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SUM RULE FOR PRODUCTS OF BESSEL FUNCTIONS: COMMENTS ON A PAPER BY NEWBERGER

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Sum rule for products of Bessel functions: Comments on a paper by Newberger

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ABSTRACT

NEWBERGER (1982) considered series of products of Bessel functions with as special case the form $\Sigma(n^j J_n^2(z))/(n+\mu)$. The interesting point is that he obtained new explicit expressions for the sum of the series. In this note we point out that some results of Newberger are not correct, especially the results obtained by the principle of analytic continuation. Our remarks include a correction for his important result for "Turkin's" function.

KEY WORDS & PHRASES: *Bessel functions, summation of series*

1. INTRODUCTION

NEWBERGER (1982) presented a sum rule for the infinite series of the form

$$(1.1) \qquad S = \sum_{n=-\infty}^{\infty} \frac{(-1)^n n^j J_{\alpha-\gamma n}(z)\, J_{\beta+\gamma n}(z)}{n+\mu},$$

where $j \in \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{C} \setminus \mathbb{Z}$, $\alpha,\beta,z \in \mathbb{C}$, $\gamma \in (0,1]$. Initially, α and β are restricted to Re $(\alpha+\beta) > -1$. Under this last restriction Newberger found interesting explicit expressions for the sum S. Afterwards he extended his results beyond this range of parameters α and β. As will be shown in this note, this last step yields incorrect results.

2. SYMMETRY RELATION FOR S; $\alpha,\beta \in \mathbb{Z}$, $\gamma = 1$.

An important observation is that S is not defined for all α and β, as stated after (1.1). This will be proved in section 4. Here we consider $\gamma = 1$ and integer values of α and β. Then the series is convergent and there is a symmetry rule. To show this we denote S of (1.1) by $S_j(\alpha,\beta,\mu)$. Then we have

$$(2.1) \qquad S_j(-\alpha,-\beta,-\mu) = (-1)^{j+\alpha+\beta+1} S_j(\alpha,\beta,\mu),$$

where we used

$$J_{-n}(z) = (-1)^n J_{-n}(z), \; n \in \mathbb{Z}.$$

"Turkin's" function

$$(2.2) \qquad T_m(z,\alpha) := \sum_{n=-\infty}^{\infty} \frac{J_n(z) J_{n-m}(z)}{n-\alpha}, \qquad m \in \mathbb{Z}$$

can be written as

$$(2.3) \qquad T_m(z,\alpha) = (-1)^m \; S_0(m,0,-\alpha).$$

Applying the symmetry rule (2.1) for this case we obtain

$$(2.4) \qquad T_{-m}(z,\alpha) = (-1)^{m+1}\, T_m(z,-\alpha).$$

Newberger found

$$(2.5) \qquad T_m(z,\alpha) = \frac{(-1)^{m+1}\pi}{\sin \alpha\pi}\, J_{m-\alpha}(z)\, J_\alpha(z), \qquad m \geq 0.$$

The addition $m \geq 0$ is not given by Newberger, but has to be made. To see this, verify the symmetry rule (2.4) for the above relation. It follows that (2.5) cannot be correct for all $m \in \mathbb{Z}$. The correct relation for negative values is

$$(2.6) \qquad T_m(z,\alpha) = -\pi \frac{J_{\alpha-m}(z) J_{-\alpha}(z)}{\sin \alpha\pi}, \qquad m \leq 0.$$

Observe also that (2.2) is an entire function of z, as are the right-hand sides of (2.5) and (2.6). For $m < 0$ (2.5) is not entire in z; for $m > 0$ (2.6) is not entire in z.

3. A RECURSION FOR $T_m(z,\alpha)$

The fact that (2.5) is no longer valid for negative values of m is also revealed by a recursion relation for $T_m(z,\alpha)$. By using the well-known identities

$$(3.1) \qquad J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$$

$$(3.2) \qquad \sum_{n=-\infty}^{\infty} J_n(z)\, J_{n-m}(z) = \delta_{m,0},$$

where Kronecker's symbol is used, we obtain

$$(3.3) \qquad T_{m-1}(z,\alpha) + T_{m+1}(z,\alpha) = \frac{2(\alpha-m)}{z} T_m(z,\alpha) + \frac{2}{z}\delta_{m,0},$$

an inhomogeneous version of (3.1). This once again shows that (2.5) cannot be true for all $m \in \mathbb{Z}$, since in that case the inhomogeneous term has to vanish. For $m \neq 0$ (3.3) gives the proper recursion for both (2.5) and (2.6). On the other hand we have, using (2.5), (2.6) and (3.1),

$$T_{-1}(z,\alpha) + T_1(z,\alpha) = \frac{\pi}{\sin \alpha\pi}[J_{-\alpha}(z)J_{\alpha-1}(z) + J_\alpha(z)J_{1-\alpha}(z)] - \frac{2\alpha\pi\ J_{-\alpha}(z)J_\alpha(z)}{z \sin \alpha\pi}.$$

Interpreting the cross product of Bessel functions as a well-known Wronskian relation for these functions (ABRAMOWITZ and STEGUN (1964), p. 360) we obtain

$$T_{-1}(z,\alpha) + T_1(z,\alpha) = \frac{2}{z} + \frac{2\alpha}{z} T_0(z,\alpha),$$

which confirms (3.3) for $m = 0$.

4. CONVERGENCE OF THE SERIES (1.1).

The convergence of the series (1.1) follows from the asymptotic expansions

$$\text{(4.1)}\qquad \begin{aligned} J_\nu(z) &\sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1), \\ J_{-\nu}(z) &\sim \frac{1}{\pi}(\tfrac{1}{2}z)^{-\nu}\ \Gamma(\nu)\ \sin \nu\pi, \end{aligned} \qquad \operatorname{Re} \nu \to \infty.$$

The second line holds for non-integer values of ν; otherwise we use $J_{-n}(z) = (-1)^n J_n(z)$, $n \in \mathbb{Z}$. Using (4.1) for the terms of (1.1) we obtain

$$\frac{n^j}{n+\mu} J_{\alpha-\gamma n}(z)\ J_{\beta+\gamma n}(z) \sim n^{j-1}(\tfrac{1}{2}z)^{\alpha+\beta}\sin(\gamma n-\alpha)\ \frac{\Gamma(\gamma n-\alpha)}{\Gamma(\gamma n+\beta)}.$$

Using $\Gamma(z+a)/\Gamma(z+b) \sim z^{a-b}$, $\operatorname{Re} z \to \infty$, we conclude that the series diverges when

$$\operatorname{Re}(\alpha+\beta) < j-1,$$

unless $\alpha,\beta \in \mathbb{Z}$, $\gamma = 1$. In general, large terms for $n \to \pm\infty$ will not cancel each other and thence there is no chance that the divergence at $n = +\infty$ combined with that at $n = -\infty$ is removed.

The series (1.1) is absolutely convergent when Re $(\alpha+\beta) > j$. This condition is sufficient to make the sum holomorphic with respect to α and β in this domain.

It follows that S, as a function of the complex parameters α and β, is defined and holomorphic for Re $(\alpha+\beta) > j$. Possibly there is an analytic continuation of $S(\alpha,\beta,\mu)$ with respect to Re $(\alpha+\beta) \leq j$, but it is not clear how this continuation looks like. For $\alpha,\beta \in \mathbb{Z}$, $\gamma = 1$, the symmetry rule (2.1) gives the value for negative α and β.

Newberger used the splitting

$$(4.2) \qquad S = (-\mu)^j S_1 + S_2$$

with S_1 equal to (1.1) with $j = 1$. He evaluated this expression in the form (see his formula (2.8)).

$$(4.3) \qquad S_1 = \frac{\pi}{\sin \mu\pi} \, J_{\alpha-\gamma\mu}(z) \, J_{\beta+\gamma\mu}(z), \quad \text{Re } (\alpha+\beta) > -1.$$

His proof is correct for the range Re $(\alpha+\beta) > 0$. The right-hand side is entire in α and β, whereas from the above remarks it follows that some combinations of α and β yield a divergent series. Extension of (4.3) to all complex α and β (and γ) is therefore not allowed.

5. A FINAL REMARK

The second part of (4.2), i.e., S_2, is also evaluated in terms of derivatives of Bessel functions. Starting point is the evaluation of

$$(5.1) \qquad \hat{S}_2 = \sum_{n=-\infty}^{\infty} (-1)^n n^p \, J_{\alpha-\gamma n}(z) \, J_{\beta+\gamma n}(z)$$

where p is an integer, $0 \leq p \leq j$.

As admitted by Newberger, the analysis for deriving the sum rule for

S_2 is quite formal, with an appeal to the theory of generalized functions. It seems worthwile to make the analysis more rigorous. For instance, application of (3.1) gives a recursion relation (denote (5.1) by $\hat{S}_2(\alpha,\beta,p)$)

$$\gamma\, \hat{S}_2(\alpha,\beta,p) = \alpha\, \hat{S}_2(\alpha,\beta,p-1) - \tfrac{1}{2}z\, [\hat{S}_2(\alpha+1,\beta,p-1) + \hat{S}_2(\alpha-1,\beta,p-1)].$$

Repeated application reduces (5.1) to the case $p = 0$. One further step makes the Fourier series in Newberger's formulas (2.12) and (2.13) convergent.

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