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AFDELING ZUIVERE WISKUNDE ZW 183/82 DECEMBER
(DEPARTMENT OF PURE MATHEMATICS)

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# LOG-CONVEX TRAPEZOIDAL APPROXIMATION OF AN ELEMENTARY INTEGRAL

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*Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.*

*The Mathematical Centre, founded 11th February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).*

1980 Mathematics subject classification: 41A55, 40A25, 33A15

Log - convex trapezoidal approximation of an elementary integral

by

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ABSTRACT

The integral $\int_0^1 x^s dx$, $s > 0$, is approximated by the canonical trapezoidal rule

$$T_n(s) = \frac{1}{2n} \{ \sum_{k=0}^{n-1} (k/n)^s + \sum_{k=1}^{n} (k/n)^s \}$$

and the log - convexity of $\{T_n(s)\}_{n=1}^{\infty}$ is studied, with s as a fixed parameter. The investigations are based on an integral representation of $T_n(s)$ and it is proved that the sequence $\{T_n(s)\}_{n=1}^{\infty}$ is log - convex (in n) for $1 < s < 3$ and $5 < s < 7$.

KEY WORDS & PHRASES: *Approximate quadrature, trapezoidal rule, convex sequences, Euler gamma function*

## 0. INTRODUCTION

We consider the *canonical trapezoidal approximations*

$$(0.1)\qquad T_n := T_n(s) := \frac{1}{2}\left(\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^s + \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\right)$$

of the integral $\int_0^1 x^s dx$, where s is any *(fixed)* positive *real* number.

In [2] it was shown that for $s > 1$ (resp. $0<s<1$) the sequence $\{T_n\}_{n=1}^{\infty}$ is decreasing (resp. increasing), whereas somewhat later it was shown in [3] that for $s = 0(1)7$ and $s \geq 8$ this sequence even has the much stronger property of being *convex*.

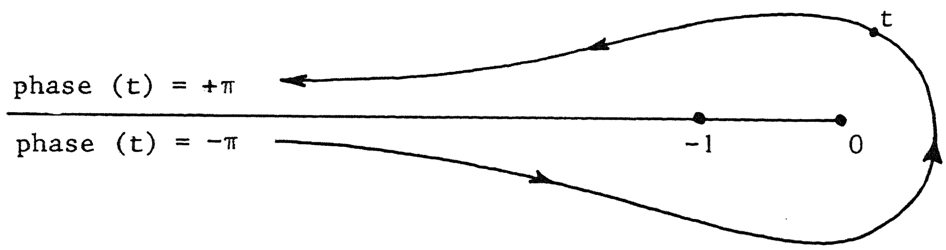
In [4; p. 8] the first named author conjectured that for all $s > 1$ the sequence $\{T_n\}_{n=1}^{\infty}$ is *logarithmically convex*, i.e. $T_n^2 \leq T_{n-1}T_{n+1}$ for all $n \geq 2$. The main goal of this note is to prove the correctness of this conjecture for the intervals $1 < s < 3$ and $5 < s < 7$.

## 1. PRELIMINARIES

Our starting point is Hankel's integral representation of the reciprocal of Euler's gamma function (cf. WHITTAKER & WATSON [6; pp. 244-245] or SANSONE & GERRETSEN [5; pp. 201-204])

$$(1.1)\qquad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\oint e^t t^{-s}dt, \qquad s \in \mathbb{C}$$

where $\oint$ denotes integration along a contour as depicted below:




For any $p > 0$ we substitute $t = pw$ in (1.1), replace s by $s + 1$ and obtain

$$(1.2) \qquad p^s = \frac{\Gamma(s+1)}{2\pi i} \oint e^{pw} w^{-s-1} dw, \qquad s \in \mathbb{C}.$$

Setting $p = \frac{k}{n}$, $k = 1(1)n$, we obtain by summation over k

$$(1.3) \qquad T_n = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} \frac{w}{2n} \frac{e^{\frac{w}{n}}+1}{e^{\frac{w}{n}}-1} w^{-s-1} dw, \qquad s > 0.$$

Letting $n \to \infty$ it follows that

$$\frac{1}{s+1} = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} w^{-s-1} dw, \qquad s > 0$$

(a result obtainable in various other ways; compare Section 4) so that (1.3) may be rewritten as

$$(1.4) \qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} H(\tfrac{w}{n}) w^{-s-1} dw, \qquad s > 0$$

where

$$H(z) = \frac{z}{2} \frac{e^z+1}{e^z-1} - 1 = z\left(\frac{1}{e^z-1} - \frac{1}{z} + \frac{1}{2}\right) .$$

It is well known that (cf. SANSONE & GERRETSEN [5; p. 88])

$$(1.5) \qquad \frac{1}{e^z-1} - \frac{1}{z} + \frac{1}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} z^{2k-1}, \qquad |z| < 2\pi$$

from which it is clear that the *(even)* function $H(z)$ has a zero of order 2 at $z = 0$. With this in mind we rewrite (1.4) as follows

$$(1.6) \qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} \left(\frac{1}{w^2} H(\tfrac{w}{n})\right) w^{1-s} dw, \qquad s > 0.$$

## 2. THE CASE $1 < s < 2$.

For $1 < s < 2$ (so that $-1<1-s<0$) we may, by the regularity of $w^{-2} H(\frac{w}{n})$ at $w = 0$, contract the contour of integration in (1.6) to the negative real axis so that by a standard argument, using the fact that H(z) is an *even* function,

$$(2.1)\qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)\sin(s-1)\pi}{\pi}\int_0^\infty \frac{1-e^{-x}}{x}\,H(\tfrac{x}{n})x^{-s-1}dx, \qquad 1 < s < 2.$$

Substituting $x = nu$ and writing $\frac{1-e^{-nu}}{nu} = \int_0^1 e^{-nuv}dv$ we may write (2.1) as

$$(2.2)\qquad T_n - \frac{1}{s+1} = \frac{\Gamma(s+1)\sin(s-1)\pi}{\pi}\int_0^\infty \left(\int_0^1 e^{-nuv}dv\right) H(u)\,u^{-s-1}du.$$

Since $\sin(s-1)\pi > 0$ for $1 < s < 2$ and $H(u) > 0$ for $u > 0$, we find, by the general theory of log-convex functions (cf. ARTIN [1]), that the sequence $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ is log-convex, a result which is even stronger than the previously announced assertion that $\{T_n\}_{n=1}^\infty$ is log-convex for all (fixed) $s \in (1,2)$.

Similarly one may show that $\{\frac{1}{s+1} - T_n\}_{n=1}^\infty$ is log-convex for all (fixed) $s \in (0,1)$.

## 3. INTERMEZZO: A SPECIAL PROPERTY OF $H(u) = u(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2})$

In the previous section we transformed (2.1) into (2.2) and then concluded that $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ is log-convex for all $s \in (1,2)$. In this section we will show that this result may also be obtained directly from (2.1) by observing that the function $H(\frac{1}{x})$, $x > 0$, has the remarkable property of being log-convex on $\mathbb{R}^+$. As a matter of fact we will prove the following

THEOREM 3.1. *There exists a constant* $\alpha_0 > 2.863$ *such that for every (fixed)* $\alpha \in (0,\alpha_0]$ *the function* $\phi_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, *defined by* $\phi_\alpha(x) := H(x^{-\alpha})$, $x > 0$, *is log-convex on* $\mathbb{R}^+$.

PROOF. In order to prove the log-convexity of $\phi_\alpha$ on $\mathbb{R}^+$ we proceed by brute force, at the same time inviting the reader to invent a nicer proof.

Writing

$$\psi(x) := \log \phi_\alpha(x) = \log H(x^{-\alpha})$$

we have

$$\psi''(x) = \alpha\, u^2 + \frac{AB-C^2}{A^2}$$

where

$$u := \frac{1}{x} ,$$

$$v := u^\alpha,$$

$$A := (e^v-1)^{-1} - \frac{1}{v} + \frac{1}{2} ,$$

$$B := 2\,\alpha^2u^2v^2e^{2v}(e^v-1)^{-3} - \alpha^2u^2v^2e^v(e^v-1)^{-2}$$

$$- \alpha(\alpha+1)u^2ve^v(e^v-1)^{-2} - \alpha(\alpha-1)\,\frac{u^2}{v} ,$$

$$C := \alpha\, uve^v(e^v-1)^{-2} - \alpha\,\frac{u}{v} .$$

It clearly suffices to show that $\psi''(x) > 0$ for all $x \in \mathbb{R}^+$ so that (since $\alpha > 0$) we may just as well prove that

$$\frac{\psi''(x)}{\alpha u^2} = 1 + \frac{A_1B_1-\alpha C_1^2}{A_1^2} > 0$$

where (u and v being defined as above)

$$A_1 := A \text{ (as defined above)},$$

$$B_1 := 2\,\alpha\, v^2e^{2v}(e^v-1)^{-3} - \alpha\, v^2e^v(e^v-1)^{-2}$$

$$- (\alpha+1)ve^v(e^v-1)^{-2} - \frac{\alpha-1}{v} ,$$

$$C_1 := ve^v(e^v-1)^{-2} - \frac{1}{v} .$$

Hence, it suffices to show that for all $x \in \mathbb{R}^+$

$$A_1^2 + A_1B_1 > \alpha\, C_1^2 .$$

Multiplying both sides of this inequality by $v^2(e^v-1)^4$ we arrive at the equivalent inequality

$$v^2(e^v-1)^2 + (v^2-2v)\ (e^v-1)^3 + (1-\frac{v}{2})^2\ (e^v-1)^4 +$$

$$+ (v+(\frac{v}{2}-1)(e^v-1))(2\alpha v^3 e^{2v} - \alpha v^3 e^v(e^v-1) - (\alpha+1)v^2 e^v(e^v-1)$$

$$- (\alpha-1)(e^v-1)^3) > \alpha(v^4 e^{2v} - 2v^2 e^v(e^v-1)^2 + (e^v-1)^4).$$

This inequality may be written in the equivalent form

$$\sum_{k=0}^{4} P_k(v)e^{kv} > 0 \tag{3.1}$$

where

$$P_0(v) = \frac{\alpha+1}{2} + \frac{v}{4} ,$$

$$P_1(v) = -\ (\alpha+1) + (3\alpha+1)v + \frac{3\alpha+1}{2}\ v^2 + \frac{\alpha}{2}\ v^3,$$

$$P_2(v) = -\ \frac{5+12\alpha}{2}\ v ,$$

$$P_3(v) = \alpha + 1 + (3\alpha+1)v - \frac{3\alpha+1}{2}\ v^2 + \frac{\alpha}{2}\ v^3,$$

$$P_4(v) = -\ \frac{\alpha+1}{2} + \frac{v}{4} .$$

Now we write the left hand side of (3.1) in the form $\Sigma_{n=0}^{\infty} c_n v^n$ and observe that $c_0 = c_1 = 0$ for all $\alpha$. For $n \geq 2$ one may verify that

$$n!c_n = (\alpha+1)(-1+3^n-2^{2n-1}) +$$

$$+ n((3\alpha+1) - (12\alpha+5)2^{n-2} + (3\alpha+1)3^{n-1} + 4^{n-2}) +$$

$$+ (3\alpha+1)\frac{n(n-1)}{2}(1-3^{n-2}) + \frac{\alpha}{2}n(n-1)(n-2)(1+3^{n-3}) =$$

$$=: \alpha\, a(n) + b(n) ,$$

where

$$a(n) := -1 + 3^n - 2^{2n-1} + 3n - 3n2^n + n3^n + \frac{n(n-1)}{2}(3-3^{n-1}) +$$

$$+ \frac{n(n-1)(n-2)}{2}(1+3^{n-3}),$$

$$b(n) := -1 + 3^n - 2^{2n-1} + n - 5n2^{n-2} + n3^{n-1} + n4^{n-2} +$$

$$+ \frac{n(n-1)}{2}(1-3^{n-2}).$$

It is a matter of routine to show that

$$a(n) = 0 \quad \text{for} \quad n \le 8,$$

$$a(n) < 0 \quad \text{for} \quad n \ge 9,$$

$$b(n) = 0 \quad \text{for} \quad n \le 6,$$

$$b(n) > 0 \quad \text{for} \quad n \ge 7,$$

and

$$\min_{n\ge 9} -\frac{b(n)}{a(n)} = -\frac{b(24)}{a(24)} = 2.863\,921\ \ldots,$$

from which it follows that for $0 < \alpha < 2.8639$ we have $c_n = 0$ for $n \le 6$ and $c_n > 0$ for $n \ge 7$, which proves the theorem.

REMARK. It is not known to us which $\alpha_0^*$ is the largest number such that $H(x^{-\alpha})$ is log-convex on $\mathbb{R}^+$ for all $\alpha \in (0,\alpha_0^*]$. Numerical computations show that $H(x^{-3})$ is *not* log-convex on all of $\mathbb{R}^+$ so that $(2.863<)\ \alpha_0^* < 3$.

## 4. FURTHER PREPARATIONS

In order to carry our analysis somewhat further we need some auxiliary formulas. In (1.2) let $p \downarrow 0$ (keeping s fixed and > 0) and it follows that

$$(4.1)\qquad \oint w^{-s-1}dw = 0, \qquad s > 0.$$

Another way of proving this formula is as follows. In (1.3) put n = 1 so that (for s>0)

$$(4.2)\qquad T_1(s) = \frac{1}{2} = \frac{\Gamma(s+1)}{2\pi i}\oint \frac{e^w+1}{2}\, w^{-s-1}dw =$$

$$= \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2}\oint e^w w^{-s-1}dw + \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2}\oint w^{-s-1}dw =$$

$$= \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2}\,\frac{2\pi i}{(s+1)} + \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2}\oint w^{-s-1}dw,$$

and it follows again that $\oint w^{-s-1}dw = 0$ for $s > 0$.
Our next important auxiliary result is

LEMMA 4.1. *For any positive integer* N *we have*

$$(4.3)\qquad H(z) = z\left(\frac{1}{e^z-1}-\frac{1}{z}+\frac{1}{2}\right) = P_N(z) + (-1)^N z^{2N+2}\,\mu_N(z)$$

*where*

$$(4.4)\qquad P_N(z) := \sum_{k=1}^{N} (-1)^{k-1}\frac{|B_{2k}|}{(2k)!}\, z^{2k}$$

*and*

$$(4.5)\qquad \mu_N(z) := \sum_{m=1}^{\infty} \frac{2}{(z^2+4\pi^2m^2)(2\pi m)^{2N}}\ .$$

PROOF. In order to prove this lemma we apply Taylor's formula as described in WHITTAKER & WATSON [6; p. 93]. We observe that (compare (1.5))

$$P_N(z) = \sum_{k=1}^{N} \frac{H^{(2k)}(0)}{(2k)!} z^{2k} \quad \text{and} \quad H^{(2k+1)}(0) = 0, \tag{4.6}$$

so that

$$(-1)^N \mu_N(z) = \frac{1}{2\pi i} \oint \frac{H(w)}{(w-z)w^{2N+2}}\, dw, \tag{4.7}$$

where $\oint$ denotes counter clockwise integration along a closed contour containing the points $w = 0$ and $w = z$ in its interior and such that it does not encircle any of the points $w = k.2\pi i$, $k \in \mathbb{Z} \setminus \{0\}$. A standard application of the calculus of residues then yields

$$\begin{aligned}(-1)^N \mu_N(z) &= \frac{1}{2\pi i} \oint \frac{\frac{1}{e^w-1} - \frac{1}{w} + \frac{1}{2}}{(w-z)w^{2N+1}}\, dw = \\ &= -\sum_{m=1}^{\infty} \left\{ \frac{1}{(2\pi im-z)(2\pi im)^{2N+1}} + \frac{1}{(-2\pi im-z)(-2\pi im)^{2N+1}} \right\} = \\ &= (-1)^N \sum_{m=1}^{\infty} \frac{2}{(z^2+4\pi^2m^2)(2\pi m)^{2N}},\end{aligned} \tag{4.8}$$

and the lemma follows.

REMARKS.

1) We note that Lemma 4.1 also holds true for $N = 0$. In this case we have the well-known formula

$$H(z) = \sum_{m=1}^{\infty} \frac{2z^2}{z^2+4\pi^2m^2}.$$

2) As an immediate consequence of Lemma 4.1 we have for any fixed $N > 0$

$$\mu_N(x) = O(x^{-2}), \qquad x \to \infty.$$

3) $\mu_N(z)$ is regular at $z = 0$.

LEMMA 4.2. *For any fixed* $N > 0$ *the function* $\mu_N(\frac{1}{x})x^{-2N-2}$ *is log-convex on* $\mathbb{R}^+$.

PROOF. In order to see this we write

$$\mu_N(\frac{1}{x})x^{-2N-2} = 2\sum_{m=1}^{\infty} \frac{x^{-2N-2}}{(x^{-2}+4\pi^2 m^2)(2\pi m)^{2N}} =$$

$$= 2\sum_{m=1}^{\infty} \frac{1}{(1+4\pi^2 m^2 x^2)(2\pi m x)^{2N}}$$

and observe that every term of this series is log-convex on $\mathbb{R}^+$. Indeed, for any (fixed) $a > \frac{1}{8}$ the function

$$\phi_a(x) = -\log(1+x^2) - 2a\log x$$

is convex on $\mathbb{R}^+$.

## 5. THE CASE $2 < s < 3$

From

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i}\oint \frac{e^w-1}{w} H(\frac{w}{n})w^{-s-1}dw$$

we obtain by means of the results of the previous section (for $2<s<3$)

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i}\oint \frac{e^w-1}{w} (H(\frac{w}{n})-P_1(\frac{w}{n}))w^{-s-1}dw +$$

$$+ \frac{\Gamma(s+1)}{2\pi i}\oint \frac{e^w-1}{w} P_1(\frac{w}{n})w^{-s-1}dw .$$

Since $P_1(z) = \frac{z^2}{12}$ we thus find that

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} (\frac{w}{n})^4 (-1)^1 \mu_1 (\frac{w}{n}) w^{-s-1} dw +$$

$$+ \frac{\Gamma(s+1)}{2\pi i} \frac{1}{12n^2} \oint (e^w-1) w^{-s} dw =$$

$$= \frac{1}{s+1} + \frac{s}{12n^2} - \frac{\Gamma(s+1)}{\pi} \sin(s-3)\pi \int_0^\infty \frac{1-e^{-t}}{t} (\frac{t}{n})^4 \mu_1 (\frac{t}{n}) t^{-s-1} dt.$$

In Section 4 it was shown that $x^{-4}\mu_1(x^{-1})$ is log-convex on $\mathbb{R}^+$ so that for any $t > 0$, $(\frac{t}{n})^4 \mu_1(\frac{t}{n})$ is log-convex as a function of $n \in \mathbb{N}$. Since $\sin(s-3)\pi < 0$ for $2 < s < 3$ it follows that $\{T_n - \frac{1}{s+1} - \frac{s}{12n^2}\}_{n=1}^{\infty}$ is log-convex (in n) for any fixed $s \in (2,3)$, a result which is even stronger than the previously announced log-convexity of $\{T_n\}_{n=1}$.

## 6. SOME REMARKS ON THE GENERAL CASE: $2N < s < 2(N+1)$

Similarly as before we have

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} P_N(\frac{w}{n}) w^{-s-1} dw +$$

$$+ (-1)^N \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} (\frac{w}{n})^{2N+2} \mu_N(\frac{w}{n}) w^{-s-1} dw =$$

$$= \frac{1}{s+1} + I_1(n) + I_2(n), \text{ say.}$$

According to the preliminaries in Section 4 we have

$$I_1(n) = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} \left( \sum_{k=1}^{N} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} (\frac{w}{n})^{2k} \right) w^{-s-1} dw =$$

$$= \sum_{k=1}^{N} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} \frac{\Gamma(s+1)}{\Gamma(s-2k+2)} \frac{1}{n^{2k}},$$

and, similarly as before,

$$I_2(n) = (-1)^N \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w - 1}{w} \left(\frac{w}{n}\right)^{2N+2} \mu_N\left(\frac{w}{n}\right) w^{-s-1} dw =$$

$$= (-1)^N \frac{\Gamma(s+1)\sin(s-2N-1)\pi}{\pi} \int_0^\infty \frac{1-e^{-t}}{t} \left(\frac{t}{n}\right)^{2N+2} \mu_N\left(\frac{t}{n}\right) t^{-s-1}\, dt,$$

the last integral being convergent at $t = 0$ since $(2N+2) - s - 1 > -1$ and at $t = \infty$ since $-1 + (2N+2) - 2 - s - 1 < -1$. We now observe that

$$\begin{array}{lll} N \text{ even and } 2N+1 < s < 2N+2 & \Rightarrow & (-1)^N \sin(s-2N-1)\pi > 0, \\ N \text{ even and } 2N < s < 2N+1 & \Rightarrow & \qquad '' \qquad < 0, \\ N \text{ odd and } 2N < s < 2N+1 & \Rightarrow & \qquad '' \qquad > 0, \\ N \text{ odd and } 2N+1 < s < 2N+2 & \Rightarrow & \qquad '' \qquad < 0. \end{array}$$

Hence, whenever we can show that $\{I_2(n)\}_{n=1}^\infty$ is log-convex then $\{T_n\}_{n=1}^\infty$ is log-convex if $(-1)^N \sin(s-2N-1)\pi > 0$. It follows that our approach can only be successful if $2N+1 < s < 2N+3$, where $N$ is *even*.


## 7. THE CASE $5 < s < 7$

We first assume $5 < s < 6$ so that

$$T_n(s) - \frac{1}{s+1} = \frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \text{log-convex (in } n).$$

Hence, in order to show the log-convexity of $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ it suffices to show the log-convexity of $\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4}\}_{n=1}^\infty$. Since $5 < s < 6$ it is easily seen that this in its turn is a consequence of the log-convexity of $\{\frac{1}{n^2} - \frac{1}{3n^4}\}_{n=1}^\infty$, the verification of which is a matter of routine. Now let $6 < s < 7$, so that by the results of Section 6 it suffices to show the log-convexity of

$$\left\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \frac{s(s-1)(s-2)(s-3)(s-4)}{42\,720n^6}\right\}_{n=1}^{\infty},$$

which, using the assumption $6 < s < 7$, is an easy consequence of the log-convexity of $\{\frac{1}{n^2} - \frac{7}{12n^4} + \frac{3}{89n^6}\}_{n=1}^{\infty}$, a (though tedious) matter of routine.

REMARK. For $9 < s < 10$ we would have to verify the log-convexity of

$$\left\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \frac{s(s-1)\ldots(s-4)}{42\,720n^6} - \frac{s(s-1)\ldots(s-6)}{1\,209\,600\,n^8}\right\}_{n=1}^{\infty}$$

whereas for still larger values of s it seems practically unfeasible (if true) to prove the log-convexity (in n) of forms of such a complexity.

REFERENCES

[1] ARTIN, E., *Einführung in die Theorie der Gammafunktion*, Teubner, Leipzig, 1931.

[2] LUNE, J. VAN DE, *Monotonic approximation of integrals in relation to some inequalities for sums of powers of integers*, Report ZW 39/75, Mathematical Centre, Amsterdam.

[3] LUNE, J. VAN DE & M. VOORHOEVE, *Convex approximation of integrals*, Report ZW 85/77, Mathematical Centre, Amsterdam.

[4] LUNE, J. VAN DE & M. VOORHOEVE, *Some problems on log-convex approximation of certain integrals*, Report ZN 85/78, Mathematical Centre, Amsterdam.

[5] SANSONE, G. & J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, Noordhoff, Groningen, 1960.

[6] WHITTAKER, E.T. & G.N. WATSON, *A course of Modern Analysis*, Cambridge, 1952.