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THE PARAMETRISATION OF THE UNSTABLE INVARIANT MANIFOLD FOR A CLASS OF HORSESHOE MAPS

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The parametrisation of the unstable invariant manifold for a class of horseshoe maps

by

H.A. Lauwerier

ABSTRACT

A few planar maps are considered for which there exists an unstable invariant manifold originating from a saddle fixed point having a parametrisation by means of easily computable analytic functions.

KEY WORDS & PHRASES: *Iterated maps, functional equations, strange attractors*

1. INTRODUCTION

Our starting-point is an iterated map of the unit interval into itself

(1.1) $$y_{n+1} = \phi(y_n) , \quad n = 0, 1, 2,...$$

where ϕ is an analytic function with

(1.2) $$\phi(0) = 0, \quad |\phi'(0)| > 1.$$

This means that $y = 0$ is a repelling fixed point. Poincaré noted that there exists an analytic function $F(z)$ such that

(1.3) $$F(az) = \phi (F(z)) ,$$

where $a = \phi'(0)$.
With the initial condition

(1.4) $$F(0) = 0 , F'(0) = 1$$

$F(z)$ is uniquely determined. If ϕ is entire then also F is an entire function. The iterated map (1.1) can be parametrised by

(1.5) $$y_n = F(a^n z_0)$$

where z_0 is determined by y_0.

The simplest non-trivial case is the map

(1.6) $$y_{n+1} = 4y_n(1-y_n)$$

for which

(1.7) $$F(z) = \sin^2 \sqrt{z}.$$

Similar functions for the more general case

$$(1.8) \qquad y_{n+1} = ay_n(1-y_n)$$

have been considered by the author elsewhere.

In this note we consider a family of horseshoe-type maps with (1.1) as its essential part. The planar map

$$(1.9) \qquad \begin{cases} x_{n+1} = bx_n(\frac{1}{2}-y_n) + y_n, \\ y_{n+1} = \phi(y_n), \end{cases}$$

with $0 < b < 1$ clearly maps the unit square $0 \le x, y \le 1$ into itself. There exists an invariant manifold J starting from the saddle (0,0) as an analytic curve

$$(1.10) \qquad x = E(t) \ , \quad y = F(t).$$

In this note we show how to compute these functions. Of particular interest is the special case where ϕ is given by (1.6). For this case the unstable manifold J is parametrised as

$$(1.11) \qquad \begin{cases} E(t) = \frac{1}{2} - \frac{1}{4}(1-\frac{b}{2}) \sum_{k=0}^{\infty} \frac{(b/4)^k \sin 2\sqrt{t}}{\sin (2^{-k}\sqrt{t})} \ , \\ F(t) = \sin^2\sqrt{t}. \end{cases}$$

This shows that J is like a sine curve folded up an infinite number of times so that it fits inside a square. It is of interest to compare a computer plot of the continuous J-line with the strange attractor plot of (1.9) with an orbit of say a thousand points.

This simple and special case for which almost everything can be computed explicitly is considered in section 2. In the subsequent section we show that the overall picture remains the same if the special map (1.6) is replaced by the more general map (1.8) or even by $y \to \phi(y)$ where $\phi(y)$ is an analytic function with $\phi(0) = 0$. The general result is that J can be described by

$$(1.12) \qquad x = E(t) \ , \quad y = F(t)$$

where F is determined by (1.3), (1.4) and E by

$$(1.13)\qquad \begin{cases} E(az) = b\,E(z)\,(\tfrac{1}{2}-F(z)) + F(z)\,, \\ E(0) = 0. \end{cases}$$

If $\phi(y)$ is a polynomial or an entire function the functions $E(z)$ and $F(z)$ are also entire but if $\phi(y)$ has a pole or a branch-point the functions $E(z)$ and $F(z)$ have generally a finite radius of convergence at $z = 0$. However, they can still be computed for a substantial range of real t-values by using a suitable combination of the power series expansions and the functional equations.

In section 4 we consider a few interesting particular cases for which the unstable manifold can be described by (simple) analytic functions. We may draw attention to the amusing map (4.1) which in spite of its simplicity is very interesting and illuminating. It demonstrates the presence of a simple invariant manifold as a space-filling Lissajous curve forming the background of an apparently fully chaotic map. The other examples illustrate the general theory. Computer plots, obtained from an HP 85 with a plotter, show the similarity between the line-plots of J and the point plots showing an orbit of some 1000 points.

2. A SIMPLE MAP

We consider the iterative map

$$(2.1)\qquad \begin{cases} x_{n+1} = bx_n(\tfrac{1}{2}-y_n) + y_n\,, \\ y_{n+1} = 4y_n(1-y_n). \end{cases}$$

with $0 < b < 1$.

As is shown in the figure 1a and 1b the unit square is mapped into itself in such a way that a vertical line $x = \xi$, $0 \le y \le 1$ is transformed into a parabolic arc

$$y = \phi\left(\frac{x-b\xi/2}{1-b\xi}\right), \quad \phi(x) = 4x(1-x).$$

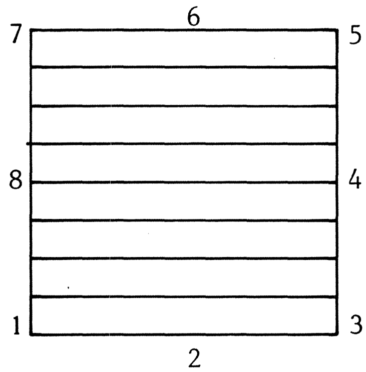

fig. 1a

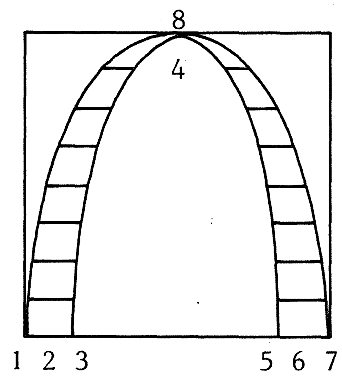

fig. 1b

There are the two fixed points (0,0) and $\left(\frac{3}{4+b}, \frac{3}{4}\right)$ both being saddles. The horizontal lines y = 0 and y = 3/4 are the stable invariant manifolds. The unstable invariant manifold J can be determined as follows.
The substitution

$$(2.2) \qquad \begin{cases} 2b\, x_n = b - (2-b)\, u_n \\ 2y_n = 1 - v_n \end{cases}$$

changes (2.1) into

$$(2.3) \qquad \begin{cases} u_{n+1} = \tfrac{1}{2}b\,(1+u_n)\, v_n\,, \\ v_{n+1} = 2v_n^2 - 2. \end{cases}$$

A simple calculation shows that

$$u_n = (b/2)v_{n-1} + (b/2)^2 v_{n-1}v_{n-2} + (b/2)^3\, v_{n-1}v_{n-2}v_{n-3} + \cdots$$
$$+ (b/2)^n v_{n-1}v_{n-2} \cdots v_0(1+u_0).$$

The sequence v_n can be parametrised by

$$(2.5) \qquad v_n = \cos\,(2^n z).$$

We note that

$$(2.6) \qquad v_0 v_1 v_2 \cdots v_{m-1} = \frac{2^{-m}\sin(2^m z)}{\sin z}\,.$$

Thus (2.4) can be written as

$$(2.7) \qquad u_n = \sum_{k=1}^{n} \frac{(b/4)^k \sin(2^n z)}{\sin(2^{n-k} z)} + \frac{(b/4)^n \sin(2^n z)}{\sin z}\, u_0 \, .$$

This shows that the unstable invariant manifold J can be described explicitly as

$$(2.8) \qquad u = \sum_{k=1}^{\infty} \frac{(b/4)^k \sin t}{\sin(2^{-k} t)} \, , \qquad v = \cos t.$$

This special case gives a means to study the phenomena attached to invariant manifolds and strange attractors in a very explicit way. This may be helpful for understanding the situation in a general case where such a parametrisation is not possible or not elementary.

Formula (2.5) shows that the sequence v_n strongly depends on the starting value v_0 but (2.7) shows that u_n for $n \to \infty$ is independent of u_0.

It is of some interest to consider the turning points of J, i.e. those points for which v = 1. In the original x,y plane they correspond to the points where J meets the line y = 0.

They are produced by the values $t = t_m$ with

$$(2.9) \qquad t_m = 2\pi m \, , \quad m = 2^q p$$

where $m > 0$, $p > 0$ and $q \geq 0$ are integers with p odd. A simple calculation shows that

$$\frac{\sin t_m}{\sin(2^{-k} t_m)} = \begin{cases} 2^k & \text{for} \quad 0 \leq k \leq q, \\ -\,2^{q+1} & \text{for} \quad k = q+1, \\ 0 & \text{for} \quad k > q+1. \end{cases}$$

Substitution in (2.8) gives

$$u(t_m) = b/2 + (b/2)^2 + \ldots + (b/2)^q - (b/2)^{q+1}.$$

For the x,y - plane this gives

$$(2.10) \qquad x(t_m) = (1-\tfrac{b}{4})(\tfrac{b}{2})^q , \qquad y(t_m) = 0.$$

This shows that the points determined by $t = t_m$ are the turning points each of an infinity of folds of the attractor J.

3. THE GENERAL MAP

For the slightly more general map

$$(3.1) \qquad \begin{cases} x_{n+1} = bx_n\,(\tfrac{1}{2}-y_n) + y_n , \\ y_{n+1} = ay_n\,(1-y_n) , \end{cases}$$

with $1 < a \le 4$ the situation is not much different. Again there are two fixed points. The origin is a saddle with multipliers a and b/2. The other fixed point

$$(3.2) \qquad x = \frac{a-1}{ab/2 + a-b} , \qquad y = \frac{a-1}{a}$$

has the multipliers $2-a$ and $b(1-a/2)/a$. It is an attractor for $1 < a < 3$ and a saddle for $3 < a \le 4$.
The substitution (2.2) changes (3.1) into

$$(3.3) \qquad \begin{cases} u_{n+1} = \tfrac{1}{2}b\,(1+u_n)\,v_n , \\ v_{n+1} = \tfrac{1}{2}a\,v_n^2 - (a-2). \end{cases}$$

Let F(z) be the Poincaré function satisfying the multiplication rule

$$(3.4) \qquad \begin{cases} F(az) = aF(z)\,(1-F(z)) , \\ F(0) = 0 , \; F'(0) = 1. \end{cases}$$

It is known [1] that F(z) is an entire function

$$(3.5) \qquad F(z) = z - \frac{z^2}{a-1} + \frac{2z^3}{(a-1)(a^2-1)} - \frac{(a+5)z^4}{(a-1)(a^2-1)(a^3-1)} + \ldots$$

They may serve as an analytical parametrisation of the sequence v_n

$$v_n = 1 - F(a^n z). \tag{3.6}$$

Again we may use the expression (2.4) and make similar conclusions as in the special case a = 4. However, there is no elementary analogue of (2.6) and (2.7).

The unstable manifold J can be described as

$$\begin{aligned} u = \frac{b}{a}\sqrt{\lambda+w} + \left(\frac{b}{a}\right)^2 \sqrt{\lambda+\sqrt{\lambda+w}}\,.\,\sqrt{\lambda+w} \;+ \\ + \left(\frac{b}{a}\right)^3 \sqrt{\lambda+\sqrt{\lambda+\sqrt{\lambda+w}}}\,.\,\sqrt{\lambda+\sqrt{\lambda+w}}\,.\,\sqrt{\lambda+w} + \dots , \end{aligned} \tag{3.7}$$

where $\lambda = a(a-2)/4$ and $w = av/2$. In (3.7) any root may take both signs. If $w = a/2$ and all roots are taken with the plus-sign we obtain

$$u = b/2 + (b/2)^2 + (b/2)^3 + \dots = b/(2-b).$$

This shows that J passes through the fixed point at the origin. If $w = 1-a/2$ and all roots are taken with the minus sign we find for u a value which shows that J also passes through the second fixed point.

Of course the expression (3.7) is not very helpful for the actual computation of J. However, it is possible even for more general maps to derive a parametrisation of the kind (2.8).

We may replace (3.1) by the more general map

$$\begin{cases} x_{n+1} = bx_n(\tfrac{1}{2}-y_n) + y_n , \\ y_{n+1} = \phi(y_n) , \end{cases} \tag{3.8}$$

where $\phi(y)$ is an analytic function holomorphic for $|y| < 1$ with $\phi(0) = 0$ and $|\phi'(0)| > 1$. Of course $\phi(y)$ should map the real interval [0,1] into itself. According to POINCARÉ [2] the sequence y_n can be parametrised by an analytic function F(z) satisfying the multiplication rule

$$F(az) = \phi(F(z)) \tag{3.9}$$

where $a = \phi'(0)$.
With the initial condition

$$F(0) = 0 , \quad F'(0) = 1$$

$F(z)$ is uniquely determined. $F(z)$ is holomorphic in some neighbourhood of $z = 0$. The functional equation (3.9) gives an analytic continuation. The first few coefficients of the power series expansion

$$(3.10) \qquad F(z) = z + c_2 z^2 + c_3 z^3 + \ldots$$

can be determined from (3.9) in a straightforward way. The combined use of (3.9) and (3.10) permit the computation of $F(z)$ for arbitrary real values of z.
Next we introduce the analytic function $K(z)$ by

$$(3.11) \qquad K(z) = \prod_{k=1}^{\infty} (1-2\, F(z/a^k))$$

We note that $K(0) = 1$. A simple calculation shows that $K'(0) = -\, 2/(a-1)$.
From (3.11) we obtain the functional equation

$$(3.12) \qquad 1 - 2\, F(z) = \frac{K(az)}{K(z)} .$$

We note that $F(z)$ and $K(z)$ are entire functions if $\phi(z)$ is entire.
The one-dimensional map $y_{n+1} = \phi(y_n)$ can be parametrised as

$$(3.13) \qquad y_n = F(a^n z).$$

The substitution (2.2) shows that

$$(3.14) \qquad v_n = \frac{K(a^{n+1} z)}{K(a^n z)} .$$

Therefore the general expression (2.7) takes the form

$$(3.15)\qquad u_n = \sum_{k=1}^{n} \frac{(b/2)^k K(a^n z)}{K(a^{n-k} z)} + \frac{(b/2)^n K(a^n z)}{K(z)}\, u_0\ .$$

obviously a generalisation of (2.7). Accordingly the invariant manifold J can be described as

$$(3.16)\qquad \begin{cases} x = \frac{1}{2} - (\frac{1}{b} - \frac{1}{2})\ K(t) \sum_{k=1}^{\infty} \frac{(b/2)^k}{K(t/a^k)} \\ y = F(t/a). \end{cases}$$

Examples

a. For $\phi(z) = 4z(1-z)$ we have $F(z) = \sin^2\sqrt{z}$ and $K(z) = \sin 2\sqrt{z}\ /(2\sqrt{z})$ according to the results in section 2.
b. For $\phi(z) = 2z\ (1-z)$ we have $F(z) = (1-\exp(-2z))/2$ and $K(z) = \exp(-2z)$. The invariant manifold is a curve connecting the fixed points (0,0) and $(\frac{1}{2},\frac{1}{2})$ determined by

$$x = \tfrac{1}{2} - (\tfrac{1}{b} - \tfrac{1}{2}) \sum_{k=1}^{\infty} (1-2y)^{(1-2^{-k})} (b/2)^k\ , \qquad 0 \le y \le \tfrac{1}{2}.$$

For the actual computation of the invariant manifold for a given map $y \to \phi(y)$ it is not necessary to compute the auxiliary function K(z). Instead we may write (3.16) as

$$(3.17)\qquad x = E(t/a)\ ,\quad y = F(t/a)$$

and determine the analytic function E(z) by using a single iteration step of (3.8)

$$\begin{cases} x' = bx\ (\frac{1}{2}-y) + y = bE(t/a)(\frac{1}{2}-F(t/a)) + F(t/a), \\ y' = \phi(F(t/a)). \end{cases}$$

Of course we should obtain

$$x' = E(t)\ ,\quad y' = F(t).$$

In this way we obtain the functional equation

$$(3.18)\qquad E(az) - \tfrac{1}{2}b\ E(z) = F(z) - b\ E(z)\ F(z).$$

Setting

$$(3.19)\qquad E(z) = \sum_{k=1}^{\infty} e_k z^k , \quad F(z) = \sum_{k=1}^{\infty} c_k z^k \text{ with } c_1 = 1$$

we obtain by comparing equal powers of z

$$(3.20)\qquad (a^n - b/2)\, e_n = c_n - b \sum_{k=1}^{n-1} c_k e_{n-k} .$$

Again the actual computation of E(z) is relatively easy. Using (3.20) we may find a few coefficients of the power series expansion. Together with (3.18) in the form

$$(3.21)\qquad E(z) = (\tfrac{1}{2} - F(z/a))\, E(z/a) + F(z/a)$$

we may cover a substantial range of z values.

4. EXAMPLES

In this section we consider a few particular cases for which an analytical description of the non-trivial invariant manifold is available. For each case we present the line-map of the invariant manifold as a continuous curve and the corresponding point-map showing an orbit with a starting point on the invariant line. We note that the latter picture is almost indistinguishable from that of an orbit with an arbitrary initial position. Not all cases considered here are covered by the general theory considered here but they all have in common the existence of an unstable invariant manifold to which most orbits are attracted.

a.

$$(4.1)\qquad \begin{cases} x_{n+1} = y_n , \\ y_{n+1} = 4x_n (1-x_n). \end{cases}$$

This rather amusing example combines an apparently chaotic point-map with a simple line-map of the attracting invariant manifold J

$$(4.2)\qquad x = \sin^2 t , \quad y = \sin^2 t\sqrt{2}.$$

This is a Lissajous curve filling the unit square in an everywhere dense way. The manifold intersects itself infinitely often at the (homoclinic) points

$$t = \tfrac{1}{2}(m/\sqrt{2} + n)\pi$$

where m and n are integers. The map (4.1) may be described as a two-dimensional interpolation of a one-dimensional map $x' = 4x(1-x)$. It is clearly a special case of another interesting class of two-dimensional maps. Figure 2 gives an impression of the first few folds of J. Figure 3 gives the corresponding point-map showing a thousand points.

b.

$$(4.3) \qquad \begin{cases} x_{n+1} = bx_n(\tfrac{1}{2}-y_n) + y_n, \\ y_{n+1} = ay_n(1-y_n). \end{cases}$$

This is the map (3.1). For the illustrations we take $b = 2/3$ with $a = 3$, 3.9 and 4. The invariant manifold J is described by (3.17) with F(z) being given by (3.5). In our computer program only six coefficients are used. The first few coefficients of E(z) are determined by the computer using (3.20). The case $a = 3$ is illustrated in figure 4. J connects the fixed point (0,0) with the attracting fixed point (3/5,2/3) which it approaches in a spiralling way. Figure 5 shows an enlargement near the latter fixed point. The corresponding point-plot is of no interest here.
The cases $a = 3.9$ and $a = 4$ are very similar. In both cases we show the line-plots of J and corresponding point-plots (cf. figs. 6,7,8,9).

4.

$$(4.4) \qquad \begin{cases} x_{n+1} = bx_n(\tfrac{1}{2}-y_n) + y_n, \\ y_{n+1} = y_n\,(3-4y_n)^2. \end{cases}$$

This case is covered by the theory presented for the map (3.8). The origin is a saddle with the multiplier $a = 9$. The one-dimensional y-map can be parametrised by the entire function

$$(4.5) \qquad \begin{aligned} F(z) &= \sin^2\sqrt{z} = (1-\cos 2\sqrt{z})/2 = \\ &= z - \frac{1}{3}z^2 + \frac{2}{45}z^3 - \frac{1}{315}z^4 + \dots . \end{aligned}$$

There are two other fixed points $(\frac{1}{2},\frac{1}{2})$ and $(2/(2+b),1)$ both saddles. The unstable invariant curve J is described by (3.17). For $t = 0$, J starts from the fixed point at the origin. We note that for $t = \pi^2/16$

$$F(9^k t) = \tfrac{1}{2} \qquad \text{for} \quad k = 0,1,2,\ldots$$

and next from (3.18)

$$E(9^k t) = \tfrac{1}{2} \qquad \text{for} \quad k = 1,2,\ldots\ .$$

This means that J passes through $(\frac{1}{2},\frac{1}{2})$ an infinite number of times which means that J forms loops at that point.
For $t_k = 9^k \pi^2/4$, $k = 0,1,2\ldots$, we have $F(t_k) = 1$ and next from (3.18)

$$E(t_{k+1}) = 1-\tfrac{1}{2}\, b\, E(t_k).$$

This shows that for $k \to \infty$ the upper points of J converge to the fixed point $(2/(2+b),1)$. Computer plots with $b = 2/3$ are given in fig. 10 and 11.

d.
(4.6)
$$\begin{cases} x_{n+1} = bx_n(\tfrac{1}{2}-y_n) + y_n, \\ y_{n+1} = ay_n\sqrt{1-y_n}. \end{cases}$$

This case is of the form (3.8) but $\phi(y)$ is only holomorphic for $|y| < 1$. The one-dimensional map

$$(4.7) \qquad y_{n+1} = ay_n\sqrt{1-y_n}$$

is of some interest in itself. Since

$$0 \le y\sqrt{1-y} \le 2/\sqrt{27} \qquad \text{for } 0 \le y \le 1$$

only the interval

$$0 < a \le \frac{3}{2}\sqrt{3} = 2.598\ldots$$

is of interest.

There are the two fixed points $y = 0$ with the multiplier a and $y = 1-1/a^2$ with the multiplier $(3-a^2)/2$. The first point is attracting for $0 < a \leq 1$, the second point is attracting for $1 \leq a \leq \sqrt{5}$. At $a = \sqrt{5} = 2.236\ldots$ there starts a Feigenbaum sequence and eventually a stable 3-cycle at $a = 2.557$.

The map (4.7) can be parametrised by the function F(z) determined by

$$(4.8)\qquad \begin{cases} F(az) = a\,F(z)(1-F(z))^{\frac{1}{2}} \\ F(0) = 0\,,\quad F'(0) = 1. \end{cases}$$

The first few terms of its power series expansion are

$$(4.9)\qquad F(z) = z - \frac{z^2}{2(a-1)} + \frac{(5-a)z^3}{8(a-1)(a^2-1)} - \ldots .$$

This function is no longer entire but for all positive real z it is defined by (4.8) and (4.9). Computer plots with $a = 2.598$, $b = 2/3$ are given in fig. 11 and 12.

e.

$$(4.10)\qquad \begin{cases} x_{n+1} = bx_n(\frac{1}{2}-y_n) + y_n, \\ y_{n+1} = \dfrac{4y_n(1-y_n)(1-k^2y_n)}{(1-k^2y_n^2)^2}\,, \qquad 0 \leq k \leq 1. \end{cases}$$

This case is of the form (3.8) but again $\phi(y)$ is only holomorphic for $|y| < k^{-1}$. The one-dimensional y-map is a generalisation of the elementary logistic map (2.1) that is obtained for $k = 0$. In fact, it is obtained from the following multiplication rule for elliptic functions

$$(4.11)\qquad \mathrm{sn}\, 2z = \frac{2\,\mathrm{sn}\,z\ \mathrm{cn}\,z\ \mathrm{dn}\,z}{1-k^2\mathrm{sn}^4 z} .$$

where k is the elliptic modulus. This means that the functional equation

$$(4.12)\qquad F(4z) = \frac{4\,F(z)(1-F(z))(1-k^2F(z))}{(1-k^2F^2(z))^2}$$

is solved by

$$(4.13)\qquad F(z) = \mathrm{sn}^2\sqrt{z} .$$

This function has a double periodic set of poles $\sqrt{z} = 2mK + (2n+1)K'$ in the usual notation. Thus the power series expansion

$$\mathrm{sn}^2\sqrt{z} = z - \frac{1+k^2}{3}z^2 + \frac{2+13k^2+2k^4}{45}z^3$$

(4.14)
$$- \frac{1+30k^2+30k^4+k^6}{315}z^4 + \ldots$$

converges for $|z| < K'^2$ where

(4.15)
$$K' = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{\cos^2\alpha + k^2\sin^2\alpha}}$$

We note the special cases

$$k = 0 \quad F(z) = \sin^2\sqrt{z} \qquad K' = \infty$$

$$k = 1 \quad F(z) = \tanh^2\sqrt{z} \qquad K' = \pi/2.$$

A simple calculation shows that the one-dimensional y-map has the trivial fixed point $y = 0$ and another real fixed point in $(0,1)$. It seems that for $0 \le k < 1$ the map is always chaotic. The special case $k = 1$ on the other hand has the secondary fixed point $y = 1$ with the multiplier -1 and which is globally attracting.

In fig. 13 and 14 computer plots are given for $k^2 = 0.5$.

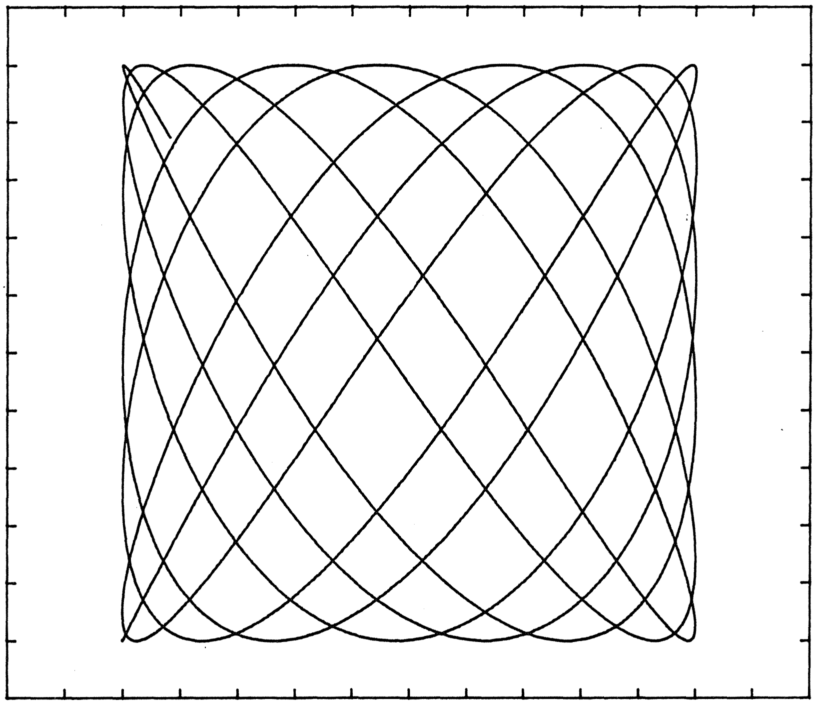

Fig. 2

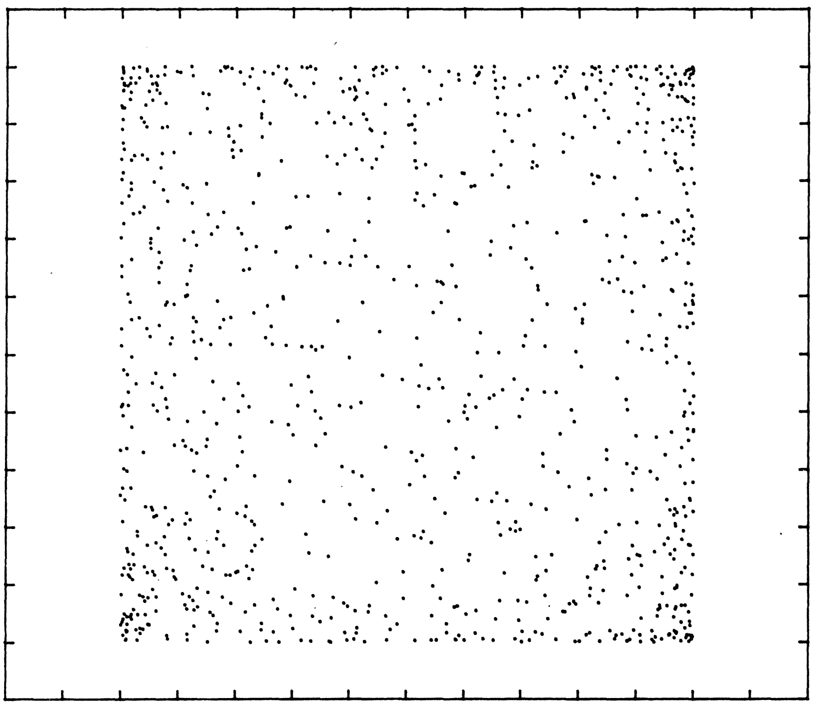

Fig. 3

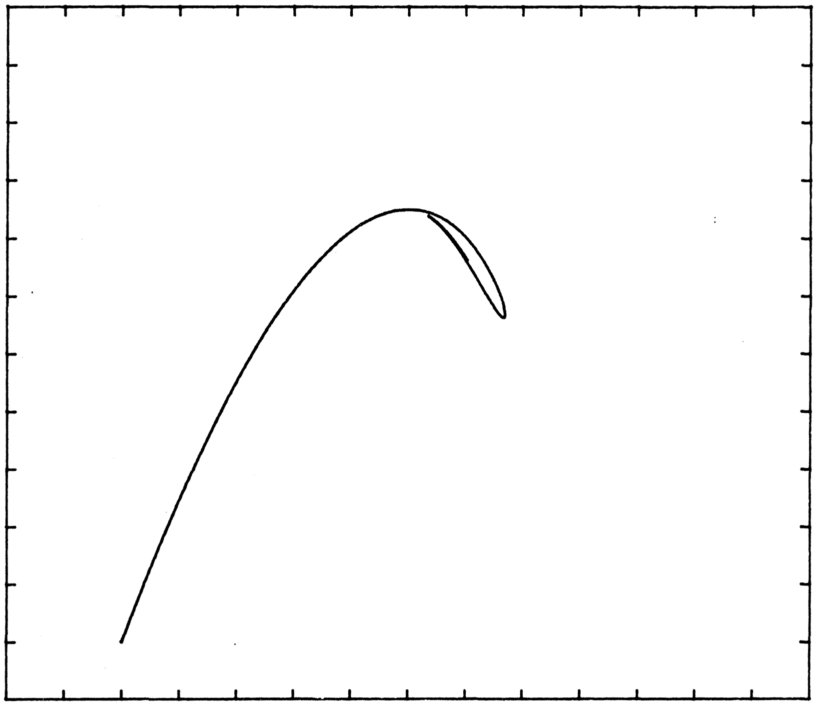

Fig. 4, a = 3

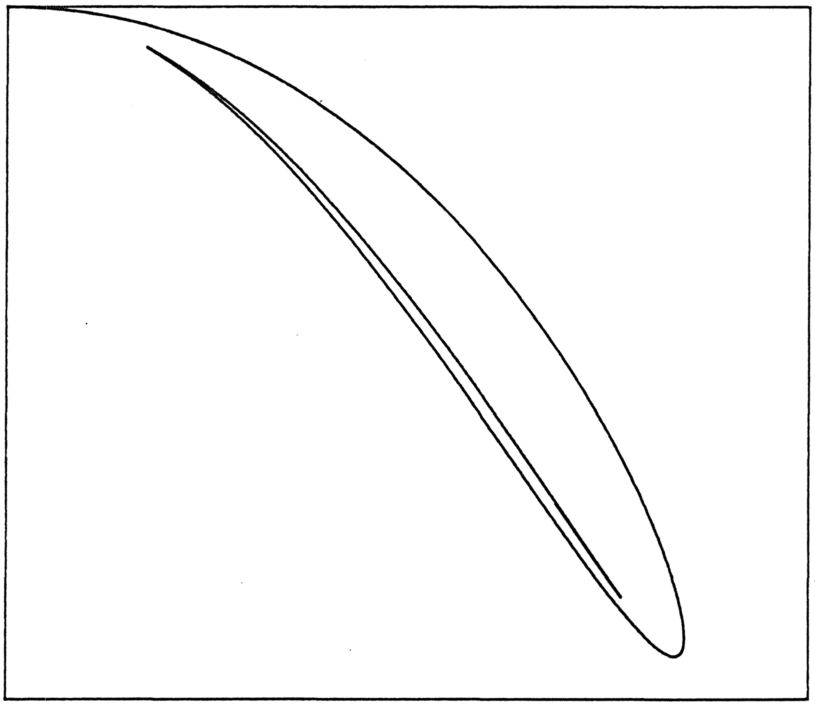

Fig. 5

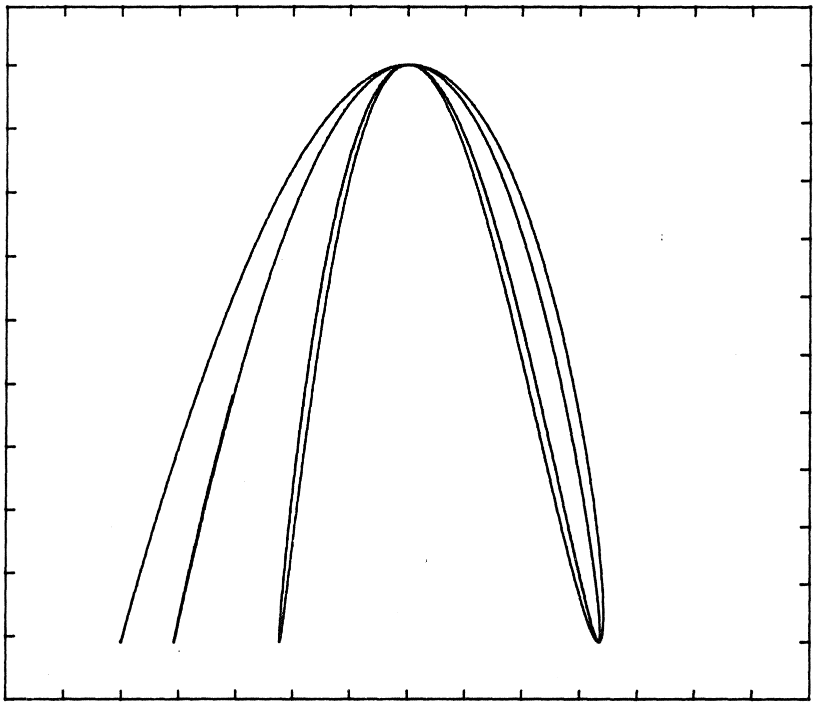

Fig. 6, a = 3.9

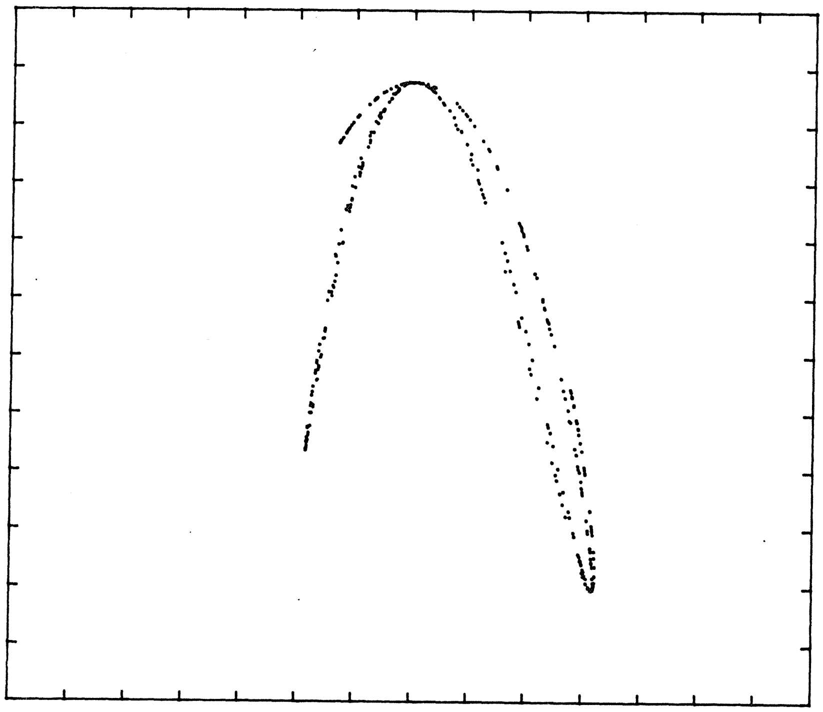

Fig. 7, a = 3.9, b = 4

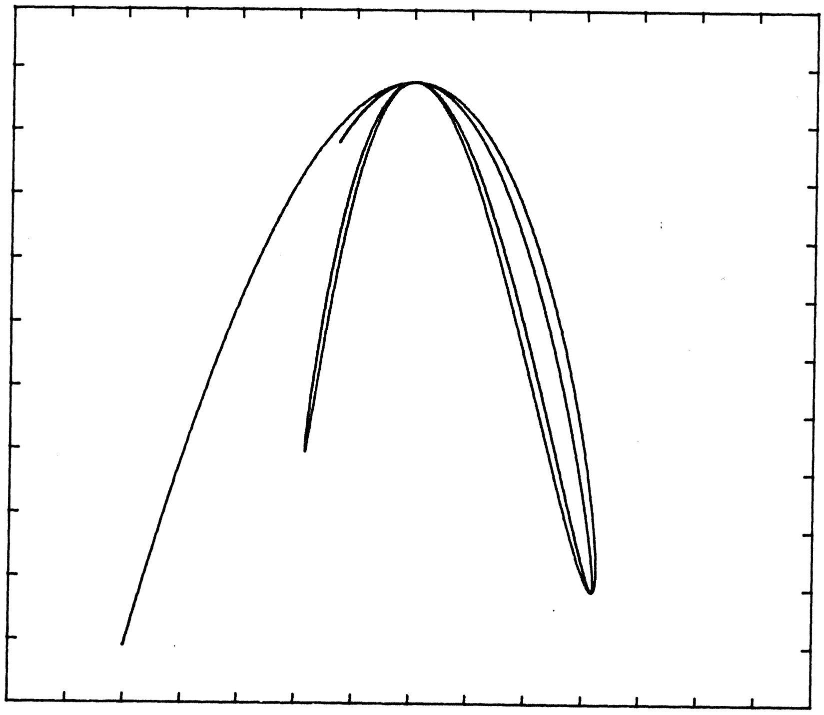

Fig. 8, a = 4, b = 2/3

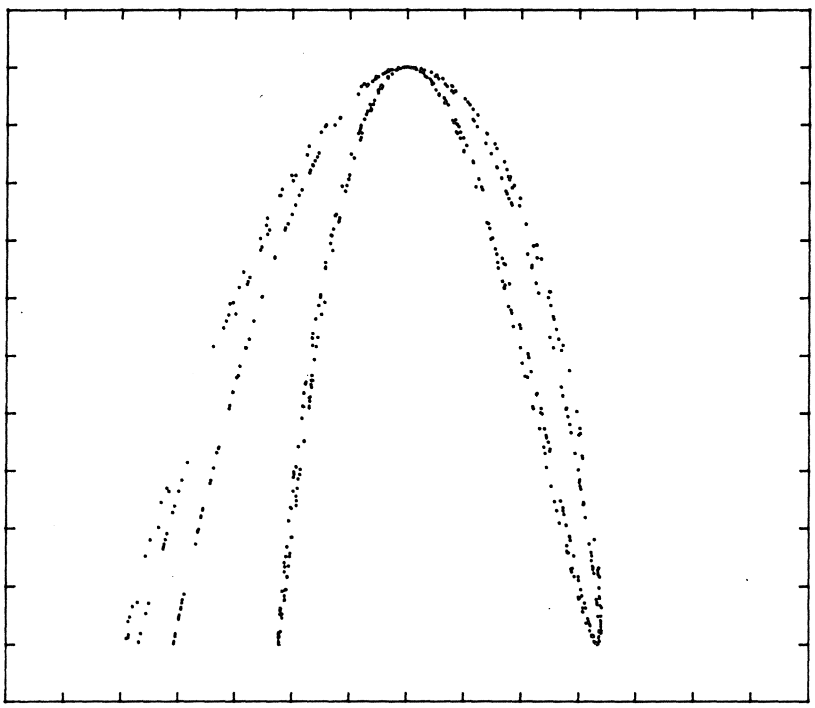

Fig. 9, a = 4, b = 2/3

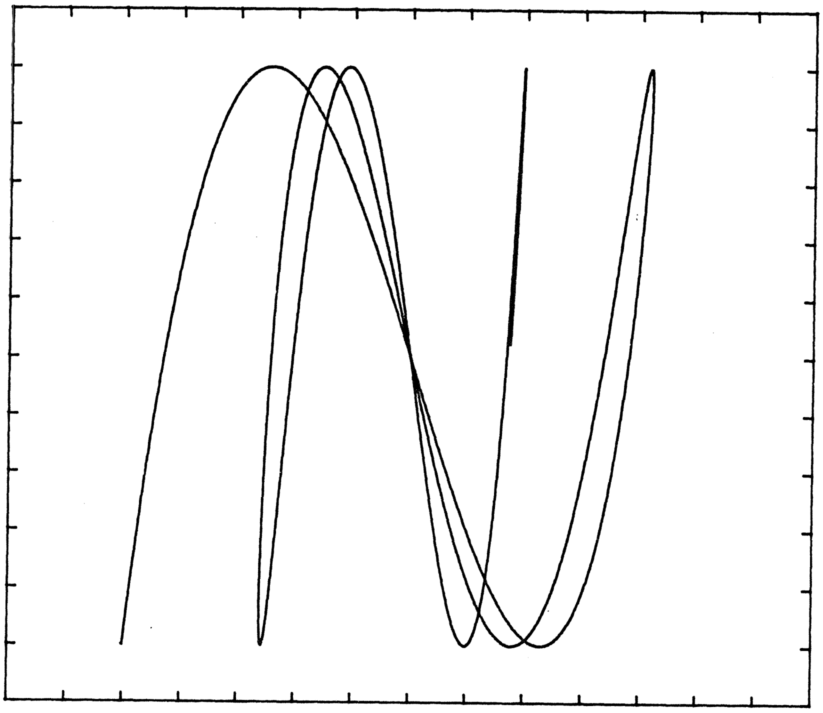

Fig. 10

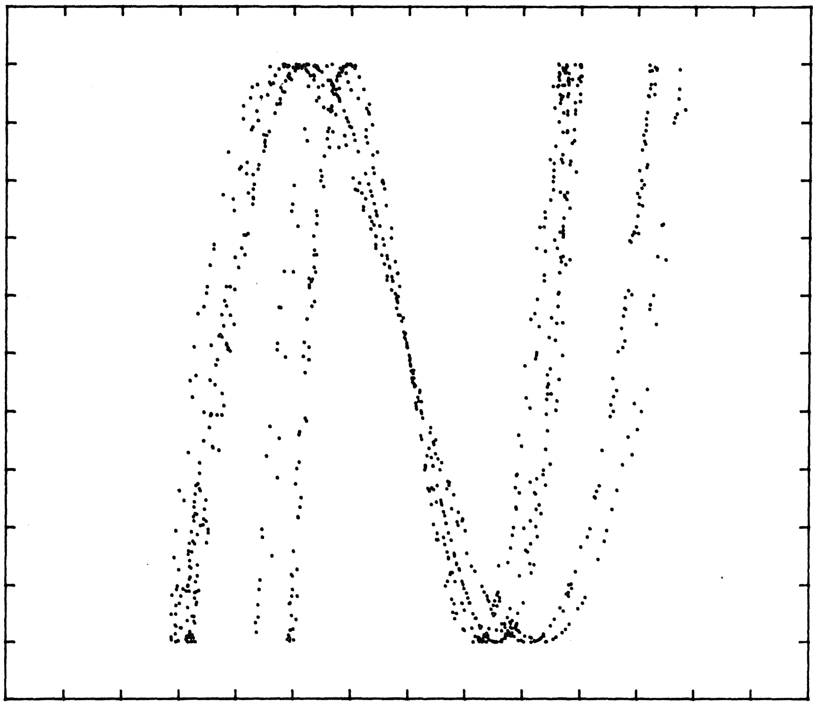

Fig. 11

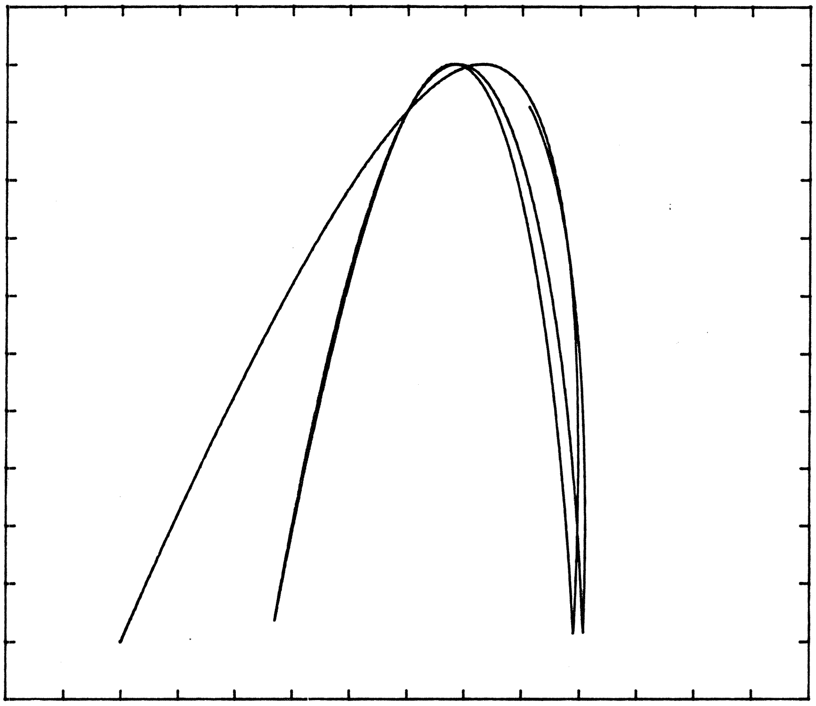

Fig. 12

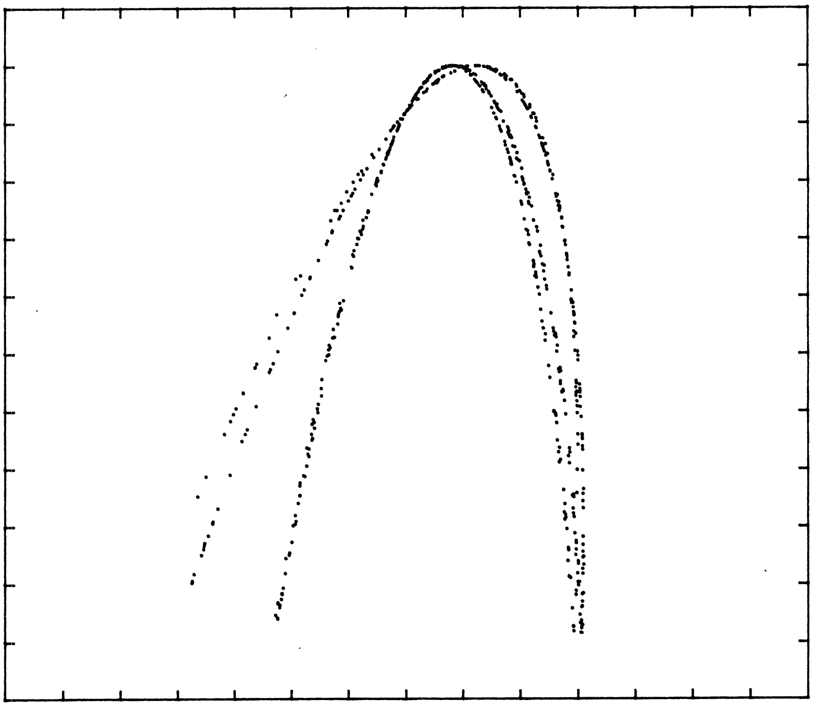

Fig. 13

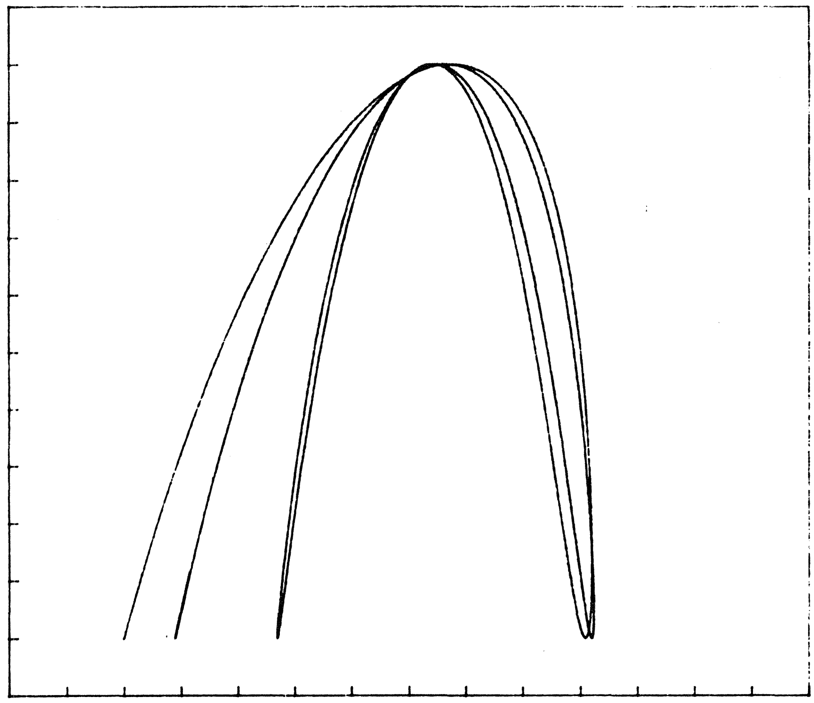

Fig. 14

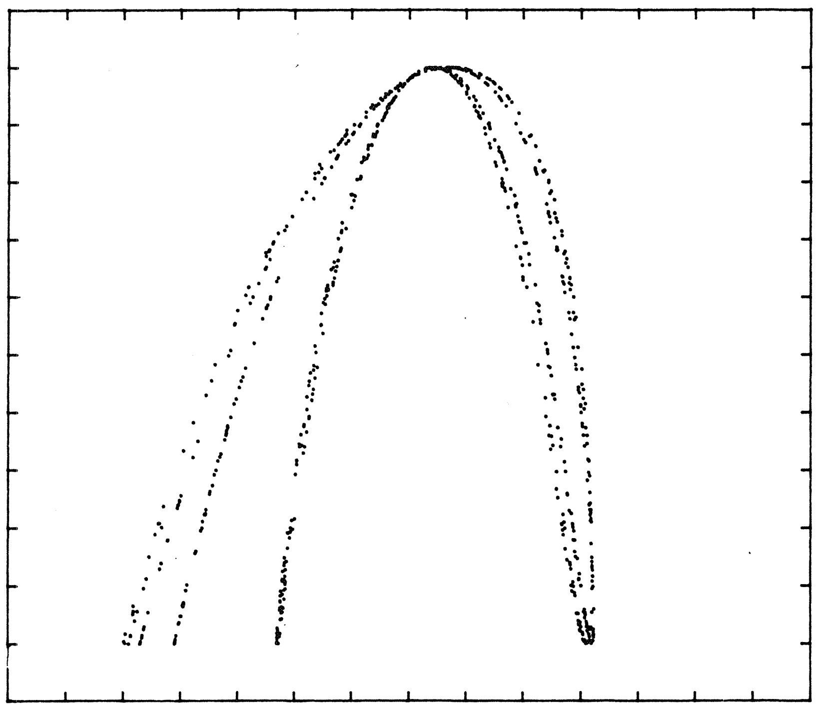

Fig. 15

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(more related literature is given in [1]).