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J. DE VRIES

PSEUDOCOMPACTNESS FOR G-SPACES

kruislaan 413 1098 SJ amsterdam

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Pseudocompactness for G-spaces

by

J. de Vries

ABSTRACT

In this note we prove that if G is a locally compact group and $\langle X,\pi \rangle$ is a Tychonov G-space, then the notions of G-pseudocompactness for $\langle X,\pi \rangle$ and pseudocompactness for X coincide. We also discuss situations where pseudocompactness is implied by the equality $\beta_G X = \beta X$.

KEY WORDS & PHRASES: G-*space*, G-*compactification*, *pseudocompact*, G-*pseudocompact*

In this note we discuss the relationship between the notion of pseudocompactness for G-spaces and two notions of G-pseudocompactness which were introduced independently by S.A. ANTONYAN [1] and the author [6], respectively. It turns out that if G is locally compact, then G-pseudocompactness according to [6] is equivalent with pseudocompactness. The notion of G-pseudocompactness according to [1] is weaker, but in certain special cases it is also equivalent with pseudocompactness. The main result of this note solves several problems of [4] in a rather obvious way: this will be discussed at the end of this note.

For a general theory of G-spaces (= topological transformation groups with acting group G) we refer to [4]. For the convenience of the reader we include here a few definitions from [4] and [6]. The symbol G stands always for a topological Hausdorff group (the Hausdorff property is rather inessential and may without restriction of generality always by assumed as long as we consider actions of G on T_1-spaces: one can always pass to G/G_0 as the acting group, where G_0 is the isotropy subgroup of G).

A G-*space* is a pair $\langle X,\pi\rangle$ where X is a topological space and π (the *action* of G on X) is a continuous mapping from $G \times X$ onto X satisfying the following conditions:

(i) $\pi(e,x) = x$ for all $x \in X$ (e is the unit element of G);

(ii) $\pi(s,\pi(t,x)) = \pi(st,x)$ for all $s,t \in G$ and $x \in X$.

Note, that these axioms imply that for each $t \in G$ the mapping $\pi^t: x \mapsto \pi(t,x): X \to X$ is a homeomorphism. For brevity, we shall write tx for $\pi(t,x)$ $(=\pi^t x)$, Ux for $\{tx: t \in U\}$, etc.. If $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, then a mapping $\phi: X \to Y$ is called *equivariant* whenever $\phi\circ\pi^t = \sigma^t\circ\phi$ for all $t \in G$, that is, $\phi(tx) = t\phi(x)$ for all $t \in G$ and $x \in X$. Every G-space $\langle X,\pi\rangle$ has an essentially unique unique *maximal* G-*compactification*

$$\phi_{\langle X,\pi\rangle}: \langle X,\pi\rangle \to \langle \beta_G X,\pi\rangle,$$

that is, an equivariant continuous mapping $\phi_{\langle X,\pi\rangle}$ from $\langle X,\pi\rangle$ to a G-space $\langle \beta_G X,\bar{\pi}\rangle$ where $\beta_G X$ is a compact Hausdorff space, which is characterized by the following property: every equivariant continuous mapping from $\langle X,\pi\rangle$ to a compact Hausdorff G-space factorizes uniquely over $\phi_{\langle X,\pi\rangle}$; cf. [4; 4.3.2(vi)]. If G is locally compact, then the mapping $\phi_{\langle X,\pi\rangle}$ is a dense equivariant embedding of X into $\beta_G X$ iff the space X is Tychonov [5]. So henceforth *we shall assume that* G *is locally compact and that every* G*-space* $<X,\pi>$ *has* X *a Tychonov space*. In that case, we shall consider X just as a dense invariant subset of $\beta_G X$, and we write

$$UC^*<X,\pi> := \{f|_X : f \in C(\beta_G X)\}.$$

The members of the function space $UC^*<X,\pi>$ can be characterized as follows [5]: if $g \in C^*(X)$ (:= the space of bounded real valued functions on X) then $g \in UC^*<X,\pi>$ iff

$$(*) \qquad \forall\varepsilon>0 \quad \exists U \in \mathcal{V}_e : |g(tx)-g(x)| < \varepsilon \text{ for all } (t,x) \in U \times X.$$

(Here $\mathcal{V}_e$ denotes the nbd filter of e in G). The set of all $g \in C(X)$ satisfying condition (*) will be denoted by $UC<X,\pi>$ and will be called the set of π-*uniformly continuous functions* (so the elements of $UC^*<X,\pi>$ are the *bounded* π-uniformly continuous functions).

A natural question to ask is, under which additional conditions one has $\beta_G X = \beta X$, where βX denotes the ordinary Stone-Čech compactification. (By the equality $\beta_G X = \beta X$ we mean that there exists a homeomorphism h of βX onto $\beta_G X$ such that $h \circ \beta_X = \phi_{<X,\pi>}$; here β_X is the canonical inclusion mapping of X into βX.) In general, one has $\beta_G X \neq \beta X$ (see [4;4.4.14 & 4.4.19]), but if, for example, the action of G on X is trivial (that is, $tx = x$ for all $t \in G$ and $x \in X$) or if G is a discrete group [4;7.3.10(iii)], then $\beta_G X = \beta X$. In [1] the following result is announced for compact groups and, unaware of this, I proved it in [6] for arbitrary k-groups:

<u>THEOREM 1</u>. *If* $<X,\pi>$ *is a Tychonov* G*-space, and* X *is pseudocompact, then* $\beta_G X = \beta X$. □

The converse is not true: in [1] is a simple example, and here is another one: let X be an arbitrary space and let π be the trivial action of G on X; then $\beta_G X = \beta X$, but X need not be pseudocompact. Actually, this shows that it is improbable to find a simple condition on G or on X which, together with the condition $\beta_G X = \beta X$ will imply that X is pseudocompact:

one needs also a certain non-triviality condition for the action. Since the above counterexample also works for non-trivial actions of *discrete* groups, one might also expect that a certain non-discreteness condition for G would help.

The following result generalizes Theorem 4.10 of [2] (local compactness of G need not be assumed).

<u>THEOREM 2</u>. *Let* $\langle X,\pi\rangle$ *be a* G-*space with* X *a* T_4-*space. If* $\beta_G X = \beta X$, *then for every net* $\{(t_\lambda,x_\lambda)\}_{\lambda\in\Lambda}$ *in* $G \times X$ *such that* $t_\lambda \rightsquigarrow e$ *in* G *one has* $\overline{\{x_\lambda : \lambda \in \Lambda\}} \cap \overline{\{t_\lambda x_\lambda : \lambda \in \Lambda\}} \neq \emptyset$.

<u>PROOF</u>. Suppose that two closed sets as indicated in the statement of the theorem are disjoint. Then they have disjoint closures in βX. By passing to a suitable subnet, we may assume that the net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to a point z in βX. Since the action of G on X extends to a continuous action of G on $\beta X(=\beta_G X$ by assumption) and the net $\{t_\lambda\}_{\lambda\in\Lambda}$ converges to e in G, it follows that $t_\lambda x_\lambda \rightsquigarrow ez = z$. This contradicts the disjointness of the closures in βX of the two sets indicated above. □

The following corollary of this theorem may be seen as a modification of Proposition 3.4 of [3] (one of the difficulties which prevent a honest generalization of that result to the present context is, that the mappings $\pi_x : t \mapsto tx : G \to X$ are in general not open). Recall, that if α is a cardinal number, then a space is called α-*pseudocompact* whenever every locally finite family of mutually disjoint, non-empty open subsets has cardinality less than α. The *local weight* of G (i.e. the least cardinal number of a local basis of G at e) will be denoted by $\ell w(G)$. Finally, recall that if $\langle X,\pi\rangle$ is a G-space, then the isotropy subgroup of x in G is the subgroup $G_x := \{t \in G : tx = x\}$. If X is a T_1-space, then G_x is always closed in G, because the mapping $\pi_x : t \mapsto tx : G \to X$ is continuous.

<u>COROLLARY 1</u>. *Let* $\langle X,\pi\rangle$ *be a* G-*space with* X *a* T_4-*space such that* $\beta_G X = \beta X$. *Then either the set*

$$X_0 := \{x \in X : G_x \text{ is open in } G\}$$

has a non-empty interior, or X *is* $\ell w(G)$*-pseudocompact.*

PROOF. Suppose the contrary: there exists a dense set of points in X, each having non-open isotropy group, and X is not $\ell w(G)$-pseudocompact. Then there exists a locally finite, disjoint family $\mathcal{W}$ of non-empty open subsets of G with cardinality $\ell w(G)$. Let $\mathcal{B}$ be a local basis at e having cardinality $\ell w(G)$, and let $U \mapsto W_U$ be an injective mapping from $\mathcal{B}$ into $\mathcal{W}$. For every $U \in \mathcal{B}$ there exists a point x_U in W_U with non-open isotropy group. So there exists $t_U \in U$ such that $t_U x_U \in W_U$ and $t_U x_U \neq x_U$. Since the family $\{W_U : U \in \mathcal{B}\}$ is locally finite, the sets $\{x_U : U \in \mathcal{B}\}$ and $\{t_U x_U : U \in \mathcal{B}\}$ are closed in X. Since they are also disjoint, this contradicts the theorem above. □

REMARK. Observe, that in the proof of this corollary local finiteness of the family $\{W_U : U \in \mathcal{B}\}$ is not very essential. Indeed the set $\{x_U : U \in \mathcal{B}\}$ is disjoint from the closure of $\{t_U x_U : U \in \mathcal{B}\}$, because the neighbourhood W_U of x_U contains only the element $t_U x_U$ of the latter set; similarly, the other way round. So it would be sufficient for the proof to guarantee that one of the sets is closed. Thus, if we define

$$s_G(X) := \sup\{\text{card } M : M \subseteq X \sim X_0 \text{ and } M \text{ is discrete and closed in } X\}$$

then a similar proof shows that

$$s_G(X) < \ell w(G).$$

It is not difficult to see, that X_0 is an invariant subset of X (indeed, for $t \in G$ and $x \in X$ we have $G_{tx} = tGt^{-1}$). Moreover, all invariant points belong to X_0. If G is connected, the only open subgroup of G is G itself, so in that case X_0 equals exactly the set of all invariant points in X. If X_0 has empty interior, then we shall say that the G-space $\langle X,\pi\rangle$ *has almost no open isotropy groups.*

COROLLARY 2. *Let* G *be locally compact and let* $\langle X,\pi\rangle$ *be a* G *space with almost no open isotropy groups. If, in addition,* X *is a separable metric space, then the equality* $\beta_G X = \beta X$ *implies that* X *is pseudocompact, hence compact.*

PROOF. We may assume that G acts effectively on X. (Otherwise, pass to the corresponding effective action of G/G_0, where $G_0 := \bigcap \{G_x : x \in X\}$; observe, that G/G_0 is locally compact, and that for given $x \in X$ the isotropy subgroup in G is open iff the corresponding isotropy subgroup in G/G_0 is open.) Then $\ell w(G) \leq w(X)$ [4;1.1.23], so by Corollary 1, X is pseudo-$w(X)$-compact. In our case, however, $w(X) = \aleph_0$, and pseudo-$\aleph_0$-compactness is the same is ordinary pseudocompactness. □

REMARK. If G is locally compact, non-discreet, and G acts freely on a metric space X (ie. $G_x = \{e\}$ for every $x \in X$) then also $\beta_G X = \beta X$ implies that X is (pseudo)compact. For still another case where $\beta_G X = \beta X$ implies pseudocompactness of X, see Corollary 4 below.

In [1], a Tychonov G-space $\langle X,\pi\rangle$ such that $\beta_G X = \beta X$ was called G-pseudocompact. Unfortunately, in [6] I introduced a different notion of G-pseudocompactness (that it is really different follows from the examples above and theorem 3 below). The notion of G-pseudocompactness according to [6] is as follows:

Let $\langle X,\pi\rangle$ be a Tychonov G-space. A finite (resp. countably infinite) collection $\mathcal{B}$ of mutually disjoint, non-empty open subsets in X is called a G-*dispersion* whenever it satisfies the following condition:

$$(**) \qquad \exists U \in V_e : \forall B \in \mathcal{B}\ \exists x_B \subseteq B : Ux_B \subseteq B.$$

The G-space $\langle X,\pi\rangle$ is called G-*pseudocompact* whenever every locally finite G-dispersion in X is finite. It is obvious, that if G is discrete or if the action of G on X is trivial, then G-pseudocompactness of $\langle X,\pi\rangle$ is exactly the same as pseudocompactness of X. Moreover, if X is pseudocompact, then $\langle X,\pi\rangle$ clearly is G-pseudocompact, but the converse was left as an open problem in [6]. In [6;5.8] I conjectured that the converse is false, but I could find no counterexample. The following theorem shows, why I couldn't; the proof is quite simple.

THEOREM 3. *Let* G *be locally compact and let* $\langle X,\pi\rangle$ *be a Tychonov* G-*space. Then* $\langle X,\pi\rangle$ *is* G-*pseudocompact iff* X *is pseudocompact.*

PROOF. "If": obvious (see also the remarks above). "Only if": let $\{W_n\}_{n\in\mathbb{N}}$

be an infinite sequence of non-empty open subsets of X, mutually disjoint. Let U be a compact symmetric neighbourhood of e in G and let $x_n \in W_n$ for every $n \in \mathbb{N}$. Since $\langle X,\pi\rangle$ is assumed to be G-pseudocompact, no sequence $\{W'_n\}_{n\in\mathbb{N}}$ with W'_k an open neighbourhood of Ux_k for every $k \in \mathbb{N}$ can be locally finite (if there would be such a sequence which is locally finite, then there would also be such a sequence which is *disjoint* and locally finite, i.e. a locally finite G-dispersion; for the straightforward proof of this, see [6;2.2(4^0)]. In particular, the sequence $\{UW_n\}_{n\in\mathbb{N}}$ is not locally finite: there exists a point x_0 in X such that every neighbourhood V of x_0 intersects infinitely many of the sets UW_n. Let V be a neighbourhood of Ux_0. Since the action of G on X is continuous as a mapping of $G \times X$ into X and U is compact, there exists a neighbourhood V' of x_0 such that $UV' \subseteq V$. For infinitely many values of $n \in \mathbb{N}$ we have now that $V' \cap UW_n \neq \emptyset$, hence $UV' \cap W_n \neq \emptyset$ (for $U^{-1} = U$), and, consequently, $V \cap W_n \neq \emptyset$. If the sequence $\{W_n\}_{n\in\mathbb{N}}$ were locally finite, then the compact set Ux_0 would have a neighbourhood, intersecting only finitely many of the sets W_n. Thus, the sequence $\{W_n\}_{n\in\mathbb{N}}$ is not locally finite. This shows, that X is pseudocompact. □

Using this theorem, we now reformulate some results from [6]; in doing so, some of the open problems of [6] are solved.

COROLLARY 3. *Let* G *and* $\langle X,\pi\rangle$ *be as in the theorem above. Consider the following properties:*

(i) *Every* $f \in UC^*\langle X,\pi\rangle$ *has a maximum and a minimum on* X;

(ii) X *is pseudocompact*

(iii) *Every* π*-uniformly continuous function on* X *is bounded.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *and* (iii) $\not\Rightarrow$ (ii).

PROOF. For (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iii) $\not\Rightarrow$ (ii), see [6;2.5]. The implication (ii) $\Rightarrow$ (i) is trivial. □

REMARK. In [6; Remark 5.11] the implication $\langle X,\pi\rangle$ is G-pseudocompact $\Rightarrow$ (i) was left open. A problem which was not considered in [6] is, under which additional conditions one has (iii) $\Rightarrow$ (ii) in the above corollary. Here is a partial solution:

COROLLARY 4. *Let* G *be a locally compact metrizable topological group, and let* $\langle X,\pi\rangle$ *be a normal Hausdorff* G*-space. Assume that there are almost no* *open isotropy subgroups. Then the following conditions are equivalent:*

(i) X *is pseudocompact*;

(ii) *Every* π*-uniformly continuous function on* X *is bounded and* $\beta_G X = \beta X$.

PROOF. For (i) $\Rightarrow$ (ii), see Corollary 3 together with Theorem 1. For (ii) $\Rightarrow$ (i), suppose f is an unbounded π-continuous function. Without restriction of generality we may suppose that $f \geq 0$. Let $\{x'_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $f(x'_{n+1}) > f(x'_n)+1$ for all n, and let $W_n := \{x \in X : |f(x)-f(x'_n)| < 1/3\}$. Since $\langle X,\pi\rangle$ has almost no open isotropy groups, for every $n \in \mathbb{N}$ there is a point $x_n \in W_n$ such that the isotropy group G_{x_n} is not open in G. Now the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a discrete, closed subset, and (as G is metrizable) we can find a sequence $\{t_n\}_{n\in\mathbb{N}}$ in G such that $t_n x_n \neq x_n$, $t_n x_n \in W_n$ for all n, and $t_n \rightsquigarrow e$ for $n \to \infty$ (cf. the proof of Corollary 1). As in the proof of Corollary 1 (see also the Remark after that proof), this contradicts Theorem 2. □

REMARKS 1. Problem 5.3 of [4] remains open.

2. The general question for necessary and sufficient conditions for the equality $\beta_G X = \beta X$ is still open. The problem whether G-speudocompactness is sufficient (cf. [6;5.10] is solved by Theorems 1 and 3 above: the answer is "yes". In this context, see also Theorem 6 in [1].

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