A NOTE ON $\Delta$-REGULAR GRAPHS
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A note on $\Gamma\Delta$-regular graphs

by

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dedicated to M. Voorhoeve & R. Tijdeman

ABSTRACT

A $\Gamma\Delta$-regular graph is a nonregular graph such that for each vertex $x$ the graphs induced on its neighbours and on its nonneighbours are both regular. We show that if $G$ is $\Gamma\Delta$-regular, $G$ and $\tilde{G}$ are connected, and $\text{diam } G = 3$ then $G$ is one of two graphs on 4 resp. 8 vertices.
GODSIL & MCKAY [1] introduced the concept of a $\Gamma\Delta$-regular graph (although they called it differently - we adopt Van Lint's terminology) - see the abstract. For the case we are interested in: $G$ and $\overline{G}$ are connected, they proved the following.

Let $n$ be the number of vertices, $\lambda$ the valency of $\Gamma(x)$ in $G$ and $\overline{\lambda}$ the valency of $\overline{\Gamma}(x)$ in $\overline{G}$.

1. In $G$ there occur exactly two valencies, $k_1$ and $k_2$, where $k_1 < k_2$.
2. $k_1 + k_2 = 4n + 2\lambda + 1$.
3. $\lambda + \overline{\lambda} = 4n - 2$.

Let $M_i = \{x \mid x \text{ has valency } k_i\}$ $(i = 1, 2)$. Write $m_i := |M_i|$.

4. Each $M_i$ (viewed as induced subgraph of $G$) is regular with valency $\alpha_i$ $(i = 1, 2)$, $\alpha_1 + \alpha_2 = 4n - 1$, $(2\alpha_1 - m_1 + 1)(k_1 - k_2) = (\lambda + 1)(n - 1) - k_1k_2$.

5. Let $x_1 \neq x_2$. Then $|\Gamma(x_1) \cap \Gamma(x_2)| = \lambda + 1 + \epsilon(k_1 - k_2)$, where
   $$\epsilon = 0 \text{ if } x_1 \in M_1, x_2 \in M_2,$$
   $$\epsilon = 1 \text{ if } x_1, x_2 \in M_1,$$
   $$\epsilon = -1 \text{ if } x_1, x_2 \in M_2.$$

6. $\text{diam } G \leq 3$.

Now suppose $G$ has diameter 3, and let $\text{dist}(x_1, x_2) = 3$. Then $\Gamma(x_1) \cap \Gamma(x_2) = \emptyset$ so that by (5) $x_1, x_2 \in M_1$ and $k_2 - k_1 = \lambda + 1$. Again by (5) points in $M_1$ do not have distance two, so that $M_1$ is a disjoint union of cliques ('sun's').

Also, no point of $M_2$ is adjacent to points of different suns but each point of $M_2$ is adjacent to some point in $M_1$ (in fact to $k_2 - k_2$ such points; $k_2 - k_2 > 0$ since $G$ is connected), so that the partition of $M_1$ into suns induces a partition of $M_2$ into 'corona's'. From (2) and $k_2 - k_1 = \lambda + 1$ we find $n = 4k_1 - 2\lambda$. On the other hand, choosing one vertex in each sun we find $n \geq (k_1 + 1)$. # of suns. Consequently the number of suns $N$ is less than four (and larger than one since $\text{diam } G = 3$), i.e. two or three.
Fix a point \( x_0 \in M_1 \) and count edges between \( \Gamma(x_0) \) and \( \Delta(x_0) \). One finds

\[
\alpha_1 (k_1 - \lambda - 1) + (k_1 - \alpha_1)(k_2 - \lambda - 1) = (n-k_1-1-(N-1)(\alpha_1+1))(k_2-k_1)
\]

(for: \( |\Gamma(x_0)| = k_1 \), \( |\Delta(x_0)| = n - k_1 - 1 \), \( |\text{sun}| = \alpha_1 + 1 \), etc.), i.e.,

\[
k_1^2 - \alpha_1(\lambda + 1) = (3k_1 - 2\lambda - 1 - (N-1)(\alpha_1+1))(\lambda + 1),
\]

or

\[
k_1^2 - 3(\lambda + 1)k_1 + (\lambda + 1)(2\lambda + 2 + (N-2)(\alpha_1+1)) = 0.
\]

Distinguish cases:

A. If \( N = 2 \) this factors as \((k_1 - (\lambda + 1))(k_1 - 2(\lambda + 1)) = 0\).

A1. \( N = 2 \) and \( k_1 = \lambda + 1 \).

Now \( k_2 = 2\lambda + 2 \), \( n = 2\lambda + 4 \), i.e. \( |\text{sun}| = 1 \), \( |\text{corona}| = \lambda + 1 \).

Considering two adjacent points in different coronas we find that they have \( 2\lambda \) common neighbours. Hence \( \lambda = 2\lambda \), i.e. \( \lambda = 0 \), \( n = 4 \) and \( G \) looks like

\[
\text{[Diagram: two points connected by edges.]}\]

A2. \( N = 2 \) and \( k_1 = 2(\lambda + 1) \).

Now \( k_2 = 3\lambda + 3 \), \( n = 6\lambda + 8 \), \( |\text{sun}| = \alpha_1 + 1 \), \( |\text{corona}| = 3\lambda + 3 - \alpha_1 \).

Count edges between sun and corona: \((\alpha_1+1)(k_1 - \alpha_1) = (3\lambda + 3 - \alpha_1)(k_2 - \alpha_2)\), but \( k_2 - \alpha_2 = 2\lambda + 2 - k_1 + \alpha_1 = \alpha_1 \) (using (2) and (4)), so that
$\alpha_1 = 1 + \frac{\lambda}{\lambda+2}$. Since $\alpha_1$ is integral this implies $\lambda = 0$, $n = 8$ and $G$
looks like

B. If $N = 3$ then $|\text{sun + corona}| = \frac{4}{3} k_1 - \frac{2}{3} \lambda = 1 + k_1 + \frac{1}{3} (k_1 - 2\lambda - 3)$.

As before it follows that $k_1 \geq 2\lambda + 3$, contradicting the equation

$$k_1^2 - 3(\lambda+1)k_1 + (\lambda+1)(2\lambda+\alpha_1 + 3) = 0.$$

This ends the proof.

Egeldonk, 80 09 25

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[1] GODSIL, C.D. & B.D. MCKAY, Graphs with regular neighbourhoods, to ap-
pear in: proceedings of Australian combinatorial conference,
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