma_{t-1}(x) \neq m_2^{\perp} \cap \Gamma_{t-1}(x)$. For suppose on the contrary, $m_1^{\perp} \cap \Gamma_{t-1}(x) = m_2^{\perp} \cap \Gamma_{t-1}(x) = M$. Let $b_i \in m_i$, $i = 1,2$. Then $y, M \subseteq b_1^{\perp} \cap b_2^{\perp}$. Then $\emptyset \neq y^{\perp} \cap M \subseteq \Gamma_{t-1}(x) \cap y^{\perp} = \emptyset$, a contradiction.

Next suppose $m_1, m_2 \in L_y(W)$ and $m_1^{\perp} \cap \Gamma_{t-1}(x) = m_2^{\perp} \cap \Gamma_{t-1}(x) = M$. Then by the previous paragraph $m_2 \subseteq m_1^{\perp}$. Set $N = \langle m_1, m_2 \rangle$. Claim $N^{\perp} \cap W = N$. Since $W \cong A_{s,2}(K)$ if $V \in \underline{\underline{V}}(W)$, then $V^{\perp} \cap W \in \underline{\underline{\text{Sing}}}$ and either $V^{\perp} \cap W \in {}_{s-1}P$ or $V^{\perp} \cap W = V$. Suppose $N^{\perp} \cap W \in {}_{s-1}P$. Since $N \in \underline{\underline{V}}$, N lies in two maximal singular subspaces, one of rank 3 and one of rank n. Since $M \subseteq \Gamma_2(y)$, $y \notin \langle M,N\rangle^{\perp}$. Since $\mathrm{rk}(\langle M,N\rangle) \geq 4$, it follows that $\mathrm{rk}(\langle M,N\rangle^{\perp}) = n$. $\langle y,N\rangle$ is a singular subspace of rank three on N and $\langle y,N\rangle \cap \langle M,N\rangle^{\perp} = N$. Therefore $\langle y,N\rangle^{\perp} = \langle y,N\rangle$. However, we are assuming $\mathrm{rk}(N^{\perp} \cap W) = s-1$. $N^{\perp} \cap W \subseteq y^{\perp}$. Then $\langle y, N^{\perp} \cap W\rangle$ is a singular subspace, $\langle y, N^{\perp} \cap W\rangle \supseteq \langle y,N\rangle$ and $\mathrm{rk}(\langle y, N^{\perp} \cap W\rangle) = s \geq 2t + 1 \geq 5$, a contradiction.

Thus, if $m_1^{\perp} \cap \Gamma_{t-1}(x) = m_2^{\perp} \cap \Gamma_{t-1}(x)$, then $\langle m_1, m_2\rangle^{\perp} \cap W = \langle m_1, m_2\rangle$. Suppose now $m_1, m_2 \in L_y(W)$, $m_2 \subseteq m_1^{\perp}$ and $\langle m_1,m_2\rangle^{\perp} \cap W = \langle m_1,m_2\rangle$. Set $M_i = m_i^{\perp} \cap \Gamma_{t-1}(x)$. We prove $N_1 = N_2$. Let $n \in L(N_1)$. Then $n \subseteq \Gamma_2(y)$, but $m_1^{\perp} \subseteq n^{\perp} \cap y^{\perp}$. However, $\mathrm{rk}(n^{\perp} \cap y^{\perp}) = 2$ and $n^{\perp} \cap y^{\perp} \subseteq W$. Also, from the type of L_a we see that $\langle y, n^{\perp} \cap y^{\perp}\rangle \in P^{-}$. Therefore $n^{\perp} \cap y^{\perp}$ is a maximal singular plane of W. However, each line of W lie in a unique singular plane of W which is maximal in W. Since $m_1 \subseteq \langle m_1,m_2\rangle$ and $\langle m_1,m_2\rangle^{\perp} \cap W = \langle m_1,m_2\rangle$, it follows that $\langle m_1,m_2\rangle = n^{\perp} \cap y^{\perp}$. Now $N_1 = m_1^{\perp} \cap n^{\perp} \supseteq m_2$. Therefore $N_1 \subseteq N_2$. Since $\mathrm{rk}(N_1) = \mathrm{rk}(N_2)$, $N_1 = N_2$ as claimed.

Now we have shown there is an injective map Φ from $V_{max}(W)$ $\{V \in \underline{\underline{V}}(W) : V^{\perp} \cap W = W\}$ into ${}_{2t-2}P(U)$. Now for $V_1, V_2 \in V_{max}(W)$, define $\Delta(V_1,V_2) = \{W \in V_{max}(W) : V_i \cap W \in L_a,\ i = 1,2\}$. Set $\lambda(V_1,V_2) = \{V \in V_{max}(W) : V \cap V' \in L_a$, for every $V' \in \Delta(V_1,V_2)\}$. If we set $\Lambda = \{\lambda(v_1,v_2) : V_1 \neq V_2 \in V_{max}(W)\}$, then $(V_{max}(W), \Lambda) \cong PG(s-2,K)$. Now ${}_{2t-2}P(U)$ is naturally isomorphic to $PG(2t-1,K)$. We finally show that Φ is a morphism of projective spaces. Since Φ is injective this will imply $s - 2 \leq 2t - 1$ from which we deduce $s \leq 2t + 1$ as desired.

Let $\lambda = \lambda(v_1 v_2) \in \Lambda$. Set $M_i = v_i^{\perp} \cap \Gamma_{t-1}(x)$. Then $M_1 \cap M_2 = \{u\}$ is a point. Then $\{y,u\}^{\perp} \subseteq W$, $\{y,u\}^{\perp} \cong A_{3,2}(K)$ and $a \in \{y,u\}^{\perp}$. It is clear to see that $V_{max}(W) \cap \underline{\underline{V}}_a(\{y,u\}^{\perp}) = \lambda(v_1,v_2)$ and from this our claim now follows.

(4.11) <u>NOTATION</u>. If $d(x,y) = k \geq 2$, set $R(x,y) = \langle x^{\perp} \cap \Gamma_{k-1}(y), x\rangle$. (So $(x, R(x,y)) \in R_{2k-1}$).

(4.12) <u>LEMMA</u>. *Let* $d(x,y) = k \geq 2$ *and* *be a geodesic from* x *to* y. *If* $X = R(x,y)$, *then* X_{π} *is defined. Moreover,* $X_{\pi} = R(y,x) = \langle y, y^{\perp} \cap \Gamma_{k-1}(x)\rangle$.

PROOF. Induction on $k \geq 2$. Suppose $k = 2$. Then $X = R(x,y) = x^{\perp} \cap S(x,y) = \langle x,\{x,y\}^{\perp}\rangle$. For $z \in \{x,y\}^{\perp}$, $X_z = z^{\perp} \cap S(x,y)$ and $y \in X_z$. Thus, if $\pi = (x,z,y)$, then X_π is defined and $X_\pi = (X_z)_y = S(x,y) \cap y^{\perp} = \langle y,\{x,y\}^{\perp}\rangle = R(y,x)$.

Assume now that the result is true for all $k \leq t$ and let $k = t+1$. Let $\pi = (x = x_0, x_1, \ldots, x_{t+1} = y)$ be a geodesic path from x to y. Set $x_1 = a$. We show that $A = R(a,y) = \langle a, \Gamma_{t-1}(y) \cap a^{\perp}\rangle \subseteq X_a$. Of course it suffices to show $\Gamma_{t-1}(y) \cap a^{\perp} \subseteq X_a$ since $X_a \in \underline{\underline{\mathrm{Sub}}}_a$. Let $b \in \Gamma_{t-1}(x) \cap a^{\perp}$, $c \in \{x,b\}^{\perp}$. Then $d(c,y) = t$ and $c \in \Gamma_t(y) \cap x^{\perp}$. Choose $c \in \Gamma_2(a)$. $c \in X = R(x,y)$ and $a^{\perp} \cap S(a,c) \subseteq X_a$. However, $S(a,c) = S(x,b)$ and hence $b \in X_a$. Now if $\rho = (a = x_1, x_2, \ldots, x_t = y)$, then by induction A_ρ is defined. Since $A \subseteq X_a$, $(x_a)\rho$ is defined. But $(X_a)\rho = X_\pi$ and hence X_π is defined. Note by induction we also have $X_\pi \supseteq y^{\perp} \cap \Gamma_{t-1}(a)$. However,

$$\bigcup_{a \in x^{\perp} \cap \Gamma_t(y)} [y^{\perp} \cap \Gamma_{t-1}(a)] = y^{\perp} \cap \Gamma_t(x).$$

Therefore, $X_\pi \supseteq \langle y, y^{\perp} \cap \Gamma_t(x)\rangle = R(y,x)$. Since both (y,X_π) and $(y,R(y,x)) \in \mathcal{R}_{2t+1}$ we have $X_\pi = R(y,x)$.

Now let $(x,X) \in \mathcal{R}_t$. suppose $d(x,y) = k > [\frac{t+1}{2}]$. Then $x^{\perp} \cap \Gamma_{k-1}(y) \cong A_{2k-1,2}$. Since $2k-1 > t$, $x^{\perp} \cap \Gamma_{t-1}(y) \not\subseteq X$. We remark that at this point it now follows diam $(P,\Gamma) = [\frac{n+1}{2}]$.

Now set

$$D(x,X) = \bigcup_{k \geq 1} \{y : d(x,y) = k,\ R(x,y) \subseteq X\} \cup \{x\}$$

(4.12) REMARK. $x^{\perp} \cap D(x,X) = X$

(4.13) LEMMA. *Let* $(x,X) \in \mathcal{R}_t$, $y \in X - \{x\}$, $Y = X_y$. Then $D(x,X) = D(y,Y)$.

PROOF. As $Y_x = X$ by (4.8) it suffices to prove $D(y,Y) \subseteq D(x,X)$. Recall

$$X_y = \bigcup_{z \in x - y^{\perp}} [S(y,z) \cap y^{\perp}].$$

Now let $z \in D(y,Y)$ with $d(y,z) = k$. Of course if $z = x$, then $z \in D(x,X)$. This leaves four cases to consider:

(i) $d(x,z) = k-1 \geq 1$;

(ii) $d(x,z) = k+1$;

(iii) $d(x,z) = k$, $d(xy,z) = k-1$;

(iv) $xy \subseteq \Gamma_k(z)$.

(i) Let $u \in x^\perp \cap \Gamma_{k-2}(x)$. Then $u \in \Gamma_2(y)$. If $v \in \{u,y\}^\perp$, then $v \in \Gamma_{k-1}(z) \cap y^\perp$. Thus $\{u,y\}^\perp \subseteq Y$ and hence $y^\perp \cap S(u,y) = \langle y,\{u,y\}^\perp \rangle \subseteq Y$. Now choose $v \in \{u,y\}^\perp \cap \Gamma_2(x)$. Then $S(y,u) = S(x,v)$. Then $x^\perp \cap S(x,v) = x^\perp \cap S(y,u) \subseteq Y_x = X$. Thus $u \in X$.

(ii) Let $u \in \Gamma_k(z) \cap x^\perp$. Suppose $u \in y^\perp$. Then $d(yu,z) = k$. Let $v \in \Gamma_{k-1}(z) \cap (yu)^\perp$. Then $v \in Y \cap \Gamma_2(x)$. $X = Y_x \supseteq x^\perp \cap S(x,v)$ and so $u \in X$. Thus assume $u \in \Gamma_2(y)$. Now let $v \in \{x,u,y\}^\perp$. $d(z,v') \leq k+1$ for each $v' \in xv$ since $v' \in y^\perp$ and $d(y,z) = k$. However, if $d(xv,z) = k+1$, then $(xv)^\perp \cap \Gamma_k(z) \in \underline{\underline{\text{Sing}}}$, contradicting $u,y \in (xv)^\perp \cap \Gamma_k(z)$. Then without loss of generality we may assume $v \in \Gamma_k(z)$. By the first part of this paragraph $v \in X$. Now Rad $(\{x,y,u\}^\perp) = \{x\}$, hence there is a $w \in \{x,y,u\}^\perp \cap \Gamma_2(z)$. Then also $w \in X$. Then $X \supseteq S(v,w) \cap x^\perp$ and so $u \in X$.

(iii) Let $w = xy \cap \Gamma_{k-1}(z)$. Let $u \in \Gamma_{k-1}(z) \cap x^\perp$. If $u \in y^\perp$, then $u \in Y \cap x^\perp \subseteq Y_x = X$. So assume $u \in \Gamma_2(y)$. As in (ii) we can find a,b with $a \in \Gamma_2(b)$, $a,b \in \{x,u,w\}^\perp \cap \Gamma_{k-1}(z)$. Then also $a,b \in y^\perp$ and so $a,b \in \Gamma_{k-1}(z) \cap y^\perp \subseteq Y$. Then $S(a,b) \cap y^\perp \subseteq Y$. As $x \in S(a,b)$ it follows that $S(a,b) \cap x^\perp \subseteq Y_x = X$. Since $u \in a^\perp \cap b^\perp \cap x^\perp$, $u \in X$.

(iv) Let $u \in \Gamma_{k-1}(z) \cap x^\perp$. If $u \in (xy)^\perp$, then $u \in Y \cap x^\perp \subseteq X$. Thus assume $u \in \Gamma_2(y)$. Now $(xy)^\perp \cap \Gamma_{k-1}(Z) \in {}_{2k-2}\mathcal{P}$ and $u \notin (xy)^\perp \cap \Gamma_{k-1}(Z)$. Clearly, we may assume $k > 1$, for otherwise $u = x$. Thus $u^\perp \cap (xy)^\perp \cap \Gamma_{k-1}(z) \in L$. Then we can find $v \in \Gamma_2(u) \cap (xy)^\perp \cap \Gamma_{k-1}(z)$. Let $a \in \{x,u,v\}^\perp$. Since $u \in (a')^\perp \cap \Gamma_{k-1}(z)$ for each $a' \in ax$, $d(z,a') \leq k$. However, if $d(xa,z) = k$ we get a contradiction : $u,v \in (ax)^\perp \cap \Gamma_{k-1}(z) \in \underline{\underline{\text{Sing}}}$. Therefore $d(xa,z) = k-1$, so without loss we may assume $a \in \Gamma_{k-1}(z)$ and $av \subseteq \Gamma_{k-1}(z)$. Let $b \in \Gamma_{k-2}(z) \cap (av)^\perp$. Since $v \in \{y,b\}^\perp$, $d(y,b) = 2$. Since $\{y,b\}^\perp \subseteq \Gamma_{k-1}(z)$, $S(y,b) \cap y^\perp \subseteq Y$. Consequently, $Y_v \supseteq S(y,b) \cap v^\perp$. Since $v \in Y \cap x^\perp$, $v \in X$. Since $b \in S(y,b) \cap v^\perp$, $b \in Y_v$. As $x \in Y \cap v^\perp$, $x \in Y_v$.

Now $d(x,b) = 2$, so $x^\perp \cap S(x,b) \subseteq (Y_v)_x = Y_x = X$ by (4.9). As $a \in \{x,b\}^\perp$, $a \in X$. However, Rad $(\{x,u,v\}^\perp) = \{x\}$, so we can find a $c \in \{x,u,v\}^\perp \cap \Gamma_{k-1}(z)$ with $c \in \Gamma_2(a)$. Then as above, $c \in X$. Then $x^\perp \cap S(a,c) \subseteq X$, and so $u \in \{a,c,x\}^\perp \subseteq S(a,c) \cap x^\perp$.

(4.14) COROLLARY. *Let* $(x,X) \in \mathcal{R}_t$, $y \in D(x,X)$ *and* π *a geodesic from* x *to* y, *then* X_π *is defined* , $X_\pi = D(x,X) \cap y^\perp$ *and if* $Y = X_\pi$, *then* $D(x,X) = D(y,Y)$

PROOF. This follows from (4.13) and induction on $d(x,y)$.

(4.15) REMARK. The corollary implies that $D(x,X) \in \underline{\underline{\mathrm{Sub}}}$ and for any $a,b \in D(x,X)$ and every geodesic path π from a to b is contained in $D(x,X)$. It follows that $D(x,X)$ satisfies the hypotheses of the main theorem. Thus, if $t < n$, then by induction $D(x,X) \cong D_{t+1,t+1}(K)$.

Now set $\overline{P_{t+1}} = \{D(x,X) : (x,X) \in \mathcal{R}_t\}$, $\overline{P} = \overline{P_{n-1}}$. For $D_1, D_2 \in \overline{P}$, define $D_1 \approx D_2$ if and only if $D_1 \cap D_2 \neq \emptyset$.

Now suppose $D_1, D_2 \in \overline{P}$, $D_1 \approx D_2$. Let $x \in D_1 \cap D_2$. By considering L_x, $L_x(D_i)$, $i = 1,2$, we see that $L_x(D_1 \cap D_2) = L_x(D_1) \cap L_x(D_2) \cong A_{n-1,2}$. Since this is true for each $x \in D_1 \cap D_2$ we have

(4.16) LEMMA. *If* $D_1, D_2 \in \overline{P}$, $D_1 \neq D_2$ *and* $D_1 \cap D_2 \neq \emptyset$, *then* $D_1 \cap D_2 \in \overline{P_{n-2}}$.

Now if $D_1 \approx D_2$, set $\ell(D_1,D_2) = \{D \in \overline{P} : D \supseteq D_1 \cap D_2\}$ and $\overline{L} = \{\ell(D_1,D_2) : D_1, D_2 \in \overline{P}$, $D_1 \approx D_2\}$. Thus we have an incidence structure $(\overline{P},\overline{L})$.

(4.17) LEMMA. *Let* $D \in \overline{P}$, $x \in P - D$. *If* $\Gamma_2(x) \cap D \neq \emptyset$, *then* $x^\perp \cap D \neq \emptyset$.

PROOF. Let $w \in \Gamma_2(x) \cap D$. $L_w(D) \cong A_{n-1,2}(K)$, $L_w(S(x,w)) \cong A_{3,2}$, let $\pi_w(D)$ be the hyperplane of π_w underlying $L_w(D)$ and $\pi_w(x)$ the three subspace underlying $L_w(S(x,w))$. Then $\pi_w(x)$ meets $\pi_w(D)$ in a least a plane so $L_w(D) \cap L_w(S(x,w))$ a contains a singular plane of L_w. Therefore ${}_3\mathcal{P}(S(x,w) \cap D) \neq \emptyset$. If $M \in {}_3\mathcal{P}(S(x,w) \cap D)$, then $M \cap x^\perp \in \underline{\underline{V}}(D)$, in particular $D \cap x^\perp \neq \emptyset$ as claimed.

(4.18) LEMMA. *If* $D \in \overline{P}$, $x \in P - D$, *then* $x^\perp \cap D \neq \emptyset$.

PROOF. Set $s = d(D,x)$. Wish to prove $s = 1$. Suppose on the contrary that $s > 1$. Choose $z \in D$ with $d(x,z) = s$ and let $x = x_0, x_1, \ldots, x_s = z$ be a geodesic from x to z. Let $y = x_{s-2}$. Then $d(x,y) = s-2$.
Since $d(s,x) = s$, $y \in P - D$. Since $\Gamma_2(y) \cap D \neq \emptyset$, by (4.17) $y^\perp \cap D \neq \emptyset$. If $w \in y^\perp \cap D$, then $w \in D$ and $d(x,w) \leq s-1$, a contradiction. Therefore $s = 1$.

(4.19) NOTATION. For $x \in P$, $\hat{x} = \{D \in \overline{P} : x \in D\}$. For $D \in \overline{P}$, $\Delta(D) = \{D' : D \approx D'\}$.

(4.20) LEMMA. $\hat{x}$, *together with its lines, is a projective space of rank* n *over* K.

PROOF. Clearly $\hat{x}$ is a singular subspace of $(\overline{P},\overline{L})$. We define a map from $\hat{x}$ to $\{X : (X : (x,X) \in R_{n-1}\}$ by $D \mapsto D \cap x^\perp$. Suppose $D_1, D_2 \in \hat{x}$. Then this map carries $\lambda(D_1,D_2)$ to $\{X : (x,X) \in R_{n-1}, X \supseteq D_1 \cap D_2 \cap x^\perp\}$. However, $(x, D_1 \cap D_2 \cap x^\perp) \in R_{n-2}$. Then $\hat{x}$, together with its lines is isomorphic to the incidence structure whose points are the hyperplanes of Π_x ($\cong PG(n,K)$) and lines are the subspaces of codimension two with inclusion as incidence. This is of course a projective space of rank n over K as claimed.

(4.21) LEMMA. *Suppose* $x \notin D \in \overline{P}$. *Then* $\hat{x} \cap \Delta(D)$ *is a hyperplane of* $\hat{x}$.

PROOF. We know $D \cap x^\perp \neq \emptyset$. Since D is geodisically closed, $x^\perp \cap D \in$ Sing. Let $y \in D \cap x^\perp$, $\pi_y(D)$ the hyperplane of π_y underlying $L_y(D)$. The line which xy is identified with meets $\pi_y(D)$. Then $\Gamma_y(xy) \cap L_y(D)$ is a singular subspace of L_y of rank n - 2 and therefore $rk(D \cap x^\perp \cap y^\perp) = n - 1$. Since $y \in D \cap x^\perp \in$ Sing, $D \cap x^\perp = D \cap x^\perp \cap y^\perp$. Set $N = D \cap x^\perp$. $rk(\langle N,x \rangle) = n$, and so $M = \langle N,x \rangle \in P^+ = {}_nP$. Then $L_x(M)$ is a maximal singular subspace of rank n - 1 and consists of all lines of Π_x lying on a point Π_D of Π_x. Now suppose $D' \in \hat{x}$ and $D \cap D' \neq \emptyset$. Then $D \cap D' \in \overline{P}_{n-2}$ and $x \in D' - (D \cap D')$. By the above $K = D \cap x^\perp \in {}_{n-2}P$ and $rk(\langle P \cap D' \cap x^\perp, x \rangle) = n - 1$. Set $K = \langle D \cap D' \cap x^\perp, x \rangle$, $L_x(K)$ is a singular subspace of L_x of rank n - 2. If $\Pi_x(D')$ is the hyperplane of Π_x corresponding to $L_x(D')$, then $\Pi_x(D')$ contain Π_D. It now follows that $\Delta(D) \cap \hat{x} = \{D' \in \hat{x} : \Pi_k(D') \supseteq \Pi_D\}$ and this is a hyperplane of $\hat{x}$.

The next two results finish the proof.

(4.22) PROPOSITION. $(\overline{P},\overline{L})$ *is a thick, non-degenerate polar space,* $D_{n+1}(K)$.

PROOF. Clearly $(\overline{P},\overline{L})$ is thick. Let $\lambda = \lambda(D_1,D_2) \in \overline{L}$, $D \in \overline{P}$. Let $x \in D_1 \cap D_2$. If $x \in D$, then $\lambda \subseteq \Delta(D)$, so assume $x \notin D$. Then $\Delta(D) \cap \hat{x}$ is a hyperplane of $\hat{x}$ by (4.2), in particular either $\lambda \subseteq \Delta(D)$ or $|\lambda \cap \Delta(D)| = 1$. Thus $(\overline{P},\overline{L})$ is a polar space. Now suppose $D \in \overline{P}$. If $y \in D$, then $L_y(D) \cong A_{n-1,2}(K)$. Since $L_y \cong A_{n,2}(K)$, $y^{\perp} \nsubseteq D$, so $D \neq P$. If $x \in P - D$, then by (4.21) $\hat{x} \nsubseteq \Delta(D)$, so $D \notin \mathrm{Rad}(\overline{P})$ and as D was arbitrary, $\mathrm{Rad}(\overline{P}) = \emptyset$. Also by (4.21), $\hat{x}$ is a maximal singular subspace of $(\overline{P},\overline{L})$ and so by (4.20), $\mathrm{rk}(\overline{P},\overline{L}) = n+1$.
To see that this is of type D it suffices to show that the residue at a point D of $\overline{P}$, $\overline{L}_D$, is of type D. The map

$$\lambda \longmapsto \bigcap_{D' \in \lambda} D' \quad \text{from } \overline{L}_D \text{ to } \overline{P}_{n-1}(D) \text{ is a bijective morphism}$$

(lines of $\overline{L}_D$ go to $\overline{P}_{n-2}(D)$, and the latter is a polar space $D_n(K)$. This completes the proposition.

THEOREM. $(P,L) \cong D_{n+1,n+1}(K)$

PROOF. The map $x \longmapsto \hat{x}$ is a map from P onto a subset of the maximal singular subspaces of $(\overline{P},\overline{L})$. Now if $\ell \in L_x$, then $\ell = \cap_{y\in\ell}\, \hat{y}$ is easily seen to have rank $k-1$ by passing to $L_x(D \in \hat{\ell}$ if and only if the hyperplane $\pi_x(D)$ contains the line "xy" of π_x). From this it follows that $\{\hat{x} : \hat{x} \in P\}$ is contained in a single class and $y \in x^{\perp}$ if and only if $\mathrm{rk}(\hat{x} \cap \hat{y}) = \mathrm{rk}(\hat{x}) -2 = \mathrm{rk}(\hat{y})-2$. Since $L_x \cong A_{n,2}(K)$ it follows that $\{\hat{x} : x \in P\}$ is an entire class and the proof is complete.

5. NEAR 2n-Gons

In this section we recall the definition of a near 2n-gons as introduced by SHULT and YANUSHKA [8], and some related notions.

(5.1) DEFINITION. An incidence structure (P,L) with point-graph (P,Δ) and metric $d(\ ,\ ) = d_{\Delta}(\ ,\ )$ is a near 2n-gon if (P,Δ) is connected with diameter n and for any pair $(x,\ell) \in P \times L$ with $d(x,\ell) = t$, there is a unique $y \in \ell$ with $d(x,y) = t$.

$y \in \ell$ with $d(x,y) = t$.

(5.2) REMARK. If (P,Δ) is a bipartite graph, then (P,Δ) is a near 2n-gon for some n. In this case lines all have two points. Conversely, a near 2n-gon with two points on each line is bipartite graph. We will refer to such near-2n-gons as thin.

(5.3) NOTATION. For $x \in P$, $\Delta(x)$ is as usual and $x^{\perp} = \Delta(x) \cup \{x\}$.

(5.4) DEFINITION. A subset X of P is 2-closed if, whenever $x, y \in X, d(x,y) = 2$, then $x^{\perp} \cap y^{\perp} \subseteq X$.

(5.5) DEFINITION. In a near 2n-gon, a quad is a subset Q of P satisfying

(i) Q is 2-closed

(ii) diam $(Q,\Delta|Q) = 2$

(iii) Q contains an ordinary quadrangle

Note a quad, together with its lines is a generalized quadrangle.

(5.6) DEFINITION. (i) In a near 2n-gon (P,L) we say quads exists if whenever $d(x,y) = 2$ there exists a quad containing x and y.

(ii) Let $x \in P$, Q a quad of (P,L). The pair (x,Q) is classical if there is a unique point $y \in Q$ with $d(x,Q) = d(x,y) = d$ and $\{z \in Q : d(x,z) = d+1\} = Q \cap y^{\perp}$.

(5.7) DEFINITION. A dual polar space is the incidence structure whose points are the maximal isotropic (singular) subspaces of a non-degenerate polar space and whose lines are the next to maximal isotropic subspaces.

Note when the polar space is of type D_n the near 2n-gon is thin.

Cumeron has the following characterization of dual polar spaces [9] .

(5.8) THEOREM. *An incidence structure* (P,L) *is a dual polar space of rank* n *if and only if the following hold*

(i) (P,L) *is a near* 2n-gon;

(ii) *quads exist;*

(iii) *every point-quad pair is classical.*

We give a proof of this in the case that (P,L) is thin using our main theorem. More precisely we prove.

(5.9) THEOREM. *Let* (P,Δ) *be a connected bipartite graph of diameter* $n \geq 3$. *Further assume*

(i) *If* $d(x,y) = 2$, *then* $|x^\perp \cap y^\perp| > 2$;

(ii) *In the near* 2n-gon (P,Δ) *quads exist and all point-quad pairs are classical.*

Then one of the following occurs

(i) $n = 3$, *there is a* skew *field* K *such that* (P,Δ) *is the dual polar space of type* $D_3(K)$; *or*

(ii) $n \geq 4$, *there is a field* K *such that* (P,Δ) *is the dual polar space of type* $D_n(K)$.

6. CHARACTERIZATION OF THIN CLASSICAL NEAR 2n-GONS

As usual $\Delta_i(x) = \{y : d(x,y) = i\}$. Let $P = \mathcal{P}_1 \cup \mathcal{P}_2$ be the partition of P as the connected components of Δ_2. If $x,y \in \mathcal{P}_i$ and $d(x,y) = 2$, then there is a unique quad on x and y which we denote by Q(x,y). Let $\mathcal{Q}$ be the collection of quads.

6.A. In this subsection we assume n = 3 and show conclusion (i) if (5.8) holds

(6.1) LEMMA. *Suppose* $Q_1, Q_2 \in \mathcal{Q}$, $Q_1 \neq Q_2$ *and* $Q_1 \cap Q_2 \neq \emptyset$. *Then* $Q_1 \cap Q_2 \in \Delta$.

PROOF. Let $x \in Q_1 \cap Q_2$. Suppose $x \in \mathcal{P}_1$. Choose $u_i \in Q_i \cap \Delta_2(x) = Q_i \cap \mathcal{P}_1$. Then $d(u_1,u_2) = 2$. Set $Q = Q(u_1,u_2)$. Now $x \notin Q$ for otherwise $Q = Q_1 = Q_2$. Therefore, the unique point $v \in Q$ with $d(v,x) = d(Q,z)$ is in $\mathcal{P}_2$ and $d(v,x) = 1$. Then $v \in x^\perp \cap u_i^\perp \subseteq Q_i$ and $\{x,v\} \in \Delta$. If $Q_1 \cap Q_2 \supsetneq \{x,v\}$, then either $|Q_1 \cap Q_2 \cap \mathcal{P}_1| > 1$ or $|Q_1 \cap Q_2 \cap \mathcal{P}_2| > 1$. In either case we get $Q_1 = Q_2$, a contradiction.

We shall for the remainder of this subsection say two distinct quads are "collinear" if they meet. If Q_1, Q_2 are collinear, let $\lambda(Q_1,Q_2) = \{Q \in \mathcal{Q} : Q \supseteq Q_1 \cap Q_2\}$. Let $\Lambda = \{\lambda(Q_1,Q_2) : Q_1 \neq Q_2 \in \mathcal{Q}, Q_1 \cap Q_2 \neq \emptyset\}$. We immediately have

(6.2) LEMMA. $(\mathcal{Q},\Lambda)$ *is a partial linear space.*

Note that lines are in one-to-one correspondence with edges in Δ. For such an edge, $\{x,a\}$, we will write $\lambda\{x,a\}$ for the corresponding line. The next lemma gives a concrete description of this line.

(6.3) LEMMA. *In* $\{x,a\} \in \Delta$, $\lambda\{x,a\} = \{Q(x,y)\ y \in \Delta(a) - \{x\}\}$

PROOF. If $y \in \Delta(a), y \neq x$, then $Q(x,y) \supseteq \{x,a\}$ and $Q(x,y) \in \lambda\{x,a\}$. On the other hand, if $Q \in \lambda\{x,a\}$, then for any $y \in Q \cap \Delta_2(x)$, $y \in \Delta(a)$ and $Q = Q(x,y)$.

(6.4) PROPOSITION. $(\mathcal{Q},\Lambda)$ *is a polar space of type* D_3.

PROOF. First we show $(\mathcal{Q},\Lambda)$ is a gamma space : let $\lambda = \lambda\{x,a\}$ for $\{x,a\} \in \Delta$ and $Q \in \mathcal{Q}$. If $Q \cap \{x,a\} \neq \emptyset$, then Q is collinear with each point of λ so we may assume $Q \cap \{x,a\} = \emptyset$. We show in this case Q is collinear with at most one point of λ. Suppose $Q \in \lambda$, $Q \cap Q_1 \neq \emptyset$. Let $Q \cap Q_1 = \{y,b\}$ where $\{a,y\}, \{b,x\} \in \Delta$. Suppose that $Q_1 \neq Q_2 \in \lambda$. Then $y \notin Q_2$, but $a \in Q_2 \cap \Delta(y)$. If $Q \cap Q_2 \neq \emptyset$, then $Q_2 \cap \Delta(y) \in Q$. Since $a \in Q$ we cannot have $Q \cap Q_2 \neq \emptyset$ as asserted. Thus $(\mathcal{Q},\Lambda)$ is a gamma space. Now consider a line $\lambda = \lambda\{x,a\}$ and a point $Q \in \mathcal{Q} \setminus \lambda$. Since diam $(P,\Gamma) = 3$, $Q \cap \Delta(a) \neq \emptyset$. By (6.3) this implies Q is collinear with some point of L and consequently $(\mathcal{Q},\Lambda)$ is a polar space. Since the induced structure on the lines of $(\mathcal{Q},\Lambda)$ contains a fixed Q is isomorphic to the dual of Q it follows from TITS [5] $(\mathcal{Q},\Gamma) \cong D_3(K)$, K a

Now it is obvious to see that for $x \in P$, $\hat{x} = \{Q \in \mathcal{Q} : x \in Q\}$ is a maximal singular subspace of the polar space $(\mathcal{Q},\Lambda)$. The result in this case follows.

6.B. Hence-forth assume $n \geq 4$. Set $\mathcal{P} = \mathcal{P}_1$ and $\Gamma = \Delta_2|\ \mathcal{P}$.

(6.5) NOTATION. If $x,y \in \mathcal{P}$, $d(x,y) = 2$(so $d_\Gamma(x,y) = 1$), set $xy = Q(x,y) \cap \mathcal{P}$. Set $\mathcal{L} = \{xy : x,y \in \mathcal{P}, d_\Gamma(x,-) = 1\}$. For $x \in \mathcal{P}$, $x^* = \Gamma(x) \cup \{x\}$.

(6.6) LEMMA. $(\mathcal{P},\mathcal{Q})$ *is a strong* Γ*-space.*

PROOF. Let $x,y,z \in \mathcal{P}$ with $y \in \Gamma(x)$, $x,y \in \Gamma_d(z)$. Set $Q = Q(x,y)$. Let $a \in Q$

$d_\Delta(z,Q) = d_\Delta(z,a)$. If $a \in P$, then $d_\Delta(z,x) - 2 = 2d-2$. In this case $\{a\} = \ell \cap \Gamma_{d-1}(z)$. If $a \in P_2$, then $d_\Delta(z,a) = 2d-1$ and $xy = P \cap Q = P \cap \Delta(a) \subseteq \Delta_{2d}(z) = \Gamma_d(z)$, and so in this case $xy \subseteq \Gamma_d(z)$.

(6.7) <u>LEMMA</u>. *Let* $\ell \in L$, $x \in P$ *and* $\ell \subseteq \Gamma_d(x)$ *with* $d \geq 2$. *Then* $\ell^* \cap \Gamma_{d-1}(x)$ *is a non-empty singular subspace of* $(P,)$. $(\ell^* = \bigcap_{y\in\ell} y^*)$.

<u>PROOF</u>. Note, if $a \in P_2$, then $\Delta(a)$ is a singular subspace of (P,L). By definition of quads, there is a unique $Q \in L$, $Q \supseteq \ell$, which we denote by $Q(\ell)$. Let $a \in Q$ such that $d_\Delta(x,a) = d_\Delta(x,Q)$. Since $\ell \subseteq \Gamma_d(x) = \Delta_{2d}(x)$, $a \in P_2$. Therefore $d_\Delta(a,x) = 2d-1$. Choose $y \in \Delta(a) \cap \Delta_{2d-2}(x)$. Then $y \in \ell^*$ since $y, \ell \subseteq \Delta(a)$. Also $y \in \Gamma_{d-1}(x)$, so $\Gamma_{d-1}(x) \cap \ell^* \neq \emptyset$.

We next show for any $y \in \Gamma_{d-1}(x) \cap \ell^*$ that $y \in \Delta(a)$ which will prove $\Gamma_{d-1}(x) \cap \ell^*$ is a Γ-clique by our first remark. Let $u,v \in \ell$. Consider $Q(y,u)$. Now $\Delta(y) \cap \Delta(u) \subseteq \Delta_{2d-1}(x)$. If $v \in Q(y,u)$, then $Q(y,u) = Q(u,v) = Q(\ell)$ contradicting $d_\Delta(x,y) = 2d-2$ and $Q \cap P \subseteq \Gamma_d(x)$. Therefore $d_\Delta(Q(y,u),v) \geq 1$. But $d_\Delta(y,v) = d_\Delta(y,u) = 2$ and so it follows that if b is the unique point of $Q(x,u)$ closest to v, then $b \in P_2$ and $d_\Delta(b,v) = 1$.
Since $b \in \Delta(y)$, $d_\Delta(b,x) \leq 2d-1$. Since $b \in \Delta(u) \cap \Delta(v)$, $b \in Q(u,v) = Q$. But $Q \cap \Delta_{2d-1}(x) = \{a\}$, so $b = a$. Since (P,Q) is a strong Γ-space $\Gamma_{d-1}(x) \cap \ell^*$ is a subspace and the lemma is proved.

(6.8) <u>LEMMA</u>. *Let* $x,y \in P$, $d_\Gamma(x,y) = 2, z \in \Gamma(x) \cap \Gamma(y)$. *Then there exists* $v \in \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)$.

<u>PROOF</u>. Let $a \in P_2 \cap Q(x,z)$, $b \in P_2 \cap Q(y,z)$. As $d_\Gamma(x,y) = 2$, $a \neq b$. Since $z \in \Delta(a) \cap \Delta(b)$ we have $d_\Delta(a,b) = 2$ and $z \in Q(a,b)$. Let $u \in Q(a,b) \cap P$, $u \neq z$. $z \notin Q(x,y) \cap Q(y,u)$. For if $z \in Q(x,y) \cap Q(y,u)$, then $Q(x,y) = Q(z,u) = Q(y,u)$. Thus $d_\Gamma(x,y) = 1$, a contradiction. Now $P \cap Q(x,y)$, $P \cap Q(y,u) \subseteq \Gamma(z)$. It follows that there is a unique a_1,b_1 in $Q(x,u) \cap \Delta(z)$, $Q(y,u) \cap \Delta(z)$, respectively, namely a and b. Let $a_2 \in \Delta(x) \cap \Delta(u)$, $a_2 \neq a$ and b_2 choosen similarly. Then $a_2,b_2 \in \Delta_3(z)$. Then $Q(a_2,b_2) \cap \Gamma(z) = \{u\}$. Now if $v \in \Delta(a_2) \cap \Delta(b_2)$, $v \neq u$, then $v \in \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)$.

(6.9) <u>LEMMA</u>. *Let* $\ell \in Q$, $x \in P$ *with* $\ell \subseteq \Gamma_2(x)$. *Then* $C(x,\ell) = x^* \cap \ell^*$ *properly contains a line.*

PROOF. Set $Q = Q(\ell)$. Let a be the unique point in $Q \cap \Delta_3(x)$. Let x,b,y, a be a geodesic from x to a. Then $Q(a,b) \cap \mathcal{P}$ is a line contained in $C(x,\ell)$. Now let $c \in Q(x,y) \cap \mathcal{P}_2$, $c \neq b$. Then $y \in Q(a,c)$ and $Q(a,c) \neq Q(a,b)$. Therefore $\mathcal{P} \cap Q(q,c) \cap Q(a,b) = \{y\}$. But $\mathcal{P} \cap Q(a,c)$ is a line in $C(x,\ell)$ and $\mathcal{P} \cap Q(a,c) \neq \mathcal{P} \cap Q(a,b)$ and (6.9) is proved.

(6.10) LEMMA. *If* $x,y \in \mathcal{P}$, $d_\Gamma(x,y) = 2$, *then* $\Gamma(x) \cap \Gamma(y)$ *is a polar space of rank three.*

PROOF. $\Gamma(x) \cap \Gamma(y)$ is a Γ-space with thick lines. By (6.8) $\Gamma(x) \cap \Gamma(y)$ is non-degenerate. From (6.7) it follows that $\Gamma(x) \cap \Gamma(y)$ is a polar space.

Now let $z \in \Gamma(x) \cap \Gamma(y)$, $u \in \Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$. Since $xz \subseteq \Gamma(u)$, $yz \subseteq \Gamma(u)$, there is a unique $b \in Q(x,z) \cap \Delta(u)$ and a unique $c \in Q(y,z) \cap \Delta(u)$. Then $u \in \mathcal{P} \cap Q(b,c)$. It follows that the lines on z in $\Gamma(x) \cap \Gamma(y)$ is a grid isomorphic to $[Q(y,z) \cap \mathcal{P}_2] \times [Q(x,z) \cap \mathcal{P}_2]$. From this it follows that maximal singular subspaces of $\Gamma(x) \cap \Gamma(y)$ are planes and $rk(\Gamma(x) \cap \Gamma(y)) = 3$.

We have now shown that (D1)-(D3) hold for $(\mathcal{P},\mathcal{L})$. Thus, either $(\mathcal{P},\mathcal{L}) \cong D_{n,n}(K)$ for some field K or $(\mathcal{P},\mathcal{L})$ is a polar space of rank 4. However, in the latter case, by the end of 6.10 and TITS [5] we have $(\mathcal{P},\mathcal{L}) \cong D_4(K) \cong D_{4,4}(K)$. Now the points in $\mathcal{P}_2$ can be identified with the maximal singular subspaces of $(\mathcal{P},\mathcal{L})$ with projective dimension n-1. From this identification it now follows that $\mathcal{P}_1 \cup \mathcal{P}_2$ can be identified with the maximal singular subspaces of an orthogonal space V of dimension $2n(\geq 8)$ over a field K, with index n, such that two are collinear if and only if they meet in an (n-1) dimensional subspace. This completes the proof of (5.10).

Acknowledgement: I would like to thank the participants in the Algebra Seminar at the Mathematisch Centrum for providing me the opportunity to present this material, and especially Arjeh Cohen for many fruitful discussions.

[1] BUEKENHOUT, F. & E.E. SHULT, *On the Foundations of Polar Geometry*, Geometriae Dedicata 3(1974), 155-170.

[2] COHEN, A.M., *On a Theorem of Cooperstein, submitted to European* J. of Combinatorics.

[3] COHEN, A.M. & B.N. COOPERSTEIN, *Some Properties of Lie Incidence Structures,* in preparation.

[4] COOPERSTEIN, B.N., *A Characterization of Some Lie Incidence Structures,* Geometrical Dedicata (1977), 205-258.

[5] TITS, J., *Building of Spherical Type and Finite* BN-*pairs,* Springer-Verlag, 1974.

[6] VEBLEN, O. & J.W. YOUNG, *Projective Geometry,* Vol. I., Blaisdell Publishing, New Young, 1938.

[7] VELDKAMP, F.D., *Polar Geometry,* I-IV. Indag. Math 21,512-551 (1959).

[8] SHULT, E.E. & A. YANUSHKA, *Near* n-gons *and line systems.*

[9] CAMERON, P.J., *Dual Polar Spaces,* Geometrial Dedicata 12(1982), 75-85.

[10] HIGMAN, D.G., *Gamma and Delta Spaces,* Abstract of talks at Hans Sur-Lesse Conference, 1979.