Bounds for binary codes of length less than 25

by

M.R. Best, A.E. Brouwer
Mathematical Centre, Amsterdam

and

F.J. MacWilliams, A.M. Odlyzko, N.J.A. Sloane
Bell Telephone Laboratories, Murray Hill, N.J.

Abstract

This paper presents improved bounds for $A(n,d)$, the maximum number of codewords in a (linear or nonlinear) binary code of word length n and minimum distance d, and for $A(n,d,w)$, the maximum number of binary vectors of length n, minimum distance d, and constant weight w, in the range $n \leq 24$ and $d \leq 10$. Some of the new values are $A(9,4) = 20$ (which was previously believed to follow from the results of Wax), $A(13,6) = 32$ (proving that the Nadler code is optimal), $A(17,8) = 36$ or 37, and $A(21,8) = 512$. The upper bounds on $A(n,d)$ are found with help of linear programming, making use of the values of $A(n,d,w)$.

1. INTRODUCTION

The main purpose of this paper is to present tables of two of the most basic functions in coding theory, namely

$A(n,d)$ = maximum number of codewords in any
(linear or nonlinear) binary code of length n
and minimum distance d between codewords,

and

$A(n,d,w)$ = maximum number of codewords in any
binary code of length n, constant weight w and
minimum distance d,

in the range $n \leq 24$, $d \leq 10$. We also give a table of the function

$T(w_1,n_1,w_2,n_2,d)$ = maximum number of codewords
in a binary code of length n_1+n_2 and minimum
distance d with exactly w_1 ones in the first
n_1 coordinates and exactly w_2 ones in the last
n_2 coordinates,

for $n_1+n_2 \leq 24$, $d = 10.$[*)]

All of the upper bounds on $A(n,d)$ outside the Plotkin range $n \leq 2d$ are obtained from modifications of Delsarte's linear programming method, making use of the values of $A(n,d,w)$ (see §3). The tables of $A(n,d,w)$ are important because they lead to bounds on $A(n,d)$, and in their own right for giving the size of the largest constant weight codes. They also give the solution to the following widely studied packing problem (ERDÖS & HANANI [17], KALBFLEISCH & STANTON [36], SCHÖNHEIM [51], STANTON, KALBFLEISCH & MULLIN [59]): What is $D(t,k,v)$, the maximum number of k-subsets of a v-set S, such

---

[*)] We would appreciate hearing of any improvements to the tables. (Send them for example to N.J.A. Sloane, Math. Research Center, Bell Labs, Murray Hill, N.J. 07974 U.S.A..)

that every t-subset of S is contained in at most one k-set? The answer is
$D(t,k,v) = A(v,2k-2t+2,k)$, so that table 2 is also a table of values of
$D(t,k,v)$.

Two recent papers which also use the linear programming approach are
BEST & BROUWER [3] and McELIECE, RODEMICH, RUMSEY & WELCH [43].

Earlier tables of bounds on $A(n,d)$ were given in JOHNSON [33], McELIECE
et al. [42] and SLOANE [53]. No table of $A(n,d,w)$ seems to have been pub-
lished before, although unpublished tables of upper bounds exist (e.g.
DELSARTE et al. [12], JOHNSON [32]). A table of $A(n,d,w)$ was promised in
STANTON et al. [59] but has never appeared. A table of upper and lower bound
on linear codes appeared in HELGERT & STINAFF [29].

The following notation is used in this paper. All codes are binary. An
(n,M,d) *code* consists of $M(\geq 1)$ binary vectors (called codewords) of length
n such that any two codewords differ in at least d places, i.e. are at
(*Hamming*) *distance* at least d apart. A code has *constant weight* w if each
codeword contains w 1's, i.e. has *weight* w. An optimal code is a code with the
maximum number of codewords for the given n and d (and for the given w, in
the case of a constant weight code).

Let C be an (n,M,d) code. The *weight distribution of C with respect to*
a vector u is the $(n+1)$-tuple of integers $(A_i(u), i=0,\ldots,n)$, where $A_i(u)$
is the number of codewords $v \in C$ such that $d(u,v) = i$. The *distance distrib-*
ution of C is the $(n+1)$-tuple of rational numbers $(A_0,A_1,\ldots,A_n)$ defined by

$$A_i = \frac{1}{M} \sum_{u \in C} A_i(u), \qquad i = 0,\ldots,n.$$

Then $A_0 = 1$, $A_i \geq 0$ and $\sum_i A_i = M \leq A(n,d)$.

2. BOUNDS ON A(n,d)

The first theorem is immediate, while the second gives $A(n,d)$ exactly
if $n \leq 2d$.

THEOREM 1.

$$A(n-1,2\delta-1) = A(n,2\delta),$$
$$A(n,d) \leq 2A(n-1,d).$$

THEOREM 2. (PLOTKIN [48] & LEVENSHTEIN [39]). *Provided certain Hadamard matrices of order $\leq n$ exist,*[*)]

$$A(n,2\delta) = 2\left\lfloor \frac{2\delta}{4\delta-n} \right\rfloor \quad if \ 4\delta > n \geq 2\delta, \tag{1}$$

$$A(4\delta,2\delta) = 8\delta, \qquad A(n,2\delta) = 1 \quad if \ n < 2\delta. \tag{2}$$

The linear programming approach is based on the following theorem.

THEOREM 3. (DELSARTE [8]-[10]). *Let C be an (n,M,d) code with distance distribution $(A_0,\ldots,A_n)$. Then the quantities $B_0,\ldots,B_n$ are nonnegative, where*

$$B_k = M^{-1} \sum_{i=0}^{n} A_i K_k(i) \quad (k = 0,1,\ldots,n),$$

and K_k is a Krawtchouk polynomial, defined by

$$K_k(t) = \sum_{j=0}^{k} (-1)^j \binom{n-t}{k-j}\binom{t}{j} \quad (k = 0,1,\ldots,n).$$

For later reference we give a short proof.

PROOF. Let w be a word in $\{0,1\}^n$ of weight i. Then it is easily checked that

$$\sum_{\substack{x\in\{0,1\}^n \\ wt(x)=k}} (-1)^{\langle w,x \rangle} = K_k(i).$$

Consequently by the definition of A_i,

$$B_k = M^{-1} \sum_{i=0}^{n} A_i K_k(i) = M^{-2} \sum_{i=0}^{n} \sum_{\substack{u,v\in C \\ wt(u-v)=i}} \sum_{\substack{x\in\{0,1\}^n \\ wt(x)=k}} (-1)^{\langle u-v,x \rangle} =$$

$$= M^{-2} \sum_{\substack{x\in\{0,1\}^n \\ wt(x)=k}} b_x^2 \geq 0, \tag{3}$$

[*)]Hadamard matrices are known to exist for all orders ≤ 264. In any case the r.h.s. is an upper bound on both (1) and (2).

where

$$b_x = \sum_{u \in C} (-1)^{<u,x>}. \tag{3a}$$

NOTE. If C is a linear code, then b_x equals M or 0 depending on whether x belongs to the dual code or not, and $B_0, \ldots, B_n$ is the weight distribution of the dual code.

To apply Theorem 3 let C be an optimal code of length n and minimum distance d. Then

$$M = A(n,d) = 1 + A_d + A_{d+1} + \ldots + A_n.$$

Suppose $L^*(n,d)$ is the optimal solution to the following linear programming problem.

Choose real variables $A_d, A_{d+1}, \ldots, A_n$ *so as to maximize*
$L = A_d + A_{d+1} + \ldots + A_n$ *subject to the constraints*

$$A_i \geq 0, \quad i = d, \ldots, n,$$
$$B_k \geq 0, \quad k = 0, \ldots, n,$$

where

$$B_k = M^{-1}(K_k(0) + \sum_{i=d}^{n} A_i K_k(i)).$$

Then plainly

$$A(n,d) \leq 1 + L^*(n,d).$$

This is the simplest version of the *linear programming bound* for binary codes (DELSARTE [8]).

Often it is possible to impose additional constraints on the B_i. Certainly

$$B_i \leq A(n,d,i), \tag{4}$$

so bounds on $A(n,d,w)$ can be used (see Table 2). Sometimes several such bounds can be combined, as the following example illustrates.

THEOREM 4. $A(13,6) = 32$, *and so the Nadler code is optimal.*

PROOF. In 1959 R.F. STEVENS & W.G. BOURICIUS [60] found (13,32,6) and (14,64,6) codes, showing that $A(13,6) \geq 32$. The former code was rediscovered by NADLER [45], and is usually referred to as the Nadler code. (See also VAN LINT [41].)

To prove $A(13,6) \leq 32$ we proceed as follows.

First observe that if we shorten a (13,M,6) code and then add an overall parity check, we get a (13,M,6) code C in which all distances are even.

If (A_i) is the distance distribution of C then $A_0 = 1$ and the remaining A_i's are zero except (possibly) for A_6, A_8, A_{10} and A_{12}. The inequalities $B_k \geq 0$ become

$$13 + A_6 - 3A_8 - 7A_{10} - 11A_{12} \geq 0.$$

$$\binom{13}{2} - 6A_6 - 2A_8 + 18A_{10} + 54A_{12} \geq 0.$$

$$\binom{13}{3} - 6A_6 + 14A_8 - 14A_{10} - 154A_{12} \geq 0.$$

$$\binom{13}{4} + 15A_6 - 5A_8 - 25A_{10} + 275A_{12} \geq 0. \tag{5}$$

$$\binom{13}{5} + 15A_6 - 25A_8 + 63A_{10} - 197A_{12} \geq 0.$$

$$\binom{13}{6} - 20A_6 + 20A_8 - 36A_{10} + 132A_{12} \geq 0.$$

Furthermore we have

$$A_{12}(u) \leq A(13,6,12) = A(13,6,1) = 1,$$
$$A_{10}(u) \leq A(13,6,10) = A(13,6,3) = 4.$$

However, these can be combined. For if $A_{12}(u) = 1$ then $A_{10}(u) = 0$. So

$$A_{10}(u) + 4A_{12}(u) \leq 4,$$

and averaging over u gives

$$A_{10} + 4A_{12} \leq 4. \tag{6}$$

Actually (6) and the first two constraints of (5) turn out to be enough, and so we consider the problem:

Maximize $A_6 + A_8 + A_{10} + A_{12}$

subject to

$$A_6 \geq 0, \ A_8 \geq 0, \ A_{10} \geq 0, \ A_{12} \geq 0$$

and

$$
\begin{aligned}
13 + A_6 - 3A_8 - 7A_{10} - 11A_{12} &\geq 0. \\
78 - 6A_6 - 2A_8 + 18A_{10} + 54A_{12} &\geq 0, \\
4 \qquad\qquad - A_{10} - 4A_{12} &\geq 0.
\end{aligned}
\tag{7}
$$

The dual problem is:

Minimize $13u_1 + 78u_2 + 4u_3$

subject to

$$u_1 \geq 0, \ u_2 \geq 0, \ u_3 \geq 0$$

and

$$
\begin{aligned}
1 + u_1 - 6u_2 \qquad &\leq 0, \\
1 - 3u_1 - 2u_2 \qquad &\leq 0, \\
1 - 7u_1 + 18u_2 - u_3 &\leq 0, \\
1 - 11u_1 + 54u_2 - 4u_3 &\leq 0.
\end{aligned}
\tag{8}
$$

Feasible solutions to these two problems are

$$A_6 = 24, \ A_8 = 3, \ A_{10} = 4, \ A_{12} = 0, \tag{9}$$

$$u_1 = u_2 = \frac{1}{5}, \ u_3 = \frac{16}{5}. \tag{10}$$

In fact since the corresponding objective functions are equal:

$$24 + 3 + 4 + 0 = 13 \cdot \frac{1}{5} + 78 \cdot \frac{1}{5} + 4 \cdot \frac{16}{5} = 31,$$

it follows that (9) and (10) are optimal solutions. (These solutions are easily obtained by hand using the simplex method - see [18] or [52].) It follows that $A(12,5) = A(13,6) \leq 32$. $\square$

REMARK. The following argument shows that (9) is the unique optimal solution. Let x_6, x_8, x_{10}, x_{12} be any optimal solution to the primal problem. The u_i of (10) are all positive and satisfy the first three constraints of (8) with equality but not the fourth. Hence from the theorem of complementary slackness (SIMONNARD [52]) the x_i must satisfy the primal constraints (7) with equality, and $x_{12} = 0$. These three equations have the unique solution

$$x_6 = 24, \quad x_8 = 3, \quad x_{10} = 4.$$

Thus (9) is the unique optimal solution. Therefore the distance distribution of a $(13,32,6)$ code in which all distances are even is unique. This result has been used by GOETHALS [19] to show that the code itself is unique and that there are exactly two nonequivalent $(12,32,5)$ codes. (cf. NADLER [45], VAN LINT [41]).

If $A(n,d) \not\equiv 0 \pmod 4$, the right hand side of the Delsarte inequalities $B_k \geq 0$ can sometimes be increased, as shown by Theorems 5 and 8.

THEOREM 5. *Let C be an (n,M,d) code with $M = A(n,d)$, and suppose that M is odd. Then*

$$B_k \geq M^{-2} \binom{n}{k} \qquad (k = 0,1,\ldots,n).$$

PROOF. If M is odd, then b_x (Eq. (3a)) is odd, and hence non-zero. From (3) we get

$$B_k \geq M^{-2} \sum_{\substack{x \in \{0,1\}^n \\ wt(x)=k}} b_x^2 \geq M^{-2} \binom{n}{k}.$$

REMARK. The first term in $B_k = M^{-1} \sum\limits_{i=0}^{n} A_i K_k(i)$ is $M^{-1} K_k(0) = M^{-1}\binom{n}{k}$. Hence Theorem 5 shows that, in all the Delsarte inequalities, the constant term may be multiplied by $(M-1)/M$. That means that – if no extra inequalities have been added – the optimal solution is simply $(M-1)/M$ times the original one, and hence $\sum\limits_{i=1}^{n} A_i < M-1$, lowering the bound by exactly one. If extra inequalities are added, the gain is in general less.

As an application we prove:

THEOREM 6. $A(9,4) = 20$.

PROOF. GOLAY [21] found a $(9,20,4)$ code, thus $A(9,4) \geq 20$. A cyclic $(8,20,3)$ code is given in SLOANE & WHITEHEAD [57]. To prove $A(9,4) \leq 20$, as usual let $\bar{C}$ be an $(8,M,3)$ code with $M = A(8,3) = A(9,4)$; and let C be the $(9,M,4)$ extended code, having distance distribution $(A_0,\ldots,A_9)$ with $A_0 = 1$ and $A_1 = A_2 = A_3 = A_5 = A_7 = A_9 = 0$.

First we maximize $A_4 + A_6 + A_8$ subject to $A_i \geq 0$, $B_k \geq 0$ and $A_8 \leq 1$, obtaining $A_4 + A_6 + A_8 \leq 20\frac{1}{3}$, hence $M \leq 21$.

Suppose $M = 21$. Then by Theorem 5 we can replace $B_k \geq 0$ by $B_k \geq \frac{1}{441}\binom{9}{k}$. Moreover, it is obvious that $A_8 \leq \frac{20}{21}$. Hence in this case, in spite of the extra inequality, all constant terms occurring in the inequalities are multiplied by $\frac{20}{21}$, so

$$M \leq 1 + \frac{20}{21} * 20\frac{1}{3} < 21.$$

Hence $A(9,4) = 20$. $\square$

If $A(n,d) \equiv 2 \pmod 4$, then a positive lower bound for B_k can be obtained by noting that b_x cannot be zero too often. For example if u_1, u_2, $u_1 + u_2$ are distinct then b_{u_1}, b_{u_2}, $b_{u_1 + u_2}$ cannot all be zero. The following linear inequality can be obtained in this way.

THEOREM 7. *Let C be an (n,M,d) code with $M = A(n,d)$, and suppose that $M \equiv 2 \pmod 4$. Then*

$$B_k \geq \frac{4}{3M^2}\binom{n}{k},$$

If (i) k *is even and* $0 \le k \le \frac{2}{3}n$, *or* (ii) *if* d *is even,* $k \equiv n$ (mod 2), *and* $\frac{1}{3}n \le k \le n$.

A slightly stronger result is:

THEOREM 8. *Let* C *be an* (n,M,d) *code with* $M = A(n,d)$, *and suppose that* $M \equiv 2$ (mod 4). *Then there exists an* $\ell \in \{0,1,\ldots,n\}$ *such that*

$$B_k \ge 2M^{-2}(\binom{n}{k} + K_k(\ell)). \qquad (k = 0,1,\ldots,n).$$

(Since $|K_k(\ell)| \le \binom{n}{k}$, this also improves Theorem 3.)

PROOF. Since M is even, b_u is even for each $u \in \{0,1\}^n$. Let e_j be the j-th unit vector in $\{0,1\}^n$. Then

$$b_x - b_{x+e_j} = \sum_{u \in C} (1-(-1)^{<u,e_j>})(-1)^{<u,x>}.$$

Hence, for fixed j, the residue class of $b_x - b_{x+e_j}$ (mod 4) is even and independent of the choice of x.

Let J be the set of those $j \in \{1,2,\ldots,n\}$ for which $b_x - b_{x+e_j} \equiv 2$ (mod 4), and let $\ell = |J|$ and $\xi = \sum_{j \in J} e_j$. Then

$$b_x - b_{x+e_j} \equiv 2 <e_j,\xi> \quad \text{(mod 4)}$$

By induction on the weight of x it follows that, since $b_0 = M \equiv 2$ (mod 4),

$$b_x \equiv 2 + 2 <x,\xi> \quad \text{(mod 4)}$$

Now for each $k \in \{0,1,\ldots,n\}$

$$B_k = M^{-2} \sum_{\substack{x \in \{0,1\}^n \\ wt(x)=k}} b_x^2 \ge 2M^{-2} \sum_{\substack{x \in \{0,1\}^n \\ wt(x)=k}} (1+(-1)^{<x,\xi>}) =$$

$$= 2M^{-2}(\binom{n}{k} + K_k(\ell)). \quad \square$$

We mention the following immediate consequence of Theorem 8, which is weaker but easier to apply.

COROLLARY. *Let C be an* (n,M,d) *code with* M = A(n,d), *and suppose that* M ≡ 2 (mod 4). *Then*

$$B_k \geq 2M^{-2}(\binom{n}{k} + \min_{\ell \in \{0,1,\ldots,n\}} K_k(\ell)).$$

E.g. $B_2 \geq \dfrac{4}{M^2}\left(\binom{n}{2} - \left[\dfrac{n^2}{4}\right]\right).$

For example, this corollary can be used to prove the upper bound in Theorem 9; the lower bound comes from [56], [57].

THEOREM 9. A(17,8) = 36 *or* 37. ☐

Table 1 gives the bounds on A(n,d). Many values come from Theorems 1 and 2. Otherwise the unmarked upper bounds are obtained by linear programming, as illustrated in Theorems 4 and 6. Other entries are explained by the key. The bounds A(9,4) ≤ 20, A(10,4) ≤ 39, A(11,4) ≤ 78, and A(12,4) ≤ 154 were claimed by WAX [63] in 1959. However, as we shall see in the next section, such bounds cannot be obtained by his method.

We conclude this section by repeating Elspas's question [16]: can A(n,d) be odd and greater than 1? From Theorem 2 and Table 1 we have:

THEOREM 10. *If* A(n,d) *is odd* (*and* > 1) *then* A(n,d) ≥ 37. *If Hadamard matrices exist of all orders, then* A(n,d) *is even whenever* n ≤ 2d. ☐

3. THE END OF THE WAX BOUND

In 1959 N.WAX [63] computed a number of upper bounds for binary codes by a method derived from sphere packing in Euclidean spaces as developed by R.A. RANKIN [49] (see also ROGERS [50]). Most of the bounds obtained were rather weak, but there were three special cases in which his "soft sphere model" seemingly yielded astonishingly good results. These were:

$$A(8,3) \leq 20,$$
$$A(9,3) \leq 39 \text{ (and hence } A(10,3) \leq 78),$$
$$A(11,3) \leq 154.$$

The first bound is confirmed by Theorem 6, but no proof of the other bounds is known.

We were unable to duplicate Wax's calculations, and in fact in this section we shall establish a lower bound on the best upper bound that can be achieved with the soft sphere model, no matter which weight function is used. Since this lower bound is inconsistent with the data found by Wax, we may conclude that Wax's results are – at least in the interesting cases mentioned above – erroneous.

We are now left with the following bounds for $A(8,3)$, $A(9,3)$, $A(10,3)$ and $A(11,3)$:

$$A(8,3) = 20$$
$$38 \leq A(9,3) \leq 40$$
$$72 \leq A(10,3) \leq 80$$
$$144 \leq A(11,3) \leq 160$$

3.1. THE SOFT SPHERE MODEL

Consider an (n,M,d) code as a subset of the vertices of the hypercube $[0,1]^n$ in Euclidean n-space $\mathbb{R}^n$. The Euclidean distance between two codepoints is at least $\sqrt{d}$. Therefore the spheres with centers in the codepoints and radii $R = \frac{1}{2}\sqrt{d}$ are disjoint. If V denotes the volume of the intersection of each sphere with the hypercube $[0,1]^n$ (by symmetry these volumes are all equal), then the number of codepoints evidently cannot exceed $1/V$. Hence $A(n,d) \leq [1/V]$.

This method, called the "hard sphere model", yields very modest results, e.g. $A(9,3) \leq 566$ (and not 56.7 as in WAX [63]) or $A(10,4) \leq 401$.

In order to sharpen the bounds, the hard spheres are replaced by larger ones with variable mass density. As basic conditions it is required that

(i) the density $\rho(r)$ associated with a single sphere is non-negative and depends only on the distance r to the center of that sphere

12

and that

(ii) in any configuration of (partly overlapping) spheres with centers at
     least 2R apart, the total density at each point does not exceed unity.
If μ is the mass of the intersection of each sphere with the hypercube [1],
we now obtain:

$$A(n,d) \leq [1/\mu].$$

The main problem is to determine a suitable density which satisfies the
basic conditions (i) and (ii), and optimizes the mass μ. R.A. RANKIN studied
this problem in [49]. In order to simplify computations, he required in
addition:

(iii) The spheres have radius $R\sqrt{2}$, i.e. $\rho(r) = 0$ if $r \geq R\sqrt{2}$.

The model described, with the conditions (i), (ii) and (iii), is called
the "soft sphere model". We shall denote the least upper bound for $A(n,d)$
that can be achieved with this model by $A_w(n,d)$. Our aim is to give a lower
bound for $A_w(n,d)$.

3.2. A LOWER BOUND FOR $A_w(n,d)$

First we derive an upper bound for ρ. We define for each positive in-
teger m:

$$y_m = \sqrt{2(m-1)/m}$$

(note: $y_1 = 0$, $y_2 = 1$), and the function $\sigma: [0,\infty] \to [0,1]$ by

---

[1]

   In case $d \leq 4$ one may instead define μ by 2^{-n} times the mass of the whole
   sphere, since the configuration may be continued with period 2 in all
   directions in $\mathbb{R}^n$. But even this extended model is included in our re-
   sults, since we estimate μ by that number.

$$\sigma(r) = \frac{1}{m} \quad \text{if } Ry_m \le r < Ry_{m+1} \qquad (m = 1,2,\ldots,n),$$

$$= \frac{1}{n+1} \quad \text{if } Ry_{n+1} \le r < R\sqrt{2},$$

$$= 0 \quad \text{if } r \ge R\sqrt{2}.$$

Then we have:

LEMMA 11. $\rho \le \sigma$.

PROOF. We have to prove that $\rho(r) \le 1/m$ if $r \ge Ry_m$ for $m = 1,2,\ldots,n+1$. Let $m \in \{1,2,\ldots,n+1\}$. Suppose m spheres with density function ρ are arranged such that their centers form the vertices of an (m-1)-dimensional regular simplex in $\mathbb{R}^n$ with edges of length 2R. Then the distance from the center of gravity of the simplex to each of the vertices equals

$$R\sqrt{2(m-1)/m} = Ry_m.$$

(Proof by induction.)

The total density at the center of gravity equals $m\rho(Ry_m)$. Hence $\rho(Ry_m) \le 1/m$ and a fortiori $\rho(r) \le 1/m$ if $r \ge Ry_m$. $\square$

This estimate for ρ immediately gives rise to an upper bound on the mass μ:

LEMMA 12. $\mu \le \left(\dfrac{\pi e R^2}{n}\right)^{\frac{1}{2}n} \dfrac{1}{\sqrt{\pi n}} \left(\displaystyle\sum_{m=1}^{n} \dfrac{1}{m(m+1)} \left(\dfrac{m}{m+1}\right)^{\frac{1}{2}n} + \dfrac{1}{n+1}\right).$

PROOF. We denote the volume of the intersection of the n-dimensional hypersphere with radius r and center 0 in $\mathbb{R}^n$ and the n-dimensional hypercube $[0,1]^n$ by B(r). The volume of the n-dimensional unit sphere will be denoted by J_n. It is well known that

$$J_n = \frac{\pi^{\frac{1}{2}n}}{(\frac{1}{2}n)!} \le \frac{\pi^{\frac{1}{2}n} e^{\frac{1}{2}n}}{(\frac{1}{2}n)^{\frac{1}{2}n}\sqrt{\pi n}} = \left(\frac{2\pi e}{n}\right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}}.$$

Hence

$$\mu = \int_0^{R\sqrt{2}} \rho(r)\, dB(r) \le \int_0^{R\sqrt{2}} \sigma(r)\, dB(r) =$$

$$= - \int_0^{R\sqrt{2}} B(r) \, d\sigma(r) \le - \int_0^{R\sqrt{2}} 2^{-n} J_n r^n \, d\sigma(r) =$$

$$= 2^{-n} J_n \left(\sum_{m=2}^{n+1} \left(\frac{1}{m-1} - \frac{1}{m} \right) (Ry_m)^n + \frac{1}{n+1} (R\sqrt{2})^n \right) \le$$

$$\le \left(\frac{R}{2} \right)^n \left(\frac{2\pi e}{n} \right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}} \left(\sum_{m=2}^{n+1} \frac{1}{(m-1)m} \left(\frac{2(m-1)}{m} \right)^{\frac{1}{2}n} + \frac{2^{\frac{1}{2}n}}{n+1} \right) =$$

$$= \left(\frac{\pi e R^2}{n} \right)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi n}} \left(\sum_{m=1}^{n} \frac{1}{m(m+1)} \left(\frac{m}{m+1} \right)^{\frac{1}{2}n} + \frac{1}{n+1} \right). \quad \square$$

This leads to the lower bound for $A_w(n,d)$:

THEOREM 13. $A_w(n,d) \ge \left[\left(\frac{4n}{\pi e d} \right)^{\frac{1}{2}n} \sqrt{\pi n} \left(\sum_{m=1}^{n} \frac{1}{m(m+1)} \left(\frac{m}{m+1} \right)^{\frac{1}{2}n} + \frac{1}{n+1} \right)^{-1} \right].$

PROOF. $R = \frac{1}{2}\sqrt{d}$ and $A_w(n,d) = [1/\mu]$ for some density function ρ. $\quad \square$

EXAMPLES. $A_w(8,3) \ge 45$, $\quad A_w(9,4) \ge 27$,

$\qquad\qquad A_w(9,3) \ge 101$, $\quad A_w(10,4) \ge 56$,

$\qquad\qquad A_w(10,3) \ge 238$, $\quad A_w(11,4) \ge 119$,

$\qquad\qquad A_w(11,3) \ge 579$, $\quad A_w(12,4) \ge 259$.

4. BOUNDS ON A(n,d,w)

The first two theorems are well-known.

THEOREM 14. *Let* d,w,n *be integers,* $d \ne 0$, $w \le n$. *Then*

(i) $\quad A(n,d-1,w) = A(n,d,w)$ *if* d *is even*,

(ii) $\quad A(n,d,w) = A(n,d,n-w)$,

(iii) $A(n,d,w) = 1$ *if* $d > 2w$,

(iv) $\quad A(n,d,w) = \left[\frac{n}{w} \right]$ *if* $d = 2w$.

THEOREM 15. *If a* 2d × 2d *Hadamard matrix exists,*

$$A(2d-2,d,d-1) = d,$$

$$A(2d-1,d,d-1) = 2d-1,$$

$$A(2d,d,d) = 4d-2.$$

Theorems 16-18 are due to JOHNSON [23], [24].

<u>THEOREM 16.</u>

$$A(n,d,w) \leq \left[\frac{dn}{dn-2w(n-w)} \right]$$

provided the denominator is positive.

A slightly stronger result is:

<u>THEOREM 17.</u> *Suppose* $A(n,d,w) = M,$ *and define* q *and* r *by*

$$wM = nq + r, \quad 0 \leq r < n.$$

Then

$$nq(q-1) + 2qr \leq (w-\tfrac{1}{2}d)M(M-1),$$

<u>THEOREM 18.</u>

$$A(n,d,w) \leq \left[\frac{n}{w} A(n-1,d,w-1) \right] \quad (n \geq w \geq 1)$$

$$A(n,d,w) \leq \left[\frac{n}{n-w} A(n-1,d,w) \right] \quad (n > w \geq 0)$$

<u>THEOREM 19.</u>

$$A(n,d,w) \leq \frac{n}{w} \cdot \frac{n-1}{w-1} \cdot \ldots \cdot \frac{n-t+1}{w-t+1} \cdot A(n-t,d,w-t) \quad (n \geq w \geq t).$$

If equality holds then any optimal constant weight code with parameters n,d,w *is a t-design. In particular*

$$A(n,2\delta,w) = \frac{n(n-1)\ldots(n-w+\delta)}{w(w-1)\ldots\delta}$$

if and only if a Steiner system S(w-δ+1,w,n) *exists.*

(For a bibliography of Steiner systems up to 1973 see DOYEN and ROSA [14]).

4.1. OPTIMAL CONSTANT WEIGHT CODES

As noted in the introduction the determination of A(n,d,w) is equivalent to determining D(t,d,v), where v = n, k = w and t = k + 1 - $\frac{1}{2}$d (if d is even). But this requires the construction of (maximal partial) Steiner t-designs, which is trivial for t = 1, while for t = 2 the recursive techniques of HANANI and WILSON are available (see e.g. WILSON [64,65]). For larger t almost nothing is known (the best studied case being t = 3, k = 4). The known results are as follows:

1. t = 1.
    This is Theorem 14 (iv): A(n,2w,w) = $\left[\frac{n}{w}\right]$.

2. t = 2.
    In this case we must look for a maximal collection of w-subsets of an n-set such that no 2-subset is covered twice (in other words: an edge-disjoint packing of w-cliques in the complete graph on n points). If a BIBD (b,v=n,r,k=w,λ=1) exists (that is, an S(2,w,n)) then obviously A(n,d,w) = b = $\binom{n}{2}/\binom{w}{2}$; otherwise we must look for the nearest approximation to this Steiner system.

2.1 d = 4, w = 3.
    It has been shown by KIRKMAN [38] in the cases n ≡ 0, 1, 2 or 3 (mod 6) and by SCHÖNHEIM [51] in the remaining cases that

$$A(n,4,3) = \begin{cases} \left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] & \text{for } n \not\equiv 5 \ (\text{mod } 6) \\ \left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] - 1 & \text{for } n \equiv 5 \ (\text{mod } 6) \end{cases}$$

(see also GUY [22], SPENCER [58] and SWIFT [61]). The cases n ≡ 1 or 3 (mod 6) correspond to Steiner triple systems.

2.2 d = 6, w = 4.
    As has been shown by HANANI [26] there exist Steiner systems S(2,4,n) iff n ≡ 1 or 4 (mod 12). In BROUWER & SCHRIJVER [7] group divisible designs GD(4,1,2;n) are constructed for each n ≡ 2 (mod 6), n ≠ 8. In BROUWER

[5] pairwise balance designs PBD($\{4,7^*\}$;n) are constructed for each n $\equiv$ 7 or 10 (mod 12), n $\neq$ 10,19. Using these and similar constructions it follows that if

$$JB(n,6,4) := \begin{cases} \left[\frac{n}{4}\left[\frac{n-1}{3}\right]\right] - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12} \\ \left[\frac{n}{4}\left[\frac{n-1}{3}\right]\right] & \text{otherwise} \end{cases}$$

then A(n,6,4) = JB(n,6,4) for all n with the exception of n = 8,9,10,11,17,19. The values of A(n,6,4) for n $\succeq$ 8,9,10,11 are easily determined by hand, that of A(17,6,4) was determined in BROUWER [4], and the lower bound on A(19,6,4) follows from a construction of H.R. PHINNEY [47].

We conjecture that for t = 2, w fixed and n sufficiently large (i.e. n$\geq$n$_0$(w)) A(n,d,w) equals the Johnson bound (obtained by applying Theorems 14 and 18) (cf. WILSON [64]).

2.3 d = 8, w = 5.

As shown by HANANI [26,27] there exist Steiner systems S(2,5,n) iff n $\equiv$ 1 or 5 (mod 20). Shortening these gives optimal codes for n $\equiv$ 0 or 4 (mod 20).

A(n,8,5) for n $\leq$ 15 follows from the following observation:

THEOREM 20. *If* d *is even,* λ = w-$\frac{1}{2}$d, M $\leq \frac{w}{\lambda}$ + 1 *then* A(n,d,w) $\geq$ M *iff* n $\geq$ wM - $\lambda\binom{w}{2}$.

Many more values of A(n,8,5) are known, but most lie outside the range of the table.

3.1 t = 3, d = 4, w = 4.

As is shown by HANANI [25] Steiner quadruple systems exist for each n $\equiv$ 2 or 4 (mod 6). Hence for these values of n we have A(n,4,4) = $\frac{1}{4}\binom{n}{3}$.

Shortening these codes once gives A(n,4,4) = $\frac{1}{24}$n(n-1)(n-3) for n $\equiv$ 1 or 3 (mod 6). Using triplewise balanced designs TBD($\{4,6\}$;n) in which the blocks of size 6 form a partition it follows that A(n,4,4) = $\frac{1}{24}$n(n^2-3n-6) for n $\equiv$ 0 (mod 6) (cf. BROUWER [6]). Exact values for the case n $\equiv$ 5 (mod 6) are not yet known.

4.2. THE LINEAR PROGRAMMING BOUND FOR A(n,d,w)

This is based on:

THEOREM 21. (DELSARTE [9],[10]). *Let* C *be an* $(n,M,2\delta)$ *code of constant weight* $w \leq \frac{1}{2}n$, *having distance distribution* $(A_0,\ldots,A_{2w})$. *Then the quantities* $B_0,\ldots,B_{2w}$ *are nonnegative, where now*

$$B_{2k} = \frac{1}{M} \sum_{i=0}^{w} A_{2i} Q_k(i,n,w) \qquad k = 0,\ldots,w,$$

the coefficients $Q_k(i,n,w)$ *are given by*

$$Q_k(i,n,w) = \frac{n-2k+1}{n-k+1} E_i(k) \binom{n}{k} / \binom{w}{i}\binom{n-w}{i}, \tag{11}$$

and $E_i(x)$ *is an Eberlein (or dual Hahn) polynomial defined by*

$$E_i(x) = \sum_{j=0}^{i} (-1)^{i-j} \binom{w-j}{i-j}\binom{w-x}{j}\binom{n-w+j-x}{j}.$$

(See DELSARTE [9], EBERLEIN [15], HAHN [23] and KARLIN & McGREGOR [37] for these polynomials.)

As in the case of A(n,d) we obtain a bound on A(n,d,w) by maximizing $A_0+A_2+\ldots+A_{2w}$ subject to the constraints

$$A_{2i} \geq 0 \qquad (i = \delta,\ldots,w), \quad A_0 = 1, \quad A_2 = A_4 = \ldots = A_{2\delta-2} = 0, \tag{12}$$

and

$$B_{2k} \geq 0 \qquad (k = 0,\ldots,w). \tag{13}$$

Additional constraints on the A_i can be expressed in terms of the function $T(w_1,n_1,w_2,n_2,2\delta)$ defined in section 1 (see Table 3). Let $u \in C$ and consider the codewords $v \in C$ such that $\text{dist}(u,v) = 2i$. By a suitable permutation of the coordinates we may assume that

```
       ←——— w ———→ ←——— n-w ———→
u =  11···1 11···1 00···0 00···0
v =  11···1 00···0 11···1 00···0
       ←—i—→ ←—i—→
```

The number of such v's is $A_{2i}(u)$, and by definition of T we have

$$A_{2i}(u) \leq T(i,w,i,n-w,2\delta), \quad i = \delta,\ldots,w,$$

so that

$$A_{2i} \leq T(i,w,i,n-w,2\delta), \quad i = \delta,\ldots,w. \tag{14}$$

Sometimes it is possible to say more, as the following example illustrates.

THEOREM 22.

$$A(17,8,7) \leq 31.$$

PROOF. Let C be a code of length 17, distance 8 and constant weight 7. Suppose C contains $M = A(17,8,7)$ codewords. For $u \in C$, the only nonzero components of the weight distribution with respect to u are $A_0(u) = 1$, $A_8(u)$, $A_{10}(u)$, $A_{12}(u)$, $A_{14}(u)$, and then

$$A_i = \frac{1}{M} \sum_{u \in C} A_i(u), \quad i = 0,8,10,12,14.$$

We have

$$A_{14}(u) \leq A(10,8,7) = A(10,8,1) = 1,$$
$$A_{12}(u) \leq T(6,7,6,10,8) = T(1,7,4,10,8) = 5.$$

These imply $A_{12} \leq 5$, $A_{14} \leq 1$ as in (14). But we can say more. For if $A_{14}(u) = 1$ then $A_{12}(u) \leq 2$. Therefore, for all $u \in C$,

$$A_{12}(u) + 3A_{14}(u) \leq 5 \text{ and } A_{14}(u) \leq 1,$$

and so

$$A_{12} + 3A_{14} \leq 5 \text{ and } A_{14} \leq 1. \tag{15}$$

Linear programming with the constraints (12), (13), (15) gives the stated result. □

Table 2 gives the bounds on $A(n,d,w)$. Upper bounds marked with an L are obtained by linear programming, as illustrated by Theorem 22. Unmarked lower and upper bounds are from Theorems 14-20. A useful technique for getting lower bounds is the following. Let C be an (n,M,d) code, and $C^* = a + C = \{a+u: u \in C\}$ any translate of C, with weight distribution $A_i(\underline{0})$. Then

$$A_i(\underline{0}) \leq A(n,d,i).$$

This technique works well for example with the (shortened) Nordstrom-Robinson and Golay codes. Other entries in the table are explained by the key. Letters on the left of an entry refer to lower bounds, on the right to upper bounds.

5. BOUNDS ON $T(w_1,n_1,w_2,n_2,d)$

$T(w_1,n_1,w_2,n_2,d)$ is the maximum number of binary vectors of length $n_1 + n_2$, having mutual Hamming distance at least d, where each vector has exactly w_1 ones in the first n_1 coordinates and exactly w_2 ones in the last n_2 coordinates. For example, $T(1,3,2,4,6) = 2$, as illustrated by the vectors 1001100, 0100011. Properties of this function are given in the following theorems.

THEOREM 23. (JOHNSON [34]).

(a) $T(w_1,n_1,w_2,n_2,d) = T(w_2,n_2,w_1,n_1,d)$.

(b) $T(w_1,n_1,w_2,n_2,d) = T(n_1-w_1,n_1,w_2,n_2,d)$,

(c) $T(0,n_1,w_2,n_2,d) = A(n_2,d,w_2)$,

(d) $T(w_1,n_1,w_2,n_2,d) \leq A(n_2,d-2w_1,w_2)$,

(e) If $d = 2w_1 + 2w_2$ then
$$T(w_1,n_1,w_2,n_2,d) = \min\left\{\left[\frac{n_1}{w_1}\right],\left[\frac{n_2}{w_2}\right]\right\},$$

(f) $T(w_1,n_1,w_2,n_2,d) \leq \left[\frac{n_1}{w_1} T(w_1-1,n_1-1,w_2,n_2,d)\right]$,

(g) $T(w_1,n_1,w_2,n_2,d) \leq \left[\frac{n_1}{n_1-w_1} T(w_1,n_1-1,w_2,n_2,d)\right]$,

(h) $T(w_1,n_1,w_2,n_2,2\delta) \leq \left[\dfrac{\delta}{\dfrac{w_1^2}{n_1} + \dfrac{w_2^2}{n_2} + \delta - w_1 - w_2} \right]$,

provided the denominator is positive.

A slightly stronger result than Theorem 23 (h) is:

THEOREM 24. *Suppose* $T(w_1,n_1,w_2,n_2,2\delta) = M$, *and define* q_i, r_i $(i = 1,2)$ *by*

$$Mw_i = q_i n_i + r_i, \qquad 0 \leq r_i < n_i.$$

Then

$$\sum_{i=1}^{2} \{n_i q_i (q_i - 1) + 2 q_i r_i\} \leq (w_1 + w_2 - \delta) M (M-1),$$

with equality if and only if all distances are 2δ.

There is also a linear programming bound for $T(w_1,n_1,w_2,n_2,2\delta)$, based on Theorem 25. Define the *left* and *right weights* of a vector $\underline{u} = (u_1,\ldots,u_{n_1+n_2})$ to be $w_L(\underline{u}) = wt(u_1,\ldots,u_{n_1})$ and $w_R(\underline{u}) = wt(u_{n_1+1},\ldots,u_{n_2})$.

THEOREM 25. *Let* C *be an* $(n_1+n_2,M,2\delta)$ *code such that* $w_L(\underline{u}) = w_1$, $w_R(\underline{u}) = w_2$ *for all* $u \in C$, *and let*

$$A_{2i,2j}(u) = |\{v \in C: w_L(u+v) = 2i, \ w_R(u+v) = 2j\}|,$$

$$A_{2i,2j} = \frac{1}{M} \sum_{u \in C} A_{2i,2j}(u).$$

Then

$$B_{2k,2\ell} = \frac{1}{M} \sum_{i=0}^{w_1} \sum_{j=0}^{w_2} A_{2i,2j} Q_k(i,n_1,w_1) Q_\ell(j,n_2,w_2) \geq 0,$$

where $Q_k(i,n,w)$ *is given in* (11).

PROOF. For $\nu = 1,2$ suppose $(X^{(\nu)}; R_0^{(\nu)},\ldots,R_{n_\nu}^{(\nu)})$ is an association scheme with intersection numbers $p_{ijk}^{(\nu)}$, incidence matrices $D_i^{(\nu)}$, idempotents $J_i^{(\nu)}$,

and eigenvalues $P_k^{(\nu)}$, $Q_k^{(\nu)}(i)$ (cf. DELSARTE [9], [10], SLOANE [54]). Then $(X^{(1)} \times X^{(2)}; R_{ij} = R_i^{(1)} \times R_j^{(2)}$, $0 \le i \le n_1$, $0 \le j \le n_2)$ is an association scheme (the *product scheme*) with intersection numbers $p_{ikr}^{(1)} \, p_{jls}^{(2)}$, incidence matrices $D_i^{(1)} \otimes D_j^{(2)}$, idempotents $J_i^{(1)} \otimes J_j^{(2)}$, and eigenvalues $P_k^{(1)}(i) P_\ell^{(2)}(j)$, $Q_k^{(1)}(i) Q_\ell^{(2)}(j)$. Hence C is a code in the product of two Johnson schemes. The result now follows from Theorem 3.3 of DELSARTE [9] and Theorem 21 above. $\square$

Table 3 gives upper bounds on $T(w_1, n_1, w_2, n_2, 10)$. Entries marked with an asterisk (*) are exact.

TABLE 1

Values of A(n,d)

n	d=4	d=6	d=8	d=10
6	4	2	1	1
7	8	2	1	1
8	$^a16$	2	2	1
9	$^d20^b$	4	2~	1
10	$^d38-40$	6	2	2
11	$^d72-80$	12	2	2
12	$^d144-160$	24	4	2
13	256	32^e	4	2
14	512	64	8	2
15	1024	128	16	4
16	$^a2048$	$^f256$	32	4
17	$^d2560-3276$	256-340	$36-37^h$	6
18	$^d5120-6552$	512-680	64-74	10
19	$^d9728-13104$	1024-1288	128-144	20
20	$^d19456-26208$	$^g2048-2372$	256-279	40
21	$^d36864-43690$	$^g2560-4096$	512	40-55
22	$^d73728-87380$	4096-6942	1024	$^j48-90$
23	$^d147456-173784$	8192-13774	2048	64-150
24	$^d294912-344636$	$^g16384-24106$	$^i4096$	$^k128-280$

KEY TO TABLE 1

a. Hamming code (HAMMING [24]).

b. Theorem 6.

d. Constructed in GOLAY [21], JULIN [35] or SLOANE & WHITEHEAD [57].

e. Theorem 4.

f. Nordstrom-Robinson code, NORDSTROM & ROBINSON [46].

g. Constructed in SLOANE, REDDY & CHEN [55].

h. Theorem 9.

i. Golay code (GOLAY [20]).

j. From a (24,48,12) Hadamard code.

k. Constructed by ALLTOP [1].

TABLE 2: A TABLE OF A(n,d,w)

Distance 4 : A(n, 4, w)

n \ w	2	3[a]	4[a]	5	6	7	8	9	10	11	12
4	2	1	1								
5	2	2	1	1							
6	3	4	3	1	1						
7	3	7	7	3	1	1					
8	4	8	14	8	4	1	1				
9	4	12	18	18	12	4	1	1			
10	5	13	30	36	30	13	5	1	1		
11	5	17	34-35	66	66	34-35	17	5	1	1	
12	6	20	51	[b]73-84	[e]132	73-84	51	20	6	1	1
13	6	26	65	[b]99-132	[d]143-182	143-182	99-132	65	26	6	1
14	7	28	91	[d]143-182	[d]210-308	[d]216-364	210-308	143-182	91	28	7
15	7	35	105	[d]213-272[f]	321-455	405-660	405-660	321-455	213-272	105	35
16	8	37	140	[d]305-336	513-725	?-1040	?-1320	?-1040	513-725	305-336	140
17	8	44	154-157	[d]424-476	792-952	?-1760	?-2210	?-2210	?-1760	792-952	424-476
18	9	48	[a]198	480-565	[d]1188-1428	?-2448	?-3960	?-4420	?-3960	?-2448	1188-1428
19	9	57	228	612-752	1428-1789	?-3876	?-5814	?-8360	?-8360	?-5814	?-3876
20	10	60	285	816-912	2040-2506	?-5111	?-9690	?-16720	?-16720	?-12920	?-9690
21	10	70	315	1071-1197	2856-3192	?-7518	?-13416	?-22610	?-27132	?-27132	?-22610
22	11	73	385	1386	3927-4389	?-10032	?-20674	?-33774	?-49742	?-54264	?-49742
23	11	83	415-419	1771	5313	?-14421	?-28542	?-52877	?-75426	?-104006	?-104006
24	12	88	498	[d]1859-2011	[e]7084	?-18216	?-43263	?-76912	?-126799	?-164565	?-208012

n \ w	2	3	4	5	6	7	8	9	10	11	12
6	1	2	1	1	1						
7	1	2	2	1	1-	1					
8	1	2	2	2	1	1	1				
9	1	3	3	3	3	1	1	1			
10	1	3	5	6	5	3	1	1	1		
11	1	3	6	11	11	6	3	1	1	1	
12	1	4	9	[h]12?	22	12	9	4	1	1	1
13	1	4	[g]13	[h]18?	[g]26	26	18	13	4	1	1
14	1	4	14	[h]28	[h]42	[d]42–51	42	28	14	4	1
15	1	5	15	[h]42	[h]70	60–88	60–88	70	42	15	5
16	1	5	20	48	[h]112	90–156	120–150	90–156	112	48	20
17	1	5	20[de]	68	112–136	112–244	120–283	120–283	90–244	112–136	68
18	1	6	[b]22	68–72	[d]144–203	128–349	[d]153–428	?–425	153–428	128–349	112–203
19	1	6	[c]25–26	68–83	[d]144–228	202–520	?–739	?–787	?–789	?–739	202–520
20	1	6	[a]30	[d]78–104	155–276	310–651	?–1199	?–1363	?–1421	?–1363	?–1199
21	1	7	[a]31	[d]102–126	210–364	465–828	?–1708	?–2364	[d]1008–2702	1008–2702	?–2364
22	1	7	[a]37	[d]132–136	294–462	615–1144	?–2277	?–3775	?–4416	[d]1288–5064	?–4416
23	1	7	[a]40	133–170	399–521	969–1518	?–3289	?–5519	?–7521	?–7953	?–7953
24	1	8	[e]42	[2]168–192	[e]532–680	[e]1368–1786	?–4554	?–8770	?–12418	?–14632	?–15906

Distance 8 : A(n,8,w)

n \ w	2	3	4	5	6	7	8	9	10	11	12
8	1	1	2	1	1	1	1				
9	1	1	2	2	1	1	1	1			
10	1	1	2	2	2	1	1	1	1		
11	1	1	2	2	2	2	1	1	1	1	
12	1	1	3	3	4	3	3	1	1	1	1
13	1	1	3	3	4	4	3	3	1	1	1
14	1	1	3	4	7	8	7	4	3	1	1
15	1	1	3	6	^{k}10	15	15	10	6	3	1
16	1	1	4	6	^{k}16	16-22	30	16-22	16	6	4
17	1	1	4	7	^{b}17	^{j}21-31^L	^{m}34-36^L	34-36	21-31	17	7
18	1	1	4	^{j}9	^{j}20-21	^{j}33-41^L	^{j}46-64	^{j}48-72	46-64	33-41	20-21
19	1	1	4	^{j}12	^{j}28	^{j}52-57	^{j}78-97	^{j}87-122^L	88-122	78-97	52-57
20	1	1	5	^{j}16	^{j}40	^{j}80	^{j}130-142	^{j}160-215	^{j}176-244^L	160-215	130-142
21	1	1	5	^{j}21	^{j}56	^{j}120	^{j}210	^{j}280-331	^{j}336-399^L	326-399	280-331
22	1	1	5	21^b	^{j}77	^{j}176	^{j}330	280-437^L	$^{'}$616-728	^{j}672-798	616-728
23	1	1	5	^{g}23	77-80	^{j}253	^{j}506	$^{'}$400-816	616-1111^L	$^{'}$1288-1417^L	1288-1417
24	1	1	6	^{e}24	77-92	253-274	^{j}759	$^{'}$640-1161^L	^{j}960-1639^L	1288-2305^L	^{j}2576

Distance 10 : A(n,10,w)

n \ w	2	3	4	5	6	7	8	9	10	11	12
10	1	1	1	2	1	1	1	1	1		
11	1	1	1	2	2	1	1	1	1	1	
12	1	1	1	2	2	2	1	1	1	1	1
13	1	1	1	2	2	2	2	1	1	1	1
14	1	1	1	2	2	2	2	2	1	1	1
15	1	1	1	3	3	3	3	3	3	1	1
16	1	1	1	3	3	3	4	3	3	3	1
17	1	1	1	3	3	5	6	6	5	3	3
18	1	1	1	3	4	6	9	10	9	6	4
19	1	1	1	3	4	8	[b]12	19	19	12	8
20	1	1	1	4	5	[n]10	[b]17-18	[g]20-24	38	20-24	17-18
21	1	1	1	4	7	13[b]	[9]21--26	[9]21--41[L]	38--49[L]	38--49	21--41
22	1	1	1	4	7	15--19	[9]22--35	[9]22--57[L]	?-74[L]	?-82[L]	?-74
23	1	1	1	4	8	[b]16--23	[9]23--50[L]	[9]2?--87[L]	?-117[L]	?-135[L]	?-135
24	1	1	1	4	9	[9]24--27	[9]24--69	?-119[L]	?-171[L]	?-223[L]	?-247[L]

KEY TO TABLE 2

a. §4.1

b. SHEN LIN [40].

c. H.R. PHINNEY [47].

d. Miscellaneous constructions.

e. From Theorem 9 and the Steiner systems S(5,6,12), S(3,5,17), S(3,6,26), S(5,6,24), S(5,7,28), S(5,8,24) (DENNISTON [13], DOYEN & ROSA [14], WITT [66]).

f. From Theorem 6 and the nonexistence of Steiner systems S(4,5,15), S(4,6,18) (MENDELSOHN & HUNG [44], WITT [66]).

g. A cyclic code.

h. From the 3-design with t = 3, v = 16, k = 6, λ = 4 obtained from the Nordstrom-Robinson code (NORDSTROM & ROBINSON [46]).

i. From translates of the (16,256,6) Nordstrom-Robinson code (NORDSTROM & ROBINSON [46]).

j. From the (24,4096,8) Golay code (GOLAY [20]).

k. From translates of the (16,32,8) Reed-Muller code.

L. From linear programming.

m. From a conference matrix (SLOANE & SEIDEL [56]).

n. A quasi-cyclic code.

q. JOHNSON [31], [34].

r. W.G. VALIANT [62].

Table 1 (top-left)

w_2	8	8	8	8	8	8	8	9	9	9	9	9
n_2	16	17	18	19	20	21	22	18	19	20	21	22
w_1 n_1												
1 2	8	12	18	24	40	52	70	20	38	52	82	114
1 3	12	18	27	36	60	78		30	57	78	123	
1 4	16	24	36	48	80			40	76	104		
1 5	20	30	45	60				50	95			
1 6	22	36	54					60				
1 7	22	38										
1 8	22											

Table (top-right)

w_2	6	6	6	6	6	6	6	7	7	7	7	7
n_2	12	13	14	15	16	17	18	14	15	16	17	18
w_1 n_1												
3 6	10	16	24	34	44	60	80	30	44	60	100	120
3 7	14	25	35	49	70	104		42	70	105	153	
3 8	21	37	56	74	101			56	100	148		
3 9	27	48	75	96				8+	129			
3 10	33	60	94					104				
3 11	44	73										
3 12	48											

Table (middle-left)

w_2	7	7	7	7	7	7	7	8	8	8	8	8
n_2	14	15	16	17	18	19	20	16	17	18	19	20
w_1 n_1												
2 4	10	14	18	30	36	48	60	24	36	54	72	120
2 5	15	22	30	50	60	80		40	60	90	120	
2 6	18	30	45	72	90			60	90	135		
2 7	21	38	63	91				76	115			
2 8	28	44	72					88				
2 9	36	49										
2 10	40											

Table (bottom)

w_2	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	6	6	6	6	6
n_2	8	9	10	11	12	13	14	15	16	10	11	12	13	14	15	16	12	13	14	15	16
w_1 n_1																					
4 8	4*	6*	8	10	14	18	21	22	28	10*	16	24	36	46	63	70	28	50	70	98	132
4 9	6*	9*	12	18	21	29	31	37		18*	24	40	54	72	93		47	76	107	151	
4 10	8	12	17	25	33	42	51			22	37	60	82	107			67	104	153		
4 11	10	18	25	35	48	55				33	55	81	108				90	139			
4 12	14	21	33	48	63					42	66	102					116				
4 13	18	29	42	55						58	89										
4 14	21	31	51							62											
5 10										36*	48	73	96	125							
5 11										48	77	106	146								
5 12										73	106	154									

UPPER BOUNDS FOR $T(w_1, n_1, w_2, n_2, 10)$, cont$\underline{d}$

TABLE 3   (cont)

* Bound is exact.

REFERENCES

[1] ALLTOP, W.O., *Personal communication.*

[2] BEST, M.R., *The Wax bound for binary codes,* Math. Centre Reports ZW72, Amsterdam, 1976. (Preprint of section 3 of this paper.)

[3] BEST, M.R. & A.E. BROUWER, *The triply shortened binary Hamming code is optimal,* Discrete Math., to appear, and Math. Centre Reports ZW32, Amsterdam, 1975.

[4] BROUWER, A.E., A(17,6,4) = 20, *or the nonexistence of the scarce design* SD(4,1;17,21), Math. Centre Reports ZW62, Amsterdam, 1975.

[5] BROUWER, A.E., *Optimal packings of* K_4*'s into a* K_n *- The case* n $\neq$ 2 *(mod 3),* Math. Centre Reports ZW76, Amsterdam, 1976.

[6] BROUWER, A.E., *Some triplewise balanced designs,* Math. Centre Reports ZW77, Amsterdam, 1976.

[7] BROUWER, A.E. & A. SCHRIJVER, *A group-divisible design* GD(4,1,2;N) *exists iff* N $\equiv$ 2 *(mod 6),* N $\neq$ 8 *(or: The packing of cocktail party graphs with* K_4*'s),* Math. Centre Reports ZW64, Amsterdam, 1976.

[8] DELSARTE, P., *Bounds for unrestricted codes, by linear programming,* Philips Research Reports, 27 (1972), 272-289.

[9] DELSARTE, P., *An algebraic approach to the association schemes of coding theory,* Philips Research Reports Supplements, No. 10, 1973.

[10] DELSARTE, P., *The association schemes of coding theory,* p.143-161 of Combinatorics, M. Hall, Jr. and J.H. Van Lint (eds.), Reidel, Dordrecht, and Mathematical Centre, Amsterdam, 1975.

[11] DELSARTE, P., *Properties and applications of the recurrence* $q^k F(n+1,k+1,a+1) = \ldots$, SIAM J. Applied Math., to appear.

[12] DELSARTE, P., W. HAEMERS & C. WEUG, *Unpublished tables,* 1974.

[13] DENNISTON, R.H.F., *Personal communication.*

[14] DOYEN, J. & A. ROSA, *A bibliography and survey of Steiner systems,* Bolletino Unione Matematica Italiana, 7 (1973), 392-419.

[15] EBERLEIN, P.J., *A two parameter test matrix,* Math. Comp., 18 (1964), 296-298.

[16] ELSPAS, B., *A conjecture on binary nongroup codes,* IEEE Trans. Info. Theory 11 (1965), 599-600.

[17] ERDÖS, P. & H. HANANI, *On a limit theorem in combinatorical analysis,* Publ. Math., Debrecen, 10 (1963), 10-13.

[18] FICKEN, F.A., *The simplex method of linear programming,* Holt, Rinehart & Winston, N.Y. 1961.

[19] GOETHALS, J.M., *The extended Nadler code is unique,* to appear.

[20] GOLAY, M.J.E., *Notes on digital coding,* Proc. IRE, 37 (1949), 657.

[21] GOLAY, M.J.E., *Binary coding,* IEEE Trans, Info. Theory 4 (1954), 23-28.

[22] GUY, R.K., *A problem of Zarankiewicz,* Research Paper No. 12, Univ. of Calgary (1967).

[23] HAHN, W., *Ueber Orthogonalpolynome, die q-Differenzengleichungen genuegen,* Math. Machr., 2 (1949), (4-34).

[24] HAMMING, R.W., *Error detecting and error correcting codes,* Bell Syst. Tech. J., 29 (1950), 147-160.

[25] HANANI, H., *On quadruple systems,* Canad. J. Math. 12 (1960) 145-157.

[26] HANANI, H., *The existence and construction of balanced incomplete block designs,* Annals Math. Stat., 32 (1961), 361-386.

[27] HANANI, H., *A balanced incomplete block design,* ibid., 36 (1965), 711.

[28] HANANI, H., *Balanced incomplete block designs and related designs,* Discrete Math. 11 (1975) 255-369.

[29] HELGERT, H.J. & R.D. Stinaff, *Minimum-distance bounds for binary linear codes,* IEEE Trans. Info. Theory, 19 (1973), 344-356.

[30] JOHNSON, S.M., *A new upper bound for error-correcting codes,* IRE Trans. Info. Theory, IT-8 (1962), 203-207.

[31] JOHNSON, S.M., *Improved asymptotic bounds for error-correcting codes,* IEEE Trans. Info. Theory, 9 (1963), 198-205.

[32] JOHNSON, S.M., *Unpublished Tables,* 1970.

[33] JOHNSON, S.M., *On upper bounds for unrestricted binary error-correcting codes,* IEEE Trans. Info. Theory, 17 (1971), 466-478.

[34] JOHNSON, S.M., *Upper bounds for constant weight error correcting codes,* Discrete Math., 3 (1972), 109-124.

[35] JULIN, D., *Two improved block codes,* IEEE Trans. Info. Theory, 11 (1965), 459.

[36] KALBFLEISCH, J.G. & R.G. STANTON, *Maximal and minimal coverings of* (k-1)-*tuples by* k-*tuples,* Pacific J. Math., 26 (1968) 131-140.

[37] KARLIN, S. & J.L. McGREGOR, *The Hahn polynomials,* formulas and an application, Scripta Math., 26 (1961) 33-46.

[38] KIRKMAN, T.P., *On a problem in combinations,* Cambridge and Dublin Math. J. 2 (1847) 191-204.

[39] LEVENSHTEIN, V.I., *The application of Hadamard matrices to a problem in coding,* Problems of Cybernetics, 5 (1964), 166-184.

[40] SHEN LIN, *Personal communication.*

[41] VAN LINT, J.H., *A new description of the Nadler code,* IEEE Trans. Info. Theory, 18 (1972), 825-826.

[42] McELIECE, R.J., E.R. RODEMICH, H.C. RUMSEY, JR. & L.R. WELCH, *Unpublished tables,* 1972.

[43] McELIECE, R.J., E.R. RODEMICH, H.C. RUMSEY, JR. & L.R. WELCH, *New upper bounds on the rate of a code via the Delsarte-MacWilliams inequalities,* to appear.

[44] MENDELSOHN, N.S. & S.H.Y. HUNG, *On the Steiner systems S(3,4,14) and* S(4,5,15), Notices Amer. Math. Soc. 18 (1971) 552-553: Abstract 71T-A78 and 792: erratum, and Utilitas Math. 1 (1972), 5-95.

[45] NADLER, M., *A 32-point* n=12, d=5 *code,* IEEE Trans. Info. Theory, 8 (1962), 58.

[46] NORDSTROM, A.W. & J.P. ROBINSON, *An optimum nonlinear code,* Info. Control 11 (1967), 613-616.

[47] PHINNEY, H.R., *Personal communication.*

[48] PLOTKIN, M., *Binary codes with specified minimum distances,* IEEE Trans. Info. Theory, 6 (1960), 445-450.

[49] RANKIN, R.A., *On the closest packing of spheres in* N *dimensions,* Annals of Math. 48 (1947), 1062-1081.

[50] ROGERS, C.A., *Packing and covering,* Cambridge, 1964.

[51] SCHÖNHEIM, J., *On maximal systems of* k-*tuples,* Stud. Sci. Math. Hungar., 1 (1966), 363-368.

[52] SIMONNARD, M., *Linear programming,* Prentice Hall, Englewood Cliffs, N.J., 1966.

[53] SLOANE, N.J.A., *A survey of constructive coding theory, and a table of binary codes of highest known rate,* Discrete Math. 3 (1972), 265-294.

[54] SLOANE, N.J.A., *An introduction to association schemes and coding theory,* p.225-260 of Theory and Application of Special Functions, R. Askey (ed.), Acad. Pr., N.Y. 1975.

[55] SLOANE, N.J.A., S.M. REDDY & C.L. CHEN, *New binary codes,* IEEE Trans. Info. Theory, 18 (1972), 503-510.

[56] SLOANE, N.J.A. & J.J. SEIDEL, *A new family of non-linear codes obtained from conference matrices,* Annals New York Acad. of Sciences, 175 (1970), 363-365.

[57] SLOANE, N.J.A. & D.S. WHITEHEAD, *A new family of single-error correcting codes,* IEEE Trans. Info. Theory 16 (1970), 717-719.

[58] SPENCER, J., *Maximal consistent families of triples,* J. Combinatorial Theory 5 (1968), 1-8.

[59] STANTON, R.G., J.G. KALBFLEISCH & R.C. MULLIN, *Covering and packing designs,* pp.428-450 of Proc. 2nd Chapel Hill Conf. on Combinatorial Mathematics and its Applications, R.C. Bose et. al. (eds.), Chapel Hill, N.C., 1970.

[60] STEVENS, R.F. & W.G. BOURICIUS, *The heuristic generation of large error correcting codes,* Unpublished memorandum, IBM Research Center, Yorktown Heights, August 1, 1959.

[61] SWIFT, J.D., *Quasi-Steiner systems,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 44 (1968), 40-44.

[62] VALIANT, W.G., A(14,6,7) < 52 *or the nonexistence of a certain constant weight code,* Math. Centre Reports ZW71, Amsterdam, 1976.

[63] WAX, N., *On upper bounds for error detecting and error correcting codes of finite length,* IRE Trans. Info. Theory, IT-5 (1959), 168-174.

[64] WILSON, R.M., *The construction of group-divisible designs and partial planes having the maximum number of lines of a given size,* Proc. 2nd Chapel Hill Conf. on Combinatorial Mathematics and its Applications, R.C. Bose et al. (eds.), Chapel Hill, N.C., 1970, pp. 488-497.

[65] WILSON, R.M., *An existence theory for pairwise balanced designs, I,* J. Combinatorial Theory (A) 13 (1972), 220-245, *II,* ibid. 13 (1972), 246-273, *III,* ibid. 18 (1975), 71-79.

[66] WITT, E., *Ueber Steinersche Systeme,* Abh. Math. Sem. Univ. Hamburg, 12 (1938), 265-275.