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THE GENERALIZED CARTAN DECOMPOSITION FOR A COMPACT LIE GROUP

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The generalized Cartan decomposition for a compact Lie group *)

by

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ABSTRACT

We prove a generalized Cartan decomposition for a compact Lie group, namely $G = KA_{pq}H$, where G is a compact semisimple real Lie group and K and H are the fixed points of two commuting involutions of G. We also prove an integral formula for this decomposition, and we give an expression for the radial part of the Laplace-Beltrami operator with respect to this decomposition.

KEY WORDS & PHRASES: *Compact Lie group, Cartan decomposition, Generalized Cartan decomposition, Integral formula,* K,H-*radial part of the Laplace-Beltrami operator*

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*) This report will be submitted for publication elsewhere.

0. INTRODUCTION

For a semisimple Lie group G one has the so called *Cartan decomposition*. That is, if (G,K) is a Riemannian symmetric pair of the compact or noncompact type, then $G = KA_pK$. Here $A_p = \exp \mathfrak{a}_p$, with $\mathfrak{a}_p$ a maximal abelian subalgebra in the -1 eigenspace $\mathfrak{p}$ of $d\theta$ in $\mathfrak{g}$ (the Lie algebra of G), where θ is the involution of G such that $(G_\theta)_0 \subset K \subset G_\theta$.

If (G,K) is of noncompact type then the above decomposition has the following generalization. Let σ be an (arbitrary) involution of G commuting with θ, put $H := (G_\sigma)_0$, and let $\mathfrak{q}$ be the -1 eigenspace of $d\sigma$ in $\mathfrak{g}$. Choose a maximal abelian subalgebra $\mathfrak{a}_{pq}$ in $\mathfrak{p} \cap \mathfrak{q}$, and put $A_{pq} := \exp \mathfrak{a}_{pq}$. Then $G = KA_{pq}H$. This decomposition, which we shall refer to as the *generalized Cartan decomposition*, was first proved in BERGER [1]. For a modern account see FLENSTED-JENSEN [2, Theorem 4.1(i)].

In this paper we shall prove a generalized Cartan decomposition for a compact Lie group. Since Flensted-Jensen's proof uses a lemma of Mostow (see [10]), which does not apply in the case of a compact Lie group, we have to follow a different, differential geometric approach. Without changes this proof also applies to the noncompact case. Thus, we are able to formulate and prove these results in a quite general way.

We also derive an integral formula corresponding to the generalized Cartan decomposition of a compact Lie group. This formula is very similar to the analogous formula for a noncompact Lie group, see FLENSTED-JENSEN [3, Theorem 2.6]. Finally we derive an expression for the radial part of the Laplace-Beltrami operator with respect to the generalized Cartan decomposition.

These results are of great importance for the analysis of the so-called *intertwining functions* on G. These are left-K-, right-H-invariant functions on G which belong to some irreducible representation of G. Recently the author proved that for a compact group G the intertwining functions can be considered as orthogonal polynomials in several variables on a region in $\mathbb{R}^\ell$ ($\ell = \dim \mathfrak{a}_{pq}$) with respect to a positive weight function. Those results will be part of the author's thesis, which is planned to appear at the University of Leiden, see also HOOGENBOOM [7].

1. NOTATION AND PRELIMINARIES

Let G be a connected real semisimple Lie group with finite center. Let θ,σ be two commuting involutions of G. We assume that either G is compact, or θ is a Cartan involution of G. Put $K := (G_\theta)_0$, $H := (G_\sigma)_0$. Let $\mathfrak{g}$ be the Lie algebra of G, and, by abuse of notation, we'll also write θ and σ for the differential of θ,σ, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ in ± 1 eigenspaces of θ, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the decomposition of $\mathfrak{g}$ in ± 1 eigenspaces of σ. Then $\mathfrak{k},\mathfrak{h}$ are the Lie algebras of K,H, respectively.

Since $\sigma\theta = \theta\sigma$ we have the following direct sum decomposition:

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}. \tag{1.1}$$

Let $\mathfrak{a}_{pq}$ be a maximal abelian subalgebra in $\mathfrak{p} \cap \mathfrak{q}$, then $\mathfrak{a}_{pq}$ necessarily consists of semisimple elements. Put $A_{pq} := \exp \mathfrak{a}_{pq}$.

2. A CARTAN DECOMPOSITION FOR H

LEMMA 2.1. $(H,(K\cap H)_0)$ *is a Riemannian symmetric pair.*

PROOF. $(K\cap H)_0 = (H_\theta)_0$. □

Lemma 2.1 enables us to use differential geometric methods, cf. eg. HELGASON [6, ch.I], for $H/(K\cap H)_0$. Therefore, introduce an H-invariant Riemannian structure on $H/(K\cap H)_0$.

LEMMA 2.2. $H = (K\cap H)_0 \exp(\mathfrak{p}\cap\mathfrak{h})$.

PROOF. By Lemma 2.1 $H/(K\cap H)_0$ is a Riemannian symmetric space. Hence, by [6, Theorem VI.3.3] and [6, Theorem I.10.3] $H/(K\cap H)_0$ is a complete Riemannian manifold. Now identify $\mathfrak{p} \cap \mathfrak{h}$ with the tangent space to $H/(K\cap H)_0$ at $o(:=e(K\cap H)_0)$, then it follows from [6, Theorem I.10.5] that $\mathrm{Exp}(\mathfrak{p}\cap\mathfrak{h}) = H/(K\cap H)_0$. □

REMARK 2.3. If G is noncompact, and σ is not a Cartan involution of G (i.e. H is noncompact) then the mapping $(k,X) \mapsto k \exp X\colon (K\cap H)_0 \times \exp(\mathfrak{p}\cap\mathfrak{h}) \to H$

is an analytic diffeomorphism. Moreover, $K \cap H$ is connected.

Let b be maximal abelian in $p \cap h$, and put $B := \exp b$.

LEMMA 2.4. $p \cap h = \bigcup_{k \in (K\cap H)_0} \mathrm{Ad}(k) \,.\, b$.

PROOF. h is a subalgebra of g, invariant under the Cartan involution θ, hence h is reductive. If h is semisimple, the lemma follows by [6, Lemma V.6.3]. So suppose h is not semisimple. Then $h = [h,h] + z(h)$, with $[h,h]$ semisimple and $z(h)$ the center of h ([8, Proposition 19.1]). The only part in the proof of [6, Lemma V.6.3] in which the semisimplicity of h would be used is $B|_{k\cap h \times k\cap h}$ is negative definite (here B denotes the Killing form on h). But if h is reductive we can argue: $B([k_0.X,H],T) = 0$ for all $T \in k \cap h$ implies $[k_0.X,H] \in z(h) \cap [h,h] = (0)$, hence $[k_0.X,H] = 0$ $(k_0 \in K\cap H, X \in p\cap h, H \in a)$. Thus the proof of [6, Lemma V.6.3] also works in the case h is reductive. □

THEOREM 2.5. $H = (K\cap H)_0 B (K\cap H)_0$.

PROOF. Let $h \in H$. Then we can write

$$(2.1) \qquad h = \ell_1 \exp X \qquad (\ell_1 \in (K\cap H)_0, X \in p\cap h),$$

and

$$(2.2) \qquad X = \mathrm{Ad}(\ell_2) H_1 \qquad (\ell_2 \in (K\cap H)_0, H_1 \in b)$$

because of Lemmas 2.2 and 2.4, respectively. Combination of (2.1) and (2.2) yields

$$h = \ell_1 \exp(\mathrm{Ad}(\ell_2)H_1) = \ell_1 \ell_2 \exp H_1 \ell_2^{-1} \in (K\cap H)_0 B (K\cap H)_0. \quad \square$$

3. THE GENERALIZED CARTAN DECOMPOSITION FOR G

Let g_0 be the +1 eigenspace of $\sigma\theta$ in g. That is $g_0 = k \cap h + p \cap q$. Let G_0 be the analytic subgroup of G with Lie algebra g_0. We shall need

Lemmas 2.1, 2.2, 2.4 and Theorem 2.5 in the cases where the pair (θ,σ) is replaced by the pair $(\theta,\sigma\theta)$. For later reference we shall state these results in a lemma. Therefore remark that $(K\cap G_0)_0 = (K\cap H)_0$.

LEMMA 3.1.

(1) $H = \exp(p\cap h).(K\cap H)$

(2) $G_0 = \exp(p\cap q).(K\cap H)$

(3) $G_0 = (K\cap H)A_{pq}(K\cap H)$.

Let Exp be the exponential mapping in the space G/K.

LEMMA 3.2. *Left multiplication with* $\exp(p\cap h)$ *leaves* $\mathrm{Exp}(p\cap h)$ *invariant.*

PROOF. $\exp(p\cap h)\exp(p\cap h) \subset H = \exp(p\cap h)(K\cap H)$, by Lemma 3.1(1). Thus $\exp(p\cap h)\,\mathrm{Exp}(p\cap h) \subset \mathrm{Exp}(p\cap h)$. □

Now Lemma 3.2 has the following corollary:

COROLLARY 3.3. $\mathrm{Exp}(p\cap h)$ *is a totally geodesic submanifold of* G/K.

N.B. Remark that Corollary 3.3 also follows from the fact that $p\cap h$ is a Lie triple system included in p, as defined in [6, p.224], by using [6, Theorem IV. 7.2].

LEMMA 3.4. $\mathrm{Exp}(p\cap h)$ *is closed in* G/K.

PROOF. H is closed in G. Because of Lemma 3.1(1) we have $\mathrm{Exp}(p\cap h) = \pi(H)$, where $\pi: G \to G/K$ is the natural projection. But π sends closed subsets of G to closed subsets of G/K, because K is compact. Hence $\mathrm{Exp}(p\cap h)$ is closed in G/K. □

PROPOSITION 3.5. $G = K\exp(p\cap q)\exp(p\cap h)$.

PROOF. We'll prove $G/K = \exp(p\cap h)\mathrm{Exp}(p\cap q)$, which implies the proposition. Let $P \in G/K$. Let $X \in p\cap h$ be such that Exp X is an element of $\mathrm{Exp}(p\cap h)$ with minimal distance to P (such an X exists because of Lemma 3.4). Let $o := \pi(e)$, and put $Q := \exp(-X)P$. Then it follows from Lemma 3.2 that o is an element of $\mathrm{Exp}(p\cap h)$ with minimal distance to Q. Let $\gamma(t) = \mathrm{Exp}\, tY$ $(Y\in p)$

be a geodesic which realizes the minimal distance between o and Q (such a γ exists because of [6, Theorem I.10.4], G/K being a complete Riemannian manifold (cf. Proof of Lemma 2.2)). We shall now prove that $Y \in p \cap q$, hence $P = (\exp X)\, Q = \exp X \operatorname{Exp} t_0 Y \in \exp(p \cap h) \operatorname{Exp}(p \cap q)$ $(t_0 \in \mathbb{R})$.

Let W be an open ball around o in ip of sufficient small radius such that Exp: $W \to V = \operatorname{Exp} W$ is a diffeomorphism and, for any $Q_1, Q_2 \in V$, Q_1 and Q_2 can be joined by precisely one geodesic of minimal length, which lies entirely in V, cf. [6, Theorem I.9.9].

Let Q' be an element of γ lying in V between o and Q. Suppose Q' has a shorter distance to $\operatorname{Exp}(p \cap h)$ than $d(Q',o)$ (d denoting the Riemannian metric in G/K), say to Exp Z $(Z \in (p \cap h))$. Then:
$d(Q,\operatorname{Exp}Z) \leq d(Q,Q') + d(Q',\operatorname{Exp}Z) < d(Q,Q') + d(Q',o) = d(Q,o)$,
a contradiction, since o was the element of $\operatorname{Exp}(p \cap h)$ with minimal distance to Q. So we may assume $Q \in V$.

V is a ball around o, hence V is σ-invariant, hence $\sigma Q \in V$. Now, let β be the unique geodesic in V which joins Q and σQ. Since β is unique, we have $\beta = \sigma\beta$. We claim $o \in \beta$. Namely, suppose $o \notin \beta$. Since $\beta = \sigma\beta$ there exists a $Q'' \in \beta$ such that $\sigma Q'' = Q''$, hence $\beta \cap \operatorname{Exp}(p \cap h) \ni Q''$. Now $Q' \neq o$, since $o \notin \beta$. Let d_β be the distance between points along β, d_γ distance along γ. β minimalizes the distance between Q and σQ, and $d(Q,o) = d(\sigma Q,o)$. Hence
$d_\beta(Q,Q'') = \frac{1}{2} d_\beta(Q,\sigma Q) < \frac{1}{2}(d_\gamma(Q,o) + d_{\sigma\gamma}(o,\sigma Q)) = d_\gamma(Q,o)$,
a contradiction. Hence $o \in \beta$, hence $\beta = \gamma$.

Now remember that $Y \in p$ is such that $\gamma(t) = \operatorname{Exp} tY$. Since $\beta = \gamma$, $\sigma\gamma(t) = \gamma(-t)$, hence $\sigma Y = -Y$, ie. $Y \in p \cap q$, which proves the proposition by the above remarks. □

<u>THEOREM 3.6</u> (*Generalized Cartan decomposition*)

$$G = KA_{pq}H.$$

<u>PROOF</u>. Let $g \in G$. Then by Proposition 3.5 there exists an $X \in p \cap q$ such that

$$g \in K \exp X \exp(p \cap h). \tag{3.1}$$

By Lemma 3.1(3) there exists an $a \in A_{pq}$ such that:

(3.2) $\quad \exp X \in (K\cap H)a(K\cap H).$

Combination of (3.1) and (3.2) gives $g \in KaH$. □

REMARK 3.7. If G is noncompact, then Theorem 3.6 can be refined such that the a in $g = kah$ ($a\in A_{pq}, g\in G, k\in K, h\in H$) becomes unique. Therefore, let Σ_0 be the set of roots of the pair $(\mathfrak{g}_0, \mathfrak{a}_{pq})$, and let W_0 be the Weyl group of Σ_0. Choose a positive Weyl chamber $\mathfrak{a}^+_{pq}$ in $\mathfrak{a}_{pq}$, and put $A^+_{pq} := \exp \mathfrak{a}^+_{pq}$. Then $G = K\overline{A^+_{pq}}H$, such that for all $g \in G$ there exists a unique $a \in \overline{A^+_{pq}}$ such that $g \in KaH$, see FLENSTED-JENSEN [2, Theorem 4.1(i)].

4. AN INTEGRAL FORMULA FOR THE GENERALIZED CARTAN DECOMPOSITION

In the case G is noncompact, FLENSTED-JENSEN [3] gives an integral formula for the generalized Cartan decomposition. Although the integral formula for G compact is very similar to the noncompact case, the proof is more involved, just as for the integral formula for the Cartan decomposition, cf. HELGASON [4, Ch.X]. Therefore, we shall treat the compact case here, and summarize the results from [3, section 2] only.

Let Σ_{pq} be the set of roots of the pair $(\mathfrak{g}_c, (\mathfrak{a}_{pq})_c)$. Then Σ_{pq} satisfies the axioms of a root system, cf. ROSSMANN [11, Theorem 5]. For $\alpha \in \Sigma_{pq}$, let $\mathfrak{g}_\alpha$ be the root space of α, and let $p_\alpha := \dim(\mathfrak{g}_\alpha \cap (\mathfrak{k}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q})_c)$, $q_\alpha := \dim(\mathfrak{g}_\alpha \cap (\mathfrak{k}\cap\mathfrak{q}+\mathfrak{p}\cap\mathfrak{h})_c)$. That is, p_α is the dimension of the set of all $X \in \mathfrak{g}_\alpha$ such that $\sigma\theta X = X$, q_α the dimension of the set of all $X \in \mathfrak{g}_\alpha$ such that $\sigma\theta X = -X$. Choose a positive system Σ^+_{pq} in Σ_{pq}.

If G is noncompact, then by a proof, similar to the proof of Lemma 4.2 we find for the density δ:

$$(4.1) \qquad \delta(X) := \Big| \prod_{\alpha\in\Sigma^+_{pq}} \operatorname{sh}^{p_\alpha}\alpha(X)\operatorname{ch}^{q_\alpha}\alpha(X) \Big| , \qquad X \in \mathfrak{a}_{pq}.$$

Put $L := K \cap H$, $M := C_L(\mathfrak{a}_{pq})$. Then with a suitable normalization of the involved measures, we have the following integral formula ([3, Theorem 2.6]):

$$(4.2)\qquad \int_G f(g)dg = \mathrm{vol}(L/M)\int_K\int_{\mathfrak{a}_{pq}^+}\int_H f(k\exp Xh)\delta(X)dhdXdk,\quad f\in C_c(G).$$

(here $\mathfrak{a}_{pq}^+$ is the positive Weyl chamber as in Remark 3.7).

From now on, let U be a compact semisimple Lie group, with analytic subgroups K,H as in section 1. Put $L := K \cap H$, $M := C_L(\mathfrak{a}_{pq})$. Define a mapping $\Phi := K/M \times A_{pq} \to U/H$ by

$$(4.3)\qquad \Phi(kM,a) := kaH,\qquad k\in K,\ a\in A_{pq}.$$

Normalize measures as follows:

$$(4.4)\qquad \int_U du = \int_K dk = \int_H dh = \int_L d\ell = \int_M dm = \int_{A_{pq}} da = 1.$$

Denote the Lie algebra of U by $\mathfrak{u}$. Now the Killing from on $\mathfrak{u}$ induces invariant measures on U/H, K/M, L/M and $\mathfrak{a}_{pq}$. Let the corresponding Riemannian measures be denoted by duH, dkM, $d\ell M$, and dX, respectively. Let ℓ, m be the Lie algebras of L,M, respectively. Let ℓ' be the orthogonal complement (with respect to the Killing form) of m in ℓ. Then we have to calculate $|\det d\Phi_{(eM,a)}|$, where $d\Phi_{(eM,a)}: \ell' + (k\cap q) + \mathfrak{a}_{pq} \to d\tau(a)(k\cap q + p\cap q)$ is the Jacobi matrix (τ defined by $\tau(u)xH := uxH$ for $u,x\in U$). Because of the fact that for $X\in\mathfrak{a}_{pq}$ $\exp X = e$ implies $\alpha(X)\in 2\pi i\mathbb{Z}$ for all $\alpha\in\Sigma_{pq}$ the following definition makes sense:

<u>DEFINITION 4.1</u>. $\delta(\exp X) := |\Pi_{\alpha\in\Sigma_{pq}^+} \sin^{p_\alpha}\alpha(iX)\cos^{q_\alpha}\alpha(iX)|$, $X\in\mathfrak{a}_{pq}$.

<u>LEMMA 4.2</u>. $|\det d\Phi_{(eM,a)}| = \delta(a)$.

<u>PROOF</u> (sketch). Let q_0 be the dimension of the zerospace of ad $\mathfrak{a}_{pq}$ in $p\cap h$, and r_0 be the dimension of the zerospace of ad $\mathfrak{a}_{pq}$ in $k\cap q$. Choose ON (:=orthonormal) bases as follows:

$$T_\alpha^1,\dots,T_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+) \text{ of } \ell'$$

$$Y_\alpha^1,\dots,Y_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+) \text{ of } p\cap q\cap a_{pq}^\perp,$$

$$X_\alpha^1,\dots,X_\alpha^{q_\alpha}(\alpha\in\Sigma_{pq}^+),\ X_0^1,\dots,X_0^{q_0} \text{ of } p\cap h,$$

and

$$Z_\alpha^1,\dots,Z_\alpha^{q_\alpha}(\alpha\in\Sigma_{pq}^+),\ Z_0^1,\dots,Z_0^{r_0} \text{ of } k\cap q$$

such that:

$$\mathrm{ad}(X)T_\alpha^j = -\alpha(iX)Y_\alpha^j,$$

$$\mathrm{ad}(X)Y_\alpha^j = \alpha(iX)T_\alpha^j,$$

$$\mathrm{ad}(X)X_\alpha^j = -\alpha(iX)Z_\alpha^j,$$

$$\mathrm{ad}(X)Z_\alpha^j = \alpha(iX)X_\alpha^j$$

for all $X\in a_{pq}$. Choose an ON basis $\{X_1,\dots,X_\ell\}$ of a_{pq}. Now we'll calculate the matrix of $d\Phi_{(eM,a)}$ with respect to the ON basis

$$T_\alpha^1,\dots,T_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+),\ Z_\alpha^1,\dots,Z_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+),\ Z_0^1,\dots,Z_0^{r_0},\ X_1,\dots,X_\ell$$

of $\ell' + (k\cap q) + a_{pq}$, and the ON basis

$$Y_\alpha^1,\dots,Y_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+),\ Z_\alpha^1,\dots,Z_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+),\ Z_0^1,\dots,Z_0^{r_0},\ X_1,\dots,X_\ell$$

of $q = (p\cap q\cap a_{pq}^\perp)+(k\cap q) + a_{pq}$. It is clear that $d\Phi_{(eM,a)}(X_j) = d\tau(a)X_j$. Now if $Y\in k\cap m^\perp$, $d\Phi_{(eM,a)}(Y)$ follows from differentiation of the 1-parameter curve

$$t\to\pi(\exp tY\exp X) = \exp X.\pi(\exp(te^{-\mathrm{ad}X}Y)),$$

where $\pi\colon U\to U/H$ denotes the canonical projection, and $X\in a_{pq}$ is such that $a = \exp X$. Thus

$$d\Phi_{(eM,a)}(Y) = d\tau(\exp X)\tfrac{1}{2}(e^{-\mathrm{ad}X}Y - e^{\mathrm{ad}X}\sigma Y),$$

hence

$$d\Phi_{(eM,a)}(T^j_\alpha) = d\tau(\exp X)\sin\alpha(iX)Y^j_\alpha,$$

$$d\Phi_{(eM,a)}(Z^j_\alpha) = d\tau(\exp X)\cos\alpha(iX)Z^j_\alpha,$$

$$d\Phi_{(eM,a)}(Z^j_0) = d\tau(\exp X)Z^j_0,$$

which proves the lemma. □

From now on the compactness of U will play an essential role.

Let $(A_{pq})_r$ be the set of elements in A_{pq} such that Φ is regular at (eM,a). That is

$$(4.5)\qquad (A_{pq})_r = \{\exp X \mid X \in \mathfrak{a}_{pq}, \alpha(X) \notin \pi i\mathbb{Z} \text{ if } p_\alpha \neq 0, \quad \alpha(X) + \tfrac{1}{2}\pi i \notin \pi i\mathbb{Z} \text{ if } q_\alpha \neq 0 \ \ \forall\alpha \in \Sigma^+_{pq}\}.$$

Let the image of $K/M \times (A_{pq})_r$ under Φ, which is an open dense subset of U/H (by Theorem 3.6), be denoted by $(U/H)_r$. Put $M_K := C_K(\mathfrak{a}_{pq})$, $M^*_K := N_K(\mathfrak{a}_{pq})$, $M_H := C_H(\mathfrak{a}_{pq})$, $M^*_H := N_H(\mathfrak{a}_{pq})$. Let W_{pq} be the Weyl group of Σ_{pq}. Then $W_{pq} = M^*_K/M_K = M^*_H/M_H$.

<u>DEFINITION 4.3</u>. Let J be the set of all pairs (s,mh) such that $m \in M^*_K$, $h \in H$, $mh \in A_{pq}$ and $s = \mathrm{Ad}(m)|_{\mathfrak{a}_{pq}} \in W_{pq}$.

<u>LEMMA 4.4</u>. *Let* $k \in K$, $h \in H$ *and* $a,b \in A_{pq}$ *be such that* $b = kah$. *Then* $b^4 = ka^4k^{-1}$.

<u>PROOF</u>. Apply σ,θ and $\sigma\theta$ to $b = kah$ and eliminate θh and σk. This gives $a^3 = hb^3k$, or $b^3 = h^{-1}a^3k^{-1}$. Thus $b^4 = b.b^3 = kah.h^{-1}a^3k^{-1} = ka^4k^{-1}$. □

Thus J is a finite set, since $J \subset W_{pq}\times(KH\cap A_{pq})$, W_{pq} is finite by definition, and $KH \cap A_{pq}$ is discrete (by Lemma 4.4) as well as compact, hence also finite. Let $j := |J|$ be the number of elements of J.

Observe that J can be given a group structure. Put, for (s_1, m_1h_1), $(s_2, m_2h_2) \in J$

(4.6) $$(s_1, m_1h_1)(s_2, m_2h_2) := (s_1s_2, m_1m_2h_2h_1).$$

Since (4.6) equals $(s_1s_2, m_1(m_2h_2)m_1^{-1}(m_1h_1))$, this is well-defined. The inverse of $(s, mh) \in J$ is given by

(4.7) $$(s, mh)^{-1} := (s^{-1}, m^{-1}h^{-1}).$$

Thus (4.6) gives J a group structure. Moreover, J acts on A_{pq} in a diffeomorphic way, via

(4.8) $$(s, mh)(\exp X) := (\exp sX)mh.$$

Let $j_1 := (s_1, m_1h_1)$, $j_2 := (s_2, m_2h_2) \in J$. Then

(4.9) $$j_1 = j_2 \Longleftrightarrow m_2^{-1}m_1 \in M \text{ and } h_2 = (m_2^{-1}m_1)h_1.$$

Thus there is a well-defined action of J on $K/M \times A_{pq}$ via

(4.10) $$(\mathrm{Ad}(m)|_{\mathfrak{a}_{pq}}, mh).(k_1M, a_1) := (k_1m^{-1}M, ma_1h)$$

(since $m \in M$ normalizes M, (4.9) implies that (4.10) is well-defined). It is clear that $\Phi \circ j = \Phi$ $\forall j \in J$.

PROPOSITION 4.5. *Φ is a regular* j*-to-one mapping of* $K/M \times (A_{pq})_r$ *onto* $(U/H)_r$.

PROOF. Regularity follows from Lemma 4.2, and the open dense subset $(U/H)_r$ is by definition the image of $K/M \times (A_{pq})_r$. So the only thing left to prove is the fact that Φ is j-to-one. Therefore, let A'_{pq} be the set of all $a \in A_{pq}$ such that the sequence $\{a^4, a^8, a^{12}, \ldots\}$ is dense in A_{pq}. Then A'_{pq} is dense in A_{pq}.

Assume $a_1 \in A'_{pq}$, $a_2 \in A_{pq}$, $k_1, k_2 \in K$ be such that $\Phi(k_1M, a_1) = \Phi(k_2M, a_2)$.

Then for certain $h_1, h_2 \in H$ we have $k_1 a_1 h_1 = k_2 a_2 h_2$. Or, by putting $k := k_2^{-1} k_1$, $h := h_1 h_2^{-1}$, $a_2 = k a_1 h$. Thus, by Lemma 4.4, we obtain $a_2^4 = k a_1^4 k^{-1}$ (hence $a_2 \in A'_{pq}$).

Let $X \in \mathfrak{a}_{pq}$. Then $\mathrm{Ad}(k)X \in \mathfrak{p}$, but also $\sigma(\mathrm{Ad}(k)X) = -\mathrm{Ad}\sigma(k)X = -\mathrm{Ad}(k)X$, hence $\mathrm{Ad}(k)X \in \mathfrak{p} \cap \mathfrak{q}$. (The last identity follows by applying $\sigma\theta$ to $a_2^4 = k a_1^4 k^{-1}$, which gives $a_2^4 = \sigma(k) a_1^4 \sigma(k^{-1})$. Hence $(k^{-1}\sigma(k)) a_1^4 (\sigma(k^{-1})k) = a_1^4$, hence $(k^{-1}\sigma(k)) a (\sigma(k^{-1})k) = a\ \forall a \in A_{pq}$, thus $\mathrm{Ad}(k)X = \mathrm{Ad}(\sigma(k))X\ \forall X \in \mathfrak{a}_{pq}$).

Moreover, $\mathrm{Ad}(k)X$ centralizes $\mathfrak{a}_{pq}$. Namely $\mathrm{Ad}(a_2^4)\mathrm{Ad}(k)X = \mathrm{Ad}(k)\mathrm{Ad}(a_1^4)X = \mathrm{Ad}(k)X$, hence $\mathrm{Ad}(a)\mathrm{Ad}(k)X = \mathrm{Ad}(k)X\ \forall a \in A_{pq}$, hence $[Y, \mathrm{Ad}(k)X] = 0\ \forall Y \in \mathfrak{a}_{pq}$. Thus $k \in M_K^*$, and $kh = k a_1^{-1} k^{-1} a_2 \in A_{pq}$. So, if $a_1, a_2 \in A'_{pq}$, $k \in K$, $h \in H$, then $a_2 = k a_1 h$ iff $k \in M_K^*$ and $kh \in A_{pq}$.

Now, let $a_1, a_2 \in A'_{pq}$, $k_1, k_2 \in K$, $h_1, h_2 \in H$ be such that $a_2 = k_1 a_1 h_1 = k_2 a_1 h_2$. Put $k := k_2^{-1} k_1$, $h := h_1 h_2^{-1}$, then $k a_1 h = a_1$, thus $k a_1^4 k^{-1} = a_1^4$, by Lemma 4.4. Thus $k a k^{-1} = a\ \forall a \in A_{pq}$, hence $\mathrm{Ad}(k)X = X\ \forall X \in \mathfrak{a}_{pq}$. Thus $\mathrm{Ad}(k_1)|_{\mathfrak{a}_{pq}} = \mathrm{Ad}(k_2)|_{\mathfrak{a}_{pq}}$, thus $k_1 h_1 = k_2 h_2$.

Thus Φ is a j-to-one mapping of $K/M \times A'_{pq}$ onto $\Phi(K/M \times A'_{pq}) =: (U/H)'$. We shall now prove that Φ is j-to-one from $K/M \times (A_{pq})_r$ onto $\Phi(K/M \times (A_{pq})_r) = (U/H)_r$. $(U/H)'$ is dense in $(U/H)_r$, because A_{pq} is dense in $(A_{pq})_r$.

Let $y \in (U/H)'$. Assume $|\Phi^{-1}(y)| > j$, $x_1, \ldots, x_{j+1} \in \Phi^{-1}(y)$. Then there is an open neighbourhood V of y, and disjunct open neighbourhoods U_i of x_i $(i = 1, \ldots, j+1)$ such that $F: U_i \to V$ is a homeomorphism. But $\exists z \in V \cap (U/H)'$, thus $\Phi^{-1}(z) \subset K/M \times A'_{pq}$, and $|\Phi^{-1}(z)| > j+1$. Contradiction.

Assume $|\Phi^{-1}(y)| < j$, ie. $\Phi^{-1}(y) = \{x_1, \ldots, x_t\}$, $t < j$. Again, take V open neighbourhood of y, and U_i open neighbourhood of x_i $(i = 1, \ldots, t)$ such that $F: U_i \to V$ is a homeomorphism. Now by the action (4.10) J acts diffeomorphic on $K/M \times A_{pq}$, and $\Phi \circ j = \Phi$, hence $j(K/M \times (A_{pq})_r) = K/M \times (A_{pq})_r$ $\forall j \in J$. Let $y_n \to y$, with $y_n \in V \cap (U/H)'$. Let $z_n \in U_1$ be such that $\Phi(z_n) = y_n$. $\exists j_n \in J$ such that $j_n . z_n \notin U_1 \cup \ldots \cup U_t$, because $J.z_n$ has cardinality $j > t$, and is mapped to y_n, since Φ is injective on each U_i $(i = 1, \ldots, t)$. Hence there is a subsequence $j_0 . z_{i_n}$, with $j_0 \in J$ fixed (because J is finite), $z_{i_n} \to x_1$, and $j_0 . z_{i_n} \to j_0 . x_1 \notin U_1 \cup \ldots \cup U_t$, and $j_0 . x_1 \in K/M \times (A_{pq})_r$ since $x_1 \in K/M \times (A_{pq})_r$. Contradiction.

Thus $|\Phi^{-1}(y)| = j$, which proves the proposition. □

<u>REMARK 4.6</u>. Let $w := |W_{pq}|$, $k := |MH \cap A_{pq}|$. Then it can be shown that $j = wk$.

THEOREM 4.7. *Let* $f \in C(U)$. *Then, with the normalization of measures* (4.4),

$$(4.11)\qquad \int_{A_{pq}} \delta(a)da \int_U f(u)du = \int_K \int_{A_{pq}} \int_H f(kah)\delta(a)dhdadk.$$

PROOF. From what is said above, it follows that we have the following expressions:

$$(4.12)\qquad \int_{U/H} f_1(uH)duH = \gamma j^{-1} \int_{A_{pq}} \int_{K/M} f_1(kaH)\delta(a)dkMda$$

$$\left(\gamma = \frac{1}{vol(A_{pq})},\ f_1 \in C(U/H)\right),$$

$$(4.13)\qquad vol(U/H)\int_U f_2(u)du = \int_{U/H} \left(\int_H f_2(uh)dh\right)duH \quad C(f_2 \in C(U)),$$

$$(4.14)\qquad vol(K/M)\int_K f_3(k)dk = \int_{K/M} \left(\int_M f_3(km)dm\right)dkM \quad C(f_3 \in C(K)).$$

Now (4.12), (4.13) and (4.14) imply (cf. HELGASON [4,p.384]) that for all $f \in C(U)$:

$$vol(U/H)\int_U f(u)du = \gamma j^{-1} vol(K/M) \int_{A_{pq}} \int_K \int_H f(kah)\delta(a)dhdkda.$$

(4.11) follows by substitution of $f \equiv 1$. □

REMARK 4.8. The evaluation of $\int_{A_{pq}} \delta(a)da$ leads to integrals of Selberg-type. See MACDONALD [9] for some explicit values and some conjectured values for integrals of this type.

5. THE K,H-RADIAL PART OF THE LAPLACE-BELTRAMI OPERATOR

In this section let G again be an arbitrary connected real semisimple Lie group. Let $\delta'(\Omega)$ denote the radial part of the Laplace-Beltrami operator acting on a K-invariant function $f \in C^\infty(G/H)$ (which we shall denote by $f \in C^\infty(K\backslash G/H)$). As in the proof of Lemma 4.2, choose a basis $X_1,\ldots,X_\ell$ of $\mathfrak{a}_{pq}$ such that $B(X_i,X_j) = \delta_{ij}$, where $B(\cdot,\cdot)$ denotes the Killing form on $\mathfrak{g}$.

Let the function δ on A_{pq} be defined as in (4.1) ($\mathfrak{g}$ noncompact), or as in Definition 4.1 ($\mathfrak{g}$ compact). For $\alpha \in \Sigma_{pq}$, let m_α be the multiplicity of α in $\mathfrak{g}$, that is $m_\alpha = p_\alpha + q_\alpha$. Put $\rho := \frac{1}{2} \sum_{\alpha\in\Sigma_{pq}^+} m_\alpha \alpha$. For $\alpha \in \Sigma_{pq}$, define A_α by $B(X,A_\alpha) = \alpha(X)$ for all $X \in \mathfrak{a}_{pq}$, and A_ρ by $B(X,A_\rho) = \rho(X)$ for all $X \in \mathfrak{a}_{pq}$.

<u>THEOREM 5.1</u>. $\delta'(\Omega) = \sum_{j=1}^{\ell} X_j^2 + 2A_\rho + 2\sum_{\alpha\in\Sigma_{pq}^+} (p_\alpha(e^{2\alpha}-1)^{-1} - q_\alpha(e^{2\alpha}+1)^{-1})A_\alpha$.

<u>PROOF</u>. (See also [2, formula (4.12)] and [3,p.307]). According to Theorem 3.6 we have $G = KA_{pq}H$. Let $f \in C^\infty(K\backslash G/H)$. Observe that according to Theorem 4.7 (or according to (4.2) if G is noncompact) we have

$$\int_{G/H} f(x)dx = c \int_{A_{pq}} f(a)\delta(a)da. \tag{5.1}$$

Then it follows from HELGASON [5, Theorem I.2.11] that

$$(\delta'(\Omega)f)(a) = \delta^{-\frac{1}{2}}\circ\Delta(\delta^{\frac{1}{2}}f)(a) - \delta^{-\frac{1}{2}}\circ\Delta(\delta^{\frac{1}{2}})(a), \tag{5.2}$$

where Δ is the Laplace-Beltrami operator on A_{pq}. Thus

$$\delta'(\Omega) = \delta^{-\frac{1}{2}}\circ\Delta\circ\delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}}\circ\Delta(\delta^{\frac{1}{2}}). \tag{5.3}$$

But if $\{X_1,\ldots,X_\ell\}$ is an orthonormal basis of $\mathfrak{a}_{pq}$, then we have

$$\Delta = \sum_{j=1}^{\ell} X_j^2.$$

Thus (5.3) becomes

$$\delta'(\Omega) = \sum_{j=1}^{\ell} \delta^{-\frac{1}{2}}\circ X_j^2\circ\delta^{\frac{1}{2}} - \sum_{j=1}^{\ell} \delta^{-\frac{1}{2}}\circ X_j^2(\delta^{\frac{1}{2}}), \tag{5.4}$$

or, by simple calculation

$$\delta'(\Omega) = \sum_{j=1}^{\ell} X_j^2 + 2\sum_{j=1}^{\ell} \delta^{-\frac{1}{2}}\circ X_j(\delta^{\frac{1}{2}})\circ X_j. \tag{5.5}$$

Substitution of the expression for the function δ gives

$$\delta'(\Omega) = \sum_{j=1}^{\ell} X_j^2 + 2A_\rho + 2\sum_{\alpha\in\Sigma_{pq}^+} (p_\alpha(e^{2\alpha}-1)^{-1} - q_\alpha(e^{2\alpha}+1)^{-1})A_\alpha. \quad \square$$

As a corollary we obtain the following expression for $\delta'(\Omega)$, acting on $f \in C^\infty(K\backslash G/H)$:

$$(5.6) \qquad (\delta'(\Omega)f)(\exp X) = (\sum_{j=1}^{\ell} X_j^2 + \sum_{\alpha\in\Sigma_{pq}^+} (p_\alpha \coth\alpha(X) + q_\alpha \operatorname{th}\alpha(X))A_\alpha).f(\exp X).$$

If G is compact (5.6) gives

$$(5.7) \qquad (\delta'(\Omega)f)(\exp X) = (\sum_{j=1}^{\ell} X_j^2 + \sum_{\alpha\in\Sigma_{pq}^+} (p_\alpha \operatorname{cotg}\alpha(iX) + q_\alpha \operatorname{tg}\alpha(iX))iA_\alpha).f(\exp X).$$

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