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VIRTUAL PERIODS AND GLOBAL CONTINUATION OF PERIODIC ORBITS

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Virtual periods and global continuation of periodic orbits *)

by

S-N. Chow **)

ABSTRACT

In this paper we improve an earlier result about global bifurcation of periodic orbits under the restriction that the phase space is three or four dimensional.

KEY WORDS & PHRASES: *ordinary differential equation, Fuller index, virtual period, Hopf bifurcation, global bifurcation*

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Consider an autonomous ordinary differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^1 \tag{1}$$

and let $x(t,a)$ denote the solution with initial point $a \in \mathbb{R}^n$. Let $\Omega \subseteq (0,\infty) \times \mathbb{R}^n$ be an open bounded set, bounded away from $\{0\} \times \mathbb{R}^n$, whose boundary $\partial\Omega$ is free of periodic solution of (1). That is

$$x(t,a) \neq a \quad \text{for all } (t,a) \in \partial\Omega \ .$$

Note that critical points are considered as periodic points with arbitrary period, hence Ω contains no such points. The Fuller degree $d(\Omega,f)$ is a rational number defined for the flow (1) and the set Ω. Moreover, if f^α, $0 \leq \alpha \leq 1$, is a homotopy of vector fields then $d(\Omega,f^\alpha)$ is independent of α provided $\partial\Omega$ is free of periodic solution of $\dot{x} = f^\alpha(x)$. In particular $d(\Omega,f) = d(\Omega,g)$ for g uniformly near f, so to give a computational formula for the degree it is sufficient to consider the generic case when all periodic solutions of (1) are hyperbolic.

In this generic case Ω contains only finitely many periodic orbits and the Fuller degree is defined as

$$d(\Omega,f) = \sum i(\Gamma) \tag{2}$$

where $\Gamma \subseteq \Omega$ is a periodic orbit and $i(\Gamma)$ is a rational number, the Fuller index. To define $i(\Gamma)$, let $\gamma \subseteq \mathbb{R}^n$ be a non-constant periodic orbit, say

$$\gamma = \{x(t,a) \mid 0 \leq t \leq T\}$$

with least period

$$T = \inf\{t > 0 \mid x(t,a) = a\} \in (0,\infty).$$

If k is a positive integer, set

$$\Gamma = \{kT\} \times \gamma \subseteq (0,\infty) \times \mathbb{R}^n$$

and define

$$i(\Gamma) = (1/k)(-1)^{\sigma} \tag{3}$$

where σ is the number of eigenvalues of $(\partial x/\partial a)(kT,a)$ in the interval $(1,\infty)$. Here σ depends on the parity of k and $i(\Gamma)$ depends on k itself. It important to note the summation (2) is taken over precisely those k and γ for which $\Gamma \subseteq \Omega$. Formula (3) refers to the generic case when all periodic solutions of (1) are hyperbolic. Observe in particular that

$$i(\Gamma) = (1/k)\mathrm{ind}(\pi^k) \tag{4}$$

where ind denotes the fixed point index and π^k is the kth iterate of the Poincaré map for y. Now for arbitrary f suppose that some orbit γ, though not necessarily hyperbolic, is isolated from periodic orbits with period near kT, for some given k. It thus corresponds to an isolated fixed point of π^k, so that $i(\Gamma)$ may be defined by (4). Also, because Γ is isolated in $(0,\infty) \times \mathbb{R}^n$ the quantity $d(\Omega_0,f)$ is defined for small enough neighborhoods Ω_0 of Γ. It is a fact, not difficult to show, that $i(\Gamma) = d(\Omega_0,f)$. This means that for any f, as long as Ω contains only finitely many orbits Γ, the Fuller degree $d(\Omega,f)$ may be calculated from (2), (4).

For more information, we refer the reader to [1], [2] and [3]. By using Fuller degree, one may prove a global version of the Hopf bifurcation theorem which was first proved in [4]. Related results may be found in [5], [6] and [7]. In the following, we state the theorem (Theorem 1) which was shown by using Fuller degree [7].

Consider a parametrized differential equation

$$\dot{x} = f(x,\alpha)$$
$$f : \mathbb{R}^n \times R \to \mathbb{R}^n \text{ smooth} \tag{5}$$
$$f(x,\alpha) = A(\alpha)x + O(|x|^2) \quad \text{near } x = 0$$

as described above. Assume the $n \times n$ matrix $A(\alpha)$ is nonsingular for all α, and let $P \subseteq \mathbb{R}$ denote the set of values α for which $A(\alpha)$ has an eigenvalue on the imaginary axis.

Fix any isolated point $\alpha_0 \in P$ and let $\pm i\omega_0$, $\omega_0 > 0$, be a pair of eigenvalues of $A(\alpha_0)$. As the integer multiples of $i\omega_0$ may also be eigenvalues of $A(\alpha_0)$, let

$m(c)$ = the generalized (or algebraic) multiplicity of $ic\omega_0$ as an eigenvalue of $A(\alpha_0)$, for $c = 1,2,3,\ldots$.

Thus $m(1) > 0$ and $m(c) = 0$ for large c. For $|\alpha - \alpha_0| \neq 0$ small there are no eigenvalues of $A(\alpha)$ on the imaginary axis near any $ic\omega_0$. Hence $ic\omega_0$ splits into various eigenvalues nearby, some in the left half plane and some in the right, but still with total multiplicity $m(c)$. For small $\varepsilon > 0$ let

$r^{\pm}(c)$ = the generalized multiplicity of those eigenvalues of $A(\alpha)$, near $ic\omega_0$, which are in the right half plane, for $0 < \pm(\alpha-\alpha_0) < \varepsilon$.

Thus $0 \leq r^{\pm}(c) \leq m(c)$, and the corresponding multiplicity of eigenvalues in the left half plane is $m(c) - r^{\pm}(c)$. Finally, set

$$r(c) = r^{+}(c) - r^{-}(c).$$

To describe the bifurcation, let

$$\left.\begin{aligned} B &= \{(T,0,\alpha) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R} \mid \alpha \in P,\ T = 2\pi k/|\omega| \text{ where } i\omega \text{ is an eigen-} \\ &\quad \text{value of } A(\alpha) \text{ and } k > 0 \text{ is an integer}\}, \\ \Lambda &= \{(T,a,\alpha) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}\ x(T,a,\alpha) = a\}, \\ K &= (\Lambda - (0,\infty) \times \{0\} \times \mathbb{R}) \cup B, \end{aligned}\right\} \quad (6)$$

where $x(t,a,\alpha)$ denotes the solution of (5) with $x = a$ at $t = 0$. Since periodic solutions near $x = 0$ can only exist when $A(\alpha)$ has eigenvalues $\pm i\omega$ on the imaginary axis, and then only with periods near $2\pi k/|\omega|$, $k = 1,2,3,\ldots$, it follows that B represents the possible bifurcation points of periodic solutions from $x = 0$. Consider the values α_0 and ω_0 chosen above and for $c = 1,2,\ldots$ let

$$p_c = (2\pi/(c\omega_0),0,\alpha_0)$$

and

K_c = the maximal connected component of $\overline{K}$ containing p_c ($K_c = \emptyset$ if $p_c \notin B$).

We may now state the main theorem.

<u>THEOREM 1.</u> *Assume*

$$\sum_{c=1}^{\infty} \frac{1}{c} \gamma(c) \neq 0. \tag{7}$$

Then either

(1) K_1 *contains a point* $(T,a,\alpha) \neq (2\pi/\omega_0, 0, \alpha_0)$ *where* $T > 0$ *and* (a,α) *is a critical point of equation (5); or*

(2) K_1 *is disjoint from* $\{0\} \times \mathbb{R}^n \times \mathbb{R} \subseteq \Lambda$ *is unbounded in* $(0,\infty) \times \mathbb{R}^n \times \mathbb{R}$, *that is, contains points* (T,a,α) *with* $T + |a| + |\alpha|$ *arbitrarily large.*

Applications to functional differential equations may be found in [3]. In [5], Theorem 1 is used to prove Liapunov center theorem. However, even though Theorem 1 is "global" but it becomes a "local" result for Liapunov center theorem. The reason is that K_1 in Theorem 1 may be unbounded in $(0,\infty) \times \mathbb{R}^n \times \mathbb{R}$, but $\{(a,\alpha): (T,a,\alpha) \in K_1\}$ may be bounded together with the least period T_0 associated with the periodic orbit through the point (a,α). To illustrate this point, consider the following example in [8]. A parametrized differential equation

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^4, \quad 0 \leq \alpha \leq 1, \tag{8}$$

is constructed in [8] with the following properties:

(1) equation (8) has an isolated periodic orbit $\gamma(\alpha)$ for all $0 \leq \alpha \leq 2/3$;

(2) $\gamma(\alpha)$, $0 \leq \alpha \leq 1$, is hyperbolic except at $\alpha = 1/3, 2/3$;

(3) $\gamma(1/3)$ has a generic period doubling bifurcation, i.e., a second family of periodic orbits $\gamma_1(\alpha)$, $1/3 \leq \alpha \leq 2/3$, bifurcates from $\alpha(1/3)$ and the least periods of $\gamma_1(\alpha)$ for α near $1/3$ are approximatily twice that of $\gamma(1/3)$;

(4) $\gamma(2/3)$ has a generic saddle-node bifurcation, i.e., $\gamma_1(\alpha)$ and $\gamma(\alpha)$ coalesce and annihilate each other at $\alpha = 2/3$.

In Figure 1, a schematic diagram of this example is shown.

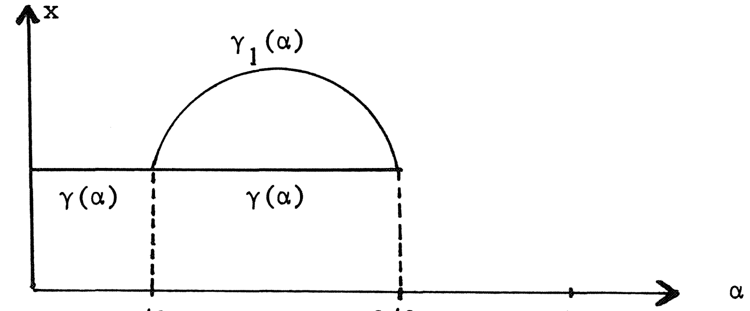

Fig. 1

If $\gamma(\alpha)$ and the periods of $\gamma(\alpha)$ were elements of K_1 in Theorem 1, then the set K_1 would be unbounded. This is shown in Figure 2, where the orbits are represented by the parameter α and their periods T and the least periods of $\gamma(\alpha)$ are assumed to be $T_0 > 0$. We note that

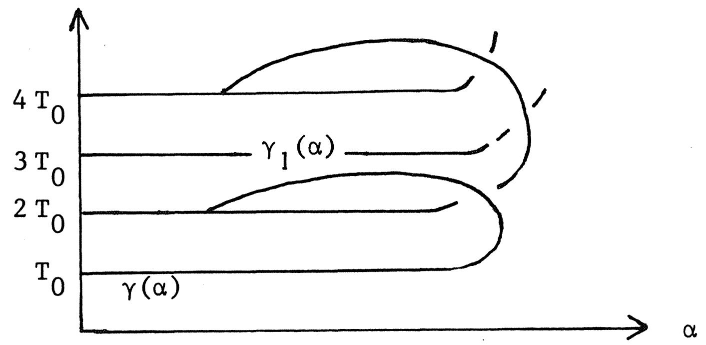

Fig. 2

the sit $\{(a,\alpha) = (T,a,\alpha) \in K_1\}$ with its least periods is bounded.
On the other hand, in order for $\gamma(\alpha)$, $0 \le \alpha < 1/3$, to have such behavior, $\gamma(\alpha)$ must have a non-orientable unstable manifold and such orbits could not be connected to a Hopf bifurcation point without any bifurcations ([9], [10]).
This indicates the possibility to extend Theorem 1 to include least periods.
In this report, we will present a theorem which says essentially Theorem 1 is true if we replace periods by least periods provided the phase space $\mathbb{R}^n$ is 3 or 4 dimensional, i.e., n = 3 or 4. This result is new and was found in collaburation with K. Alligood, J. Mallet-Paret and J. Yorke.
The following definition is essential in our approach.

DEFINITION 2. Let γ be a periodic orbit of (1) with least period $T_0 > 0$ and π be its Poincaré map at $a \in \mathbb{R}^n$. Let $A = D\pi(a)$ be the derivative of π at a and

$$M = \{m \ge 1: \text{there exists } x \in \mathbb{R}^{n-1} \text{ with } x, Ax, \dots, A^{m-1}x \text{ distinct, but } x = A^m x\}.$$

We say T is a *virtual period* of γ if $T = m\,T_0$ for some $m \in M$.

DEFINITION 3. γ is said to be a *nice periodic orbit* of (1) if the Poincaré map π of γ at satisfies the condition that a is an isolated fixed point for each iterate π^k, k = 1,2,3,..., though the neighborhood of isolation may depend on k.
The following theorems indicate the role of virtual periods.

THEOREM 4 [10]. *Let* γ *be a nice periodic orbit and* π *be the Poincaré map of* γ *at* a. *Let* $\tilde{\pi}$ *be* C'*-close to* π. *Then a necessary condition for there to exists* b *close to* a *with* b, $\tilde{\pi}(b)$, ..., $\tilde{\pi}^{m-1}(b)$ *distinct, but* $b = \tilde{\pi}^m(b)$ *is that* $m \in M$, *where* M *is as in Definition 2.*

THEOREM 5 [10]. *Let* γ *be a nice periodic orbit and* π *be the Poincaré map of* γ *at* $a \in \mathbb{R}^n$. *Let* k_m *denote the fixed point index of* π^m, $m \geq 1$. *Then the vector* $k = (k_1, k_2, \ldots)$ *has the form*

$$k = \begin{cases} \sum_{m \in M} c_m j_m & \sigma- = \text{even} \\ \sum_{m \in Me} c_m j_m + \sum_{m \in M_0} c_m (j_m - j_{2m}), & \sigma- = \text{odd} \end{cases}$$

where $\sigma-$ *is the number of eigenvalues of the derative* $D\pi(a) = A$, *counting multiplicity, in* $(-\infty, -1)$, M *is the set in Definition 2,* c_m *are integers,* $Me = \{m: m \in M, m \text{ is even}\}$, $M_0 = M \backslash Me$, *and* j_m *is the vector* $j_m = (j_{ma})_{a=1}^{\infty}$ *with*

$$j_{ma} = \begin{cases} m, & \text{if } m \text{ divides } a, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5 says that the following definition is well-defined.

DEFINITION 6. Let γ be a nice periodic orbit of (1). The ϕ *index of* γ, $\phi(\gamma)$, is defined by

$$\phi(\gamma) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} k_m$$

where k_m is the fixed point index of π^m, the m th iterate of the Poincaré map π. It is not difficult to see from Theorem 5 that the following are true.

PROPOSITION 7. *The* Φ*-index of a nice periodic orbit is an integer.*

PROPOSITION 8. *If* γ *is a nice periodic orbit with a non-orientable unstable manifold, then* $\phi(\gamma) = 0$.

We have the following "generalization" of Theorem 1 in terms of virtual periods.

THEOREM 9 [11]. *Let* γ_0 *be a nice periodic orbit of (5) for* $\alpha = \alpha_0$. *If the* ϕ*-index,* $\phi(\gamma_0)$, *is nonzero and if* Γ *is the component of periodic orbits of (5) containing* γ_0, *then either of the following conditions hold:*

(a) $\Gamma - (\gamma_0 \times \{\alpha_0\})$ *is connected or*

(b) *each of the two components* Γ_i, $i = 1,2$, *satisfies one of the following:*

(1) Γ_i *is unbounded in* (x,α)-*space,*
(2) $\bar{\Gamma}_i$ *contains a center, i.e., a generalized Hopf bifurcation point;*
(3) *the virtual periods of orbits in* Γ_i *are unbounded.*

REMARK 10. By Proposition 8, the ϕ-index of the periodic orbit $\gamma(\alpha)$ in Figure 1, for any $0 \le \alpha < 1/3$, is zero. This shows that the assumption, $\phi(\gamma_0) \ne 0$, is necessary in Theorem 9.

We are now ready to state and prove our main result.

THEOREM 10. *If the phase space* $\mathbb{R}^n$ *is 3 or 4 dimensional, then under the hypotheses of Theorem 9 condition (b3) may be replaced with the following stronger condition*

(b3') the least periods of orbits in Γ_i *are unbounded.*

LEMMA 11. *A periodic orbit* γ *in* $\mathbb{R}^3$ *or* $\mathbb{R}^4$ *has at most one virtual period in addition to the least period* $T_0 > 0$ *of* γ.

PROOF. Let $\mu_1, \ldots, \mu_k$ denote characteristic multipliers of γ (k=2 or 3). Note that $2T_0$ is a virtual period if and only if $\mu_i = -1$ for some i; mT_0, $m \ge 3$, is a virtual period if and only if for some $i \ne j$, $\mu_i = \bar{\mu}_j$, $\mu_i^m = 1$ but $\mu_i^p \ne 1$ for any $1 \le p < m$. If the phase space $\mathbb{R}^n$ is 3-dimensional, then there exists at most one virtual period since $k = 2$. If $\mathbb{R}^n$ is 4-dimensional and there are two distinct virtual periods in addition to the least period T_0, then we may assume $\mu_1 = -1$, $\mu_2 = e^{-i\theta}$, $\mu_3 = e^{i\theta}$. The product $\mu_1\mu_2\mu_3 = -1$. This contradicts that the Poincaré map is orientation preserving.

PROOF OF THEOREM 10. Suppose no other conditions in Theorem 9 are satisfied except (b3) for Γ_i. We will show that (b3') is satisfied by Γ_i.
It can be shown as in [11] that if $a_2 > a_1$, are sufficiently large, there exists a compact connected set $Q \in \Gamma_i$ such that $(\gamma,\alpha) \in Q$ implies the virtual period of γ lies in $[a_2, 2a_1]$. Furthermore, for each $a \in [a_1, a_2]$, there exists $(\gamma,\alpha) \in Q$ such that the virtual period of γ is in $[a, 2a)$.
Suppose (b3') is false. Then there exist $T_2 > T_1 > 0$ such that $(\gamma,\alpha) \in \Gamma_i$ implies the least period of γ is in $[T_1,T_2]$. We may assume

$$a_1 > T_2, \quad a_2 > \frac{2T_2}{T_1} a_1 .$$

By Lemma 11, there is at most one virtual period for γ. Denote the least periods and virtual periods by $T_0(\gamma,\alpha)$ and $m(\gamma,\alpha)T_0(\gamma,\alpha)$ for $(\gamma,\alpha) \in Q$. By the property of Q, there exist (γ_1,α_1), $(\gamma_2,\alpha_2) \in Q$ such that $m(\gamma_j,\alpha_j)T_0(\gamma_j,\alpha_j) \in [a_j,2a_j)$, $j = 1,2$. This implies $m(\gamma_1,\alpha_1) < m(\gamma_2,\alpha_2)$. We will obtain a contradiction by showing $m(\gamma,\alpha)$ is constant for $(\gamma,\alpha) \in Q$.

Since Q is compact and connected, if suffices to show that $m(\gamma,\alpha)$ is continuous on Q. This amounts to showing the least period $T_0(\gamma,\alpha)$ is continuous on Q. If $(\gamma_1,\alpha_1) \in Q$, then $T_0(\gamma_1,\alpha_1)$ is near $T_0(\tilde{\gamma},\tilde{\alpha})$ or $m(\tilde{\gamma},\tilde{\alpha})T_0(\tilde{\gamma},\tilde{\alpha})$ for $(\tilde{\gamma},\tilde{\alpha})$ near (γ_1,α_1). But the latter is impossible, because $m(\tilde{\gamma},\tilde{\alpha})T_0(\tilde{\gamma},\tilde{\alpha}) \geq a_1 > T_2$, violationg the bounds on the least periods. Thus $T_0(\gamma,\alpha)$ is continuous on Q.

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