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INTEGRATION OF THE LINEAR FILTERING PROBLEM BY MEANS OF CANONICAL TRANSFORMATIONS

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Integration of the linear filtering problem by means of canonical transformations*)

by

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ABSTRACT

In this note we dwelve some more into the formal analogy of quantum mechanics and filtering theory, and we integrate the DMZ-equation by transforming it into a Schroedinger equation that can be integrated in the standard way.

KEY WORDS & PHRASES: *linear filtering-canonical transformations; harmonic oscilator*

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1. INTRODUCTION AND PRELIMINARIES

In this note we exploit the formal analogy between quantum (and classical) mechanics and filtering problems by showing how one can solve the DMZ (Duncan-Mortenson-Zakai) - equation.

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} - \frac{\omega^2}{2}x^2\rho + x\eta\rho \tag{1.1}$$

where ω is a real number and η should be thought of as "Stratonovitch derivative" of the observation process. See [1] or [2] for the filtering background.

Equation (1.1) can be converted, by defining $\psi(x,t) = \rho(x,ti)$, into

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 4}{\partial x^2} + \frac{\omega^2 x^2}{2} + x\,\xi\,\psi \tag{1.2}$$

where $\xi(t) = -\eta(ti)$, and (1.2) can be solved using the theory of canonical transformations [3] - [4].

To do this it is easier to start from the classical system, seek the canonical transformation there and then implement it (or realize it) as a unitary change of representation for the quantum system described by (1.2). This is carried out in section 2. In section 3 we rapidly cover the many-dimensional case and in section 4 we make a few comments on how this procedure is related to the work presented in [2]. Disappointingly little seems to come out in this direction.

The results of this paper "simplify" a bit some of the standard computations and allow for a general initial density. Also, they add more to the work of MITTER in [5].

The origin of this paper stems from a conversation with M. Hazewinkel to whom I mentioned that (1.1) should be integrable by means of canonical transformations and he told me what the real questions behind (1.1) where.

2. SOLUTION OF 1.2 (and (1.1)).

Consider the mechanical system described by the Hamiltonian

$$H(p,x) = \tfrac{1}{2}(p^2 + \omega^2 x^2) + \xi(t)x. \tag{2.1}$$

The Hamiltonian equations describing the dynamics of it are

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = p \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\omega^2 x - \xi(t) \tag{2.2}$$

and the corresponding quantom evolution equation is (1.2).

Observe now that

$$F(x,p,t) = px + pf - x\dot{f} + \phi(t) \tag{2.3}$$

generates the canonical transformation [3]

$$p = \frac{\partial F}{\partial x} = P - \dot{f} \qquad Q = \frac{\partial F}{\partial p} = x - \dot{f} \tag{2.4}$$

changing $H(p,x)$ into

$$\tilde{H}(P,Q) = \tfrac{1}{2}(P^2 + \omega^2 Q^2) \tag{2.5}$$

if f and ϕ are chosen, satisfying zero initial conditions, such that

$$\ddot{f} + \omega^2 f = \xi \qquad \frac{\partial \phi}{\partial t} = (\dot{f}^2 - \omega^2 f)\,/\,2.$$

In integrating $f + \omega^2 f = \xi$ one should remember that $\xi(t)$ is a "Stratonovitch differential". With zero initial conditions

$$f = \frac{1}{\omega}\int^{t} \sin \omega(t-s)\, \xi(s)ds.$$

In the (P,Q) coordinates, equations (2.2) and (1.2) become, respectively, (2.6) $$\frac{dQ}{dt} = P \qquad \frac{dP}{dt} = -\omega^2 Q$$

(2.7) $$i\frac{\partial\tilde{\psi}}{\partial t} = -\frac{1}{2}\frac{\partial^2\tilde{\psi}}{\partial Q^2} + \frac{\omega^2 Q^2}{2}\tilde{\psi}$$

the integration of the first is trivial and that of the second can be found in any text of elementary quantum mechanics. It happens to be

$$\tilde{\psi}(Q,t) = \sum \alpha_n e^{-i\varepsilon_n t}\tilde{\psi}_n(Q)$$

where $\varepsilon_n = \omega(n+\frac{1}{2})$, $-\frac{1}{2}\frac{\partial^2\tilde{\psi}}{\partial Q^2} + \frac{\omega^2 Q^2}{2}\tilde{\psi}_n = \varepsilon_n\tilde{\psi}_n$,

and $\alpha_n = (\tilde{\psi}_n, \tilde{\psi}(\cdot,0)) = \int \tilde{\psi}_n(Q)\,\tilde{\psi}(\cdot,0)\,dQ$.

Note that

$$\tilde{\psi}_0(Q) = \left(\frac{\omega}{2\pi}\right)^{\frac{1}{2}} \exp -\frac{\omega Q^2}{2}$$

is the eigenfunction corresponding to $\varepsilon_0 = \frac{\omega}{2}$, a fact that we use below.

All that is needed now is to obtain $\psi(x,t)$ from $\tilde{\psi}(Q,t)$. Well, it so happens (see [9]) that

(2.8) $$\psi(x,t) = \int \langle x|Q)\,\tilde{\psi}(Q,t)\,dQ$$

where the transformation function $\langle x|Q)$ can be obtained from

$$\langle x|P) = (2\pi)^{-\frac{1}{2}} \exp i\,F(x,P,t)$$

by means of

(2.9) $$\langle x|Q) = \int \langle x|P)e^{-iPQ}\frac{dP}{(2\pi)^{\frac{1}{2}}} = \exp i(\phi - xf)\,\delta(Q-x-f)$$

which plugged back into (2.8) gives

(2.10) $$\psi(x,t) = \exp i(\phi - xf)\,\tilde{\psi}(x+f,t)$$

Since the initial condition was originally given for $\psi(x,0)$, it is easy to see, from our choice of initial conditions for f and ϕ, that $\psi(x,0) = \tilde{\psi}(Q,0)$ and therefore, for arbitrary initial condition, in terms of the eigenfunction expansion, (2.10) reads

$$\psi(x,t) = \sum_n \alpha_n \exp i\{\phi - xf) - \varepsilon_n t\} \tilde{\psi}_n(x+f)$$

from which the solution to the original equation is

$$\rho(x,t) = \psi(t/i) = \sum_n a_n \exp i(\phi(t/i) - xf(t/i)) e^{-\varepsilon_n t} \tilde{\psi}_n(x+f(t/i)) \tag{2.11}$$

Also, when $\psi(x,0) = \psi_0(Q)$, the expression above reduces to the exponential

$$\rho(x,t) = \exp[-\frac{\omega t}{2} + \frac{\omega}{2}(x+f(t/i))^2 + i(\phi(+t/i) - x f(t/i)) \tag{2.12}$$

a rather known result.

Actually, the solution of (2.7) can be written as

$$\tilde{\psi}(Q,t) = \int K(Q,t;Q_0;0)\, \tilde{\psi}_0(Q_0)\, dQ_0$$

where

$$K(Q,t;Q_0,t_0) = \left(\frac{m\omega}{2\pi i n \operatorname{Sin} \omega(t-t_0)}\right)^{\frac{1}{2}} \exp\{\frac{im\,\omega}{2n \operatorname{Sin} \omega(t-t_0)}[(Q^2+Q_0^2)\cos \omega(t-t_0) - 2QQ_0]$$

a result which can be found in [6]. Changing $t \to t/i$ and multiplying by $e^{-t/2}$ one obtains the transition kernel for the oscillator process [7]. In any case $\rho(x,t)$ can be obtained as follows, first put $\tilde{\psi}(Q,t/i) = \tilde{\rho}(Q,t)$ where.

$$\tilde{\rho}(Q,t) = \int G(Q,t;Q_0)\, \rho_0(Q_0)\, dx_0^Q \tag{2.13}$$

where $G(Q,t;Q_0) = K(Q,t/i;Q_0)$. From this one obtains

$$\rho(x,t) = \exp i(\phi(t/i) - x f(t/i))\, \tilde{\rho}(x+f(t/i),t) \tag{2.14}$$

and these last two identities express the solution to (1.1) in terms of the initial conditions in a nicer way than (2.11), but part of the comments above were easier verifiable with it.

3. THE MANY-DIMENSIONAL CASE

Consider the filtering problem (see [1] or [2])

$$dx_i = \sum \alpha_{ij} dw_i \qquad dy_i = \sum_i c_{ij} x_j dt + dv_i$$

for which the DMZ – equation is

(3.1) $$\frac{\partial \rho}{\partial t} = \{\tfrac{1}{2} \sum_{\omega,k} \alpha_{ik}\alpha_{jt} \frac{\partial^2}{\partial x_i \partial x_j} - \tfrac{1}{2} \sum_{ijk} c_{kt} c_{kj} x_i x_j\} \rho + \sum \xi_i x_i \rho$$

where $\xi_i = \Sigma_k c_{ik} \frac{dy_k}{dt}$, and again $\frac{dy_k}{dt}$ is to be understood formaly as a "Stratonovitch derivative".

If we define the matrices μ and Ω by

$$\mu = \alpha\alpha^+ \qquad \Omega = c^+ c$$

we could consider, in analogy with section 2, the mechanical system with Hamiltonian

(3.2) $$H(p,x) = \tfrac{1}{2} (p^+ \mu p + x^+ \Omega x) + \xi^+ x$$

where vectors are supposed to be column vectors and of course $^+$ denotes the transpose.

To (3.2) one has associated the classical Hamiltonian equations

(3.3) $$\frac{dx}{dt} = \mu p \qquad \frac{dp}{dt} = - \Omega x + \xi$$

and the Schroedinger equation (obtainable from (3.1) by putting $\psi(x,t) = \rho(x,t_i)$)

(3.4) $$i \frac{\partial t}{\partial t} = - \tfrac{1}{2} \sum_{ij} \mu_{ij} \frac{\partial^2 t}{\partial x_i \partial x_j} + \tfrac{1}{2}(x^t \Omega x) \psi + (\xi^+ x)\psi.$$

Note first, that the canonical transformation, generated by

$F = \sum_{ij} (\alpha^{+})^{-1}_{ij} P_i x_j$ transforms (3.3) and (3.4) into

$$\frac{dQ}{dt} = P \qquad \frac{dP}{dt} = - \tilde{\Omega}\, Q - \hat{\xi} \tag{3.5}$$

and

$$i\frac{\partial\tilde{\psi}}{\partial t} = -\tfrac{1}{2}\sum \frac{\partial^2 \tilde{\psi}}{\partial Q^2} + \tfrac{1}{2}(Q^{+} \tilde{\Omega}\, Q)\, \tilde{\psi} + \hat{\xi}^{+} Q\, \tilde{\psi} \tag{3.6}$$

where

$$\tilde{\psi} = \tilde{\psi}\,(Q,t), \qquad \tilde{\Omega} = \alpha\, \Omega\, \alpha^{+} \qquad \hat{\xi} = \alpha^{+} \xi$$

and the associated Hamiltonian is

$$H = \tfrac{1}{2}\{P^{+}P + Q^{+} \tilde{\Omega}\, Q\} + \hat{\xi}^{+} Q. \tag{3.7}$$

Let now D be an orthogonal matrix bringing $\tilde{\Omega}$ to diagonal form, i.e. $(D^{+} \tilde{\Omega}\, D)_{ij} = \omega_i^2 \delta_{ij}$. Let us now consider the canonical transformation generated by $F' = \Sigma\, D_{ji} P_i Q'_j$. With this transformation (3.7) is transformed into

$$H = \sum_i H_i = \sum_i \tfrac{1}{2}(P'^2_i + \omega_i^2 (Q'_i)^2) + \hat{\xi}'_i\, Q_i \tag{3.8}$$

where

$$\hat{\xi}'_i = \sum D_{ji}\, \hat{\xi}_j, \quad Q'_i = \sum D_{ji}\, Q_i, \text{ etc. }.$$

What we have done, is to separate variables in (3.4), preserving the Hamiltonian structure, i.e. (3.4) becomes

$$i\, \frac{\partial\psi'}{\partial t} = \sum_i \{-\tfrac{1}{2}\, \frac{\partial^2\psi'}{\partial Q'_i} + \frac{\omega^2_i}{2}\, (Q'_i)^2\, \psi' + \xi'_i\, Q'_i\, \psi'_i\}$$

Now proceeding like in section 2, we see that

$$\psi'(Q',t) = \sum_{k_1,\ldots,k_n} a(k_1,\ldots,k_n)\exp - i \sum_i \varepsilon_{k_i} t \prod_1^n \psi'_{k_i}(Q'_i+f_i) \ \Pi \exp i(\phi_i(t)-Q'_i f'_i)$$

with all of the simbols having the same meaning as in section 2 and

$$a(k_1,\ldots,k_n) = \int \prod_1^n \psi_{k_i}(Q'_i) \ \psi'(Q'_1,\ldots,Q'_n,0)\, dQ'_n \,.$$

We have leave for the interested reader to supply in the transformation of variables expressing $\psi(x,t)$ in terms of $\psi'(Q't)$ and then making $t \to t/i$ to obtain $\rho(x,t)$.

4. CONCLUDING COMMENTS

There does not seem to exist an obvious connection between this method and the standard formulation. This is due to the fact that the equation $i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2}\,\frac{\partial^2\psi}{\partial x^2} + \frac{x^2\psi}{2}$ or its "associated" difusion equation $\frac{\partial\rho}{\partial t} = \frac{\partial^2\rho}{\partial x^2} - \frac{x^2\rho}{2}$ does not seem to relate to a filtering problem.

This is rather unfortunate, because all the algebraic structure associated to filtering problems, discussed in [2] for example is lost.

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