stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZW 191/83 MEI
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER, A.M. COHEN & H.A. WILBRINK

NEAR POLYGONS WITH LINES OF SIZE THREE AND FISCHER SPACES

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 51E20, 51B25, 51F25

Near polygons with lines of size three and Fischer spaces *)

by

A.E. Brouwer, A.M. Cohen & H.A. Wilbrink

ABSTRACT

We show that certain near polygons carry in a natural way the structure of a Fischer space, and determine all near hexagons with lines of length 3 and with quads, having at least one GQ(2,4) quad. In particular, we prove the uniqueness of the $PO_6^-(3)$ near hexagon on 567 vertices and describe a new near hexagon on 243 vertices.

KEY WORDS & PHRASES: *Near hexagon, Near polygon, Fischer space*

---

*) This report will be submitted for publication elsewhere.

0. INTRODUCTION

A *near polygon* is a connected partial linear space (X,L) such that given a point x and a line L there is a unique point on L closest to x (where distances are measured in the point graph: two points are adjacent iff they are distinct and collinear). For more details on near polygons, see Brouwer & Wilbrink [2] - properties derived there will be assumed known without further mention.

A *Fischer space* is a linear space (E,L) such that

(i) all lines have size 2 or 3

(ii) for any point x the map $\sigma_x: E \to E$ fixing x and all lines through x and interchanging the two points distinct from x on a line of length 3 through x is an automorphism.

One can show that $\sigma_x\sigma_y$ has order 2 or 3 whenever the line xy has size 2 or 3 (respectively). For more details on Fischer spaces, see Buekenhout [4].

We shall call two distinct points x, y of E *adjacent* (and write $x \sim y$) when they are on a 3-line. Usually, we do not mention L and talk about the near polygon X or the Fischer space E.

1. BIG SUBSPACES OF A NEAR POLYGON

A geodetically closed proper sub- near polygon Y of a near polygon X with the property that each point of X has distance at most one to Y is called a *big subspace* of X.

<u>LEMMA 1</u>. *Let* Y *be a big subspace and* L *a line of* X *such that* $Y \cap L = \emptyset$. *Then* πL *is a line of* Y *(where* π *denotes the projection onto* Y*).*

<u>PROOF</u>. The projection $\pi: X \to Y$ is well defined by $\pi(y) = y$ if $y \in Y$ and $\pi(x) \sim x$ if $x \notin Y$ since Y is a geodetically closed subspace and X does not contain triangles of lines. Let a, b $\in$ L. Since Y is geodetically closed and $\pi(a)$ a b $\pi(b)$ is a path connecting $\pi(a)$ and $\pi(b)$, we have $d(\pi(a),\pi(b)) \leq 2$. Now $\pi(a)$ has distance at most two to both b and $\pi(b)$ so is adjacent to a point of the line b $\pi(b)$ which must be $\pi(b)$. This shows that πL is a clique hence contained in a line L', but since $L' /\!/ \pi L$ it follows that $\pi L = L'$. □

LEMMA 2. *Let* Y, Y' *be two disjoint big subspaces of* X. *Then* π: Y' $\rightarrow$ Y *is an isomorphism.* □

LEMMA 3. *Let* Y, Y' *be two big subspaces of* X *with* $Y \cap Y' \neq \emptyset$. *Then any line meeting both* Y *and* Y' *also meets* $Y \cap Y'$.

PROOF. Let $L = yy'$ be a line with $L \cap Y = \{y\}$, $L \cap Y' = \{y'\}$, $y \neq y'$. Let $z \in Y \cap Y'$. Let $i = d(z,y)$, $i' = d(z,y')$. If $i = i'$ then there is a point $y'' \in L$ with $d(z,y'') = i-1$, and by geodetic closure of Y and Y' we find $y'' \in Y \cap Y'$, so we are done. Thus we may assume $i > i'$. But now again by geodetic closure $y' \in Y$. This settles the Lemma. □

2. THE FISCHER SPACE OF THE BIG SUBSPACES OF A NEAR POLYGON WITH s = 2

THEOREM. *Let* X *be a near polygon with lines of size* 3. *For any big subspace* Y *define an involution* σ_Y *by*

$$\sigma_Y(x) = \begin{cases} x \text{ if } x \in Y \\ z \text{ if } \{x,y,z\} \text{ is a line meeting } Y \text{ in } y. \end{cases}$$

Then σ_Y *is well defined and an automorphism of* X. *If* Y *and* Y' *are two big subspaces then*

(i) *if* Y *meets* Y' *then* σ_Y *and* $\sigma_{Y'}$ *commute* ;

(ii) *if* $Y \cap Y' = \emptyset$ *then* $Y'' = \sigma_Y(Y') = \sigma_{Y'}(Y)$ *is a third big subspace and* $\sigma_{Y''} = \sigma_Y\sigma_{Y'}\sigma_Y = \sigma_{Y'}\sigma_Y\sigma_{Y'}$.

PROOF. $\sigma = \sigma_Y$ is well defined since Y is geodetically closed. We have to show that σ preserves lines. This is clear for lines meeting Y, so let $L = \{a,b,c\}$ be a line disjoint from Y. The point $\sigma(a)$ is adjacent to some point of the line $b\,\pi(b)$ (by the near polygon property), and since this line has only three points we find $\sigma(a) \sim \sigma(b)$. Thus $\sigma(L)$ is a line and σ is an automorphism. If Y' is another big subspace then if $Y \cap Y' \neq \emptyset$ we see from Lemma 3 that σ leaves Y' invariant; now if $x \in X\backslash(Y \cup Y')$ and $x \sim y \in Y$, $x \sim y' \in Y'$ then the lines xy and xy' determine a 3×3 grid, and inside this lattice one immediately sees that $\sigma_Y(\sigma_{Y'}(x)) = \sigma_{Y'}(\sigma_Y(x))$. If $Y \cap Y' = \emptyset$ then $Y'' = \sigma(Y')$ is disjoint from both Y and Y', and

$Y \cup Y' \cup Y''$ is isomorphic to the direct product $L \times Y$ where L is a 3-line. □

<u>Corollary</u>. *Let* X *be a near polygon with lines of size* 3. *Let* E *be the collection of big subspaces of* X, *and let* L_E *be the collection of subsets* $\{Y, Y', \sigma_Y(Y')\}$ *of* E. *Then* (E, L_E) *is a Fischer space.* □

2. THE KNOWN EXAMPLES

The classical examples of near polygons are dual polar spaces (cf. CAMERON [5]), in our case $O_{2d+1}(2)$, $Sp_{2d}(2)$ and $U_{2d}(2^2)$ where d is the diameter of the near polygon. Here big subspaces correspond to points of the corresponding polar space, and big subspaces are adjacent iff the (polar) points are nonadjacent (i.e., not orthogonal).

Direct products of near polygons correspond to direct sums of Fischer spaces (points in different summands are joined by 2-lines).

Other near polygons known with lines of size 3 do not possess big subspaces:

d	v	group	t_2+1	t_3+1	$t+1$	Reference
3	729	$3^6.M_{12}$	2	-	12	Shult & Yanushka [9]
3	759	M_{24}	3	-	15	Shult & Yanushka [9]
4	315	HJ.2	1	4	5	Cohen [6]

(The third possesses sub near hexagons $G_2(2)$ such that every point has a unique neighbour in each of them. These are not geodetically closed, but are Cameron sets, and there are no involutions σ_Y, e.g., because $t_2=0$.)

However, recently Aschbacher [1] described a near hexagon on 567 points with group $PO_6^-(3)$ which is not regular but contains big subspaces. It is constructed as follows:

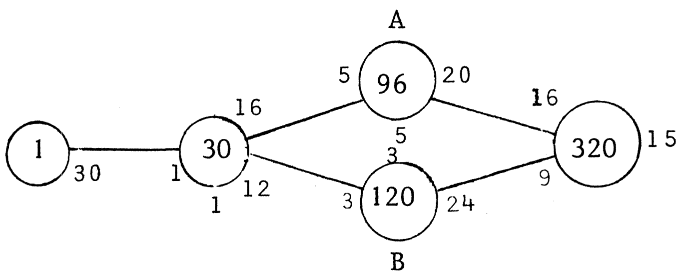
Let V be a vector space of dimension 6 over $\mathbb{F}_3$ equipped with a nondegenerate quadratic form Q of Witt index 2. Let N be the set of 126 projective points of norm 1. The points of the near hexagon are the 6-tuples of projective points forming an orthonormal basis of V:

$$X = \{B \subset N \mid |B| = 6 \text{ and } b \in (B\setminus\{b\})^{\perp} \text{ for } b \in B\}.$$

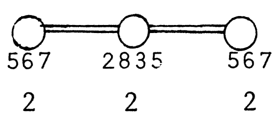
Two points B, B' of X are called adjacent if $|B \cap B'| = 2$, or, equivalently, lines are pairs of projective points with norm one and mutually orthogonal:

$$L = \left\{\{b,b'\} \subset N \mid (b,b') = 0\right\}.$$

Each line has size 3, and there are $\binom{6}{2} = 15$ lines on each point.
The parameter diagram of this near hexagon has the form



Thus $s+1 = 3$, $t+1 = 15$, $t_2+1 \in \{3,5\}$. Any two points at distance two determine a unique quad, either a 15-pt quad $(O_5(2))$ or a 27-pt quad $(O_6^-(2))$. The 126 27-pt quads correspond to the 126 points of N. The 567 15-pt quads correspond to certain sets of 30 points of N partitioned into 15 pairs such that the 15 pairs and 15 bases in this 30-set carry the structure of a GQ(2,2); they can be found from the bases $B' \subset N' = Q^{-1}(2)$: the $\binom{6}{2} = 15$ lines meeting B' in two points have 2 points in N and 2 points in N'.
In this way we find the required 15 pairs on 30 points of N.
Points, lines and 15-pt quads yield a Buekenhout geometry with diagram:



$PO_6^-(3)$ has an outer automorphism interchanging N and N', yielding a polarity of this geometry. (For more details, see KANTOR [8].)
The 27-pt quads are big quads, and the corresponding Fischer space is the

linear space induced on N by the lines of PG(5,3): the elliptic lines become 2-lines and the tangents become 3-lines. The main purpose of this note is to characterize this near hexagon.
In a subsequent note we shall determine all near hexagons with quads and lines of size 3 having big 15-point quads.

3. NEAR HEXAGONS WITH A BIG QUAD

In this section we consider near hexagons (X,L) (i.e., near polygons of diameter 3) with lines of size 3 and containing a quad Q of type GQ(2,4).

Since points at distance two from Q would determine an ovoid in Q, while GQ(2,4) does not possess ovoids, we have:

(1) Q is a big subspace.

Now also assume that any two points at distance two from each other have at least two common neighbours; then our near hexagon 'has quads', and there is a constant number t+1 of lines through each point. Let $v = |X|$

(2) $v = 54t - 189$.

For, if $x \in Q$ then x is on t+1 lines, 5 of which are in Q and t-4 of which have two points in $X \setminus Q$. Thus $v = 27+27.(t-4).2$. □

The lines and quads on a fixed point x form a linear space L_x with t+1 points, and lines of size 2,3 or 5. (For, the possible quads are of types GQ(2,1), GQ(2,2) and GQ(2,4), and these have 2,3 or 5 lines on each point.)
If $\pi: X \to Q$ is the projection onto Q (defined by $\pi|_Q = \text{id.}$ and $\pi x \sim x$ for $x \in X \setminus Q$) then the projection of a line in $X \setminus Q$ is a line in Q. If $x_1 \in X \setminus Q$ then all lines on x, projecting to a fixed line L_1 of Q are in the quad $Q(x_1,\pi x_1,L_1)$, and thus there are at most 4 such lines (note that the line $x_1\pi x_1$ does project to a point).
More precisely: suppose that the line $x_1\pi x_1$ is in A quads of type GQ(2,1), in B quads of type GQ(2,2), and in C quads of type GQ(2,4). Then we have

(3) $t+1 = A + 2B + 4C + 1$

but also, since no quad can meet Q in a single point:

(4) $A + B + C = 5$.

It follows that $6 \le t+1 \le 21$, and that if $t+1 = 21$ then all quads are of type GQ(2,4) so that we have the (unique) classical dual polar space with v=891 belonging to $U(6,2^2)$. Let us consider the possibilities for the linear space L_x.

It satisfies the following two axioms:

(i) all lines have size 2,3 or 5

(ii) all lines meet every 5-line.

Also, we may assume that (for some x) (at least) two different line sizes occur, for otherwise the near hexagon is regular and hence known.

A. *Each point of* L_x *is on at most three 5-lines.*
Suppose some $u \in L_x$ is on (at least) four 5-lines. All lines not on u meet each of these four lines, and hence also are 5-lines. It follows immediately that $L_x \cong PG(2,4)$ and t+1 = 21, the classical case.

B. *Each point of* L_x *is on at most two 5-lines.*
Suppose some $u \in L_x$ is on exactly three 5-lines L_j (j=1,2,3). If there is a point $v \notin L_1 \cup L_2 \cup L_3$ then v is on four 5-lines and we are in case A. Thus t+1 = 13, and all remaining lines of L_x are 3-lines meeting all L_j (j=1,2,3).

Now if t+1 = 13 then it is impossible that some L has exactly two 5-lines M_1, M_2 on a point v. For otherwise there would be a line joining two of the four remaining points and not passing through v and hence be another 5-line M_3. Now M_1, M_2 and M_3 form a triangle covering 12 points, and the remaining point lies on a 4-line, contradiction.

Let Q_1 be a 15-point quad on x, and let $x \sim y \in Q_1$. (Then xy is not the point u of the local space L_x.) Consider the local space L_y. On the point v = xy we see one 5-line M and four 3-lines, and if there are no other 5-lines then we see a resolution of the edges of K_8 (namely $L_y \backslash M$) in five parallel classes, which is ridiculous. Thus L_y has the same structure as L_x.

Now a global counting argument will kill this case. Consider the partition of X into Q_1, $\Gamma_1(Q_1)$ and $\Gamma_2(Q_1)$. We have $|Q_1| = 15$, $|\Gamma_1(Q_1)| = 2.10.15 = 300$, $|\Gamma_2(Q_1)| = (54.12-189) - 15 - 300 = 144$. Now Q_1 does not possess partitions into ovoids, so $\Gamma_2(Q_1)$ does not contain lines, and all 13 lines on a point of $\Gamma_2(Q_1)$ have a unique point in $\Gamma_1(Q_1)$. Thus there are exactly 144.13/2 = 946 lines meeting both $\Gamma_1(Q_1)$ and $\Gamma_2(Q_2)$. On the other hand, given a point $q \in Q_1$, we know that L_q has three 5-lines and, apart from these, 3-lines only, so that the 20 points of $\Gamma_1(Q_1)$ adjacent to q are on the average in $\frac{1}{10}(1*(4+4+4)+9*(4+2+2)) = 8\,^2\!/5$ lines entirely contained within $\Gamma_1(Q_1)$ and therefore on $13 - 1 - 8\,^2\!/5 = 3\,^3\!/5$ lines meeting $\Gamma_2(Q)$. Counting lines between $\Gamma_1(Q_1)$ and $\Gamma_2(Q_1)$ we find $300.3\ ^3/5 = 13.144/2$, a contradiction.

C. *If* Q_1 *is a 15-pt quad, then on the average each point of* $\Gamma_1(Q_1)$ *is in* $\bar{\alpha} = \frac{3}{5}t + \frac{6}{5} + \frac{24}{5(t-2)}$ *lines contained entirely within* $\Gamma_1(Q_1)$, *and* $t \leq 14$.
Let us formalise the counting argument of the preceding paragraph. Suppose Q_1 is a 15-pt quad. Then $|Q_1| = 15$, $|\Gamma_1(Q_1)| = 30(t-2)$, $|\Gamma_2(Q_1)| = 24(t-6)$. The number of lines meeting $\Gamma_2(Q_1)$ is $12(t-6)(t+1)$ so that on the average each point of $\Gamma_1(Q_1)$ is in $\bar{\alpha} := t - \frac{12(t-6)(t+1)}{30(t-2)} = \frac{3}{5}t + \frac{6}{5} + \frac{24}{5(t-2)}$ lines contained entirely within $\Gamma_1(Q_1)$.
This will turn out to be a very strong restriction: if forces the occurrence of relatively many big quads. As a first application, note that the lines on $q \in \Gamma_1(Q_1)$ contained entirely within $\Gamma_1(Q_1)$ are in the union of three quads, and each quad contributes 1 or 2 or 4 lines. Also, by B, these three quads cannot all be big, so we find $\frac{3}{5}t + \frac{6}{5} + \frac{24}{5(t-2)} \leq 4 + 4 + 2 = 10$ and it follows that (if a 15-pt quad occurs, then) $t \leq 14$.
In fact a solution with $t = 14$ exists and is unique - it is Aschbacher's near hexagon and will be discussed below.

D. *If* L_x *contains some 5-line but no 3-lines, then* $t = 5$ *or* 8.
If $t = 5$, *then the near hexagon is a direct product* $L \times Q$ *of a line and a big quad.*
Suppose that L_x does not contain 3-lines, but contains some 5-line. Then clearly L_x contains either one or two 5-lines and so $t + 1 \in \{6,9\}$, $v \in \{81,243\}$. If $t + 1 = 6$ then one verifies easily that each point of X is in a unique big quad, and our near hexagon is the product $L \times Q$ of a 3-line and a big quad.
Also the case $t+1 = 9$ will give a unique solution, see below. (In the sequel, we may assume that points x for which L_x contains a 3-line do exist, and in particular, that $t \geq 6$.)

E. $t \neq 13$.
If $t = 13$ then since the average $\bar{\alpha} = \frac{3}{5}.13 + \frac{6}{5} + \frac{24}{5.11} = 9 + \frac{24}{55} > 4 + 2 + 2$ it follows that for some x, L_x contains two 5-lines on some point u. The complement of the union of these two 5-lines consists of 5 points, and the remaining lines on u cover at most two edges on these 5 points. The remaining edges must be covered by 5-lines, but one immediately sees that this is impossible.

F. $t \neq 12$.
If $t = 12$ then $\bar{\alpha} > 8$, but now we find a contradiction as under B.

G. $t \neq 11$.
If $t = 11$ then $\bar{\alpha} = 8\frac{1}{3}$ so that for some x, L_x contains two 5-lines on some

point u. As before we conclude that there must be a third 5-line so that L_x is the union of three 5-lines. Now such a point $x \in Q_1$ contributes $\frac{1}{9}(3*(4+4+1)+6*(4+2+2)) = 8\frac{1}{3}$ to the average, while points x for which L_x contains at most one 5-line contribute at most $4+2+2 = 8$, so it follows that L_x is the union of three 5-lines for every point $x \in Q_1$ and (by D) for every point $x \in X$.

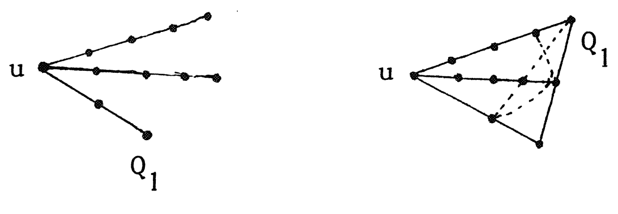
Thus we have the following situation:

$v = 405$; each L_x contains 3 5-lines (and 9 3-lines and 9 2-lines), so that there are $\frac{405.3}{27} = 45$ big quads altogether.

Each big quad meets $\frac{27.2}{3} = 18$ other big quads and hence is disjoint from 26 others. Now on the set of 45 big quads we find a Fischer space with 13 3-lines on each point, but one easily checks that such a Fischer space does not exist (either by first checking that its group must act primitively and concluding that we have a rank-3 graph with $v = 45$ and $k = 18$, but such a graph does not exist - or by some simple ad hoc arguments).

H. $t \neq 10$.

If $t = 10$ then $\bar{\alpha} = 7\frac{4}{5}$, but no L_x contains three 5-lines, and if some contains two, then the line joining the remaining two points must be a 3-line passing through the point of intersection u of the 5-lines. On the other hand, an L_x without 5-lines contributes at most $2+2+2 = 6$ to $\bar{\alpha}$ and an L_x with exactly one 5-line (is isomorphic to the unique $B(\{3,5^*\};11)$ design - the 5-edge coloring of K_6) contributes exactly $\frac{1}{8}(4*(4+2+2)+4*(2+2+2)) = 7$ to $\bar{\alpha}$. Thus, the high value of $\bar{\alpha}$ must be caused by the L_x with two 5-lines; there are two possibilities, depending on the position of the quad Q_1 within L_x:



In the first case (the line Q_1 of L_x passes through u) we have $\bar{\alpha} = \frac{1}{8}(8*(4+2+2)) = 8$, in the second case $\bar{\alpha} = \frac{1}{8}((4+4+2)+(2+2+2)+2*(4+2+2)+4*(4+2+1)) = 7\frac{1}{2}$. Now L_x contains nine 3-lines, so averaging over all possible Q_1 we find that L_x contributes $\frac{1}{9}(8*7\frac{1}{2}+8) = 7\frac{5}{9} < 7\frac{4}{5}$ to the average value of

$\bar{\alpha}$ (watch this!), a contradiction.

I. $t \neq 9$.

If $t = 9$ then $\bar{\alpha} = 7\frac{2}{7}$. A local linear space L_x with at most one 5-line contributes less than 7 to $\bar{\alpha}$, while the uniquely determined L_x with two 5-lines contributes exactly $\frac{1}{7}((4+4+1)+6*(4+2+1)) = 7\frac{2}{7}$, so all local spaces have two 5-lines (or 2-lines only, but if L_x has two 5-lines and L_y has 2-lines only, then $x \not\sim y$ and by connectivity we see that the latter case does not occur). We have $v = 54.9 - 189 = 297$ and the total number of 15-point quads is $\frac{297.4}{15}$ which is not an integer. Contradiction.

J. *If* $t = 8$ *then there are no* 15-*point quads in the near hexagon.*

If $t = 8$ then $\bar{\alpha} = 6\frac{4}{5}$. But any L_x without 5-lines contributes at most 6, and any L_x with one 5-line contributes $\frac{1}{6}(4*4+10*2+4*1) = 6\frac{2}{3} < 6\frac{4}{5}$, a contradiction.

This shows that no local space L_x contains a 3-line. Below we shall see that there is a unique near hexagon with $t+1 = 9$, $v = 243$.

K. $t \neq 7$.

If $t = 7$ then $\bar{\alpha} = 6\frac{9}{25}$, but the unique L_x (with a 5-line) contributes $\frac{1}{5}(4*4+7*1+4*2) = 6\frac{1}{5}$, contradiction.

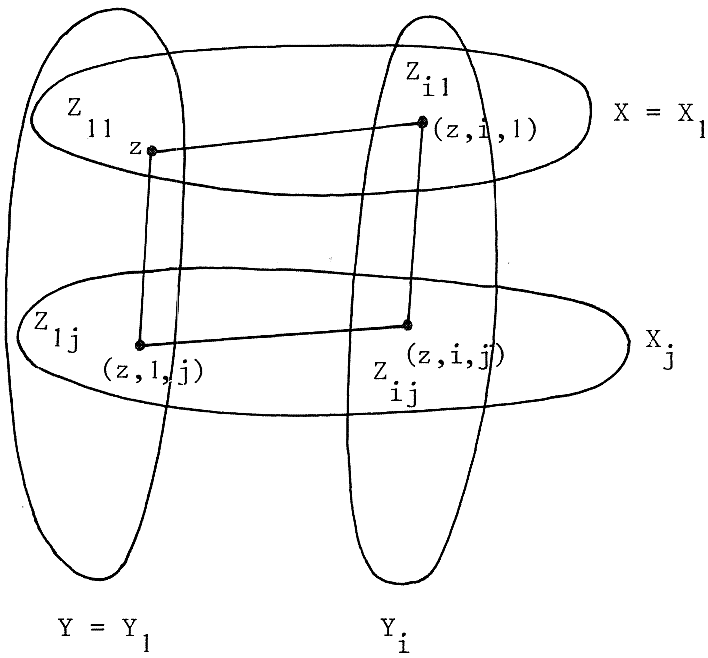
L. $t \neq 6$.

If $t = 6$ then ($\Gamma_2(Q_1)=\emptyset$ so) $\bar{\alpha} = 6$ and the unique L_x with a 5-line (the union of a 5-line and a 3-line) contributes $\frac{1}{4}(4*(4+1+1)) = 6$, and the only L_y without a 5-line contributing 6 or more is the Fano plane which contributes exactly 6. Thus, all local spaces are of one of these two types, or contain 2-lines only. However, two adjacent points have the same type of local space, and if always the second or third type occurs we have a regular near polygon. Thus, we always have the first type so each point lies in a unique 15-pt quad and in a unique 27-pt quad, a kind of direct product construction, and we shall see in the next section that this is impossible. (This can also be seen directly: $v=135$, so there are exactly five 27-pt quads, partitioning X. They yield a Fischer space on 5 points with lines of size 3 only - ridiculous.)

4. A PRODUCT CONSTRUCTION

We still have to discuss the cases $t+1 = 9$ and $t+1 = 15$. In this section we treat the first case.
Thus, we have the situation $t+1 = 9$, $v = 243$, each point in two big quads, 18 big quads altogether, falling into two families of 9, each partitioning X. Each quad from one family meets each quad from the other family in a line. This leads to a kind of amalgamated product.

The general construction is as follows:
Let X and Y be near polygons with partitions $\{Z_{i1} \mid 1 \le i \le a\}$ and $\{Z_{1j} \mid 1 \le j \le b\}$ into big subspaces, where $X \cap Y = Z_{11}$.
Write $I_c = \{i \in \mathbb{N} \mid 1 \le i \le c\}$.
Define a partial linear space Z (sometimes denoted by $X \times_{Z_{11}} Y$) as follows: let the point set be $Z_{11} \times I_a \times I_b$; define projections $\pi_X : Z \to X$ and $\pi_Y: Z \to Y$ by $z \sim \pi_X(z,i,j) \in Z_{i1}$ $(i \ne 1)$ and $z = \pi_X(z,1,j)$, and similarly for π_Y; let the lines be those subsets $L \subset Z$ such that both $\pi_X(L)$ and $\pi_Y(L)$ are either a singleton or a line, and either $\pi_X(L)$ is contained in some Z_{i1} or $\pi_Y(L)$ is contained in some Z_{1j} or both, and where $|L| > 1$.



Write $X_j = Z_{11} \times I_a \times \{j\}$ and $Y_i = Z_{11} \times \{i\} \times I_b$. Then we can identify X with X_1 and Y with Y_1 and thus regard X and Y as subspaces of Z. (This uses Lemmas 1 and 2 for X,Y.) Now note that π_X and π_Y commute.

The X_j and Y_i are subspaces of Z and $\pi_X|_{X_j}$ and $\pi_Y|_{Y_i}$ are isomorphisms.

(For: suppose L is a line with two points in X_j but also meeting X_k, $k \ne j$.

Then $\pi_Y(L)$ meets both Z_{1j} and Z_{1k}, so by definition of line in Z we have $\pi_X(L) \subset Z_{i1}$ for some i. But Z_{1j} is a subspace, so $\pi_Y(L)$ meets Z_{1j} in one point only, and all points in $L \cap X_j$ have the same z,i,j coordinates, i.e., coincide. Thus X_j is a subspace. If L is a line in X_j and $\pi_X(L)$ is a singleton then all points of L have the same coordinates, i.e., $|L| = 1$, impossible. Conversely, if L' is a line in X then we can find a line $L \subset X_j$ with $\pi_X(L) = L'$: if $L' \subset Z_{11}$ this is possible since $\pi_X|_{Z_{1j}} : Z_{1j} \to Z_{11}$ is an isomorphism, if L' meets Z_{11} in a single point z then $L = \{(z,i,j) \mid L' \cap Z_{i1} \neq \emptyset\}$ satisfies the requirements, finally if $L' \cap Z_{11} = \emptyset$ then let $M' = \pi_Y(L')$, a line in Z_{11}, and let $M \subset Z_{1j}$ be the line with $\pi_X(M) = M'$. Then $L = \{(z,i,j) \mid \pi_X(z,i,j) \in L'\}$ is a line since $\pi_Y(L) = M$. Thus $\pi_X|_{X_j} : X_j \to X$ is an isomorphism.)

Write $Z_{ij} = Y_i \cap X_j$.

<u>PROPOSITION 1</u>. *Let* $(Z,\mathcal{L})$ *be a near polygon with two partitions* $\{Y_i \mid 1 \le i \le a\}$ *and* $\{X_j \mid 1 \le j \le b\}$ *into big subspaces such that* $Z_{ij} := Y_i \cap X_j \neq \emptyset$ *for all* i,j. *Then* $Z \cong X_1 \times_{Z_{11}} Y_1$ *as defined above.*

<u>PROOF</u>. First note that Z_{ij} is a big subspace of both Y_i and X_j (by Lemma 3). Define $X = X_1$, $Y = Y_1$ and π_X, π_Y in the obvious way. Then π_X and π_Y commute (by Lemma 1) so that we can label the points (z,i,j) as above. Clearly (by Lemma 3) all lines of the near polygon are lines of the partial linear space, and the converse holds since we know in both structures that $\pi_X|_{X_j} : X_j \to X$ is an isomorphism. □

<u>PROPOSITION 2</u>. *Let* $(Z,\mathcal{L})$ *be a near polygon as above and assume that for each line* $M \subset Z_{11}$ *there is a line* $\bar{M} \subset Y$ *such that* $\bar{M}$ *is not contained in any* Z_{1j} *and* $\pi_X(\bar{M}) = M$. *Define for lines* L *in* Z, $I_L := \{i \mid L \cap Y_i \neq \emptyset\}$. *If for two lines* L, L' *in* X *we have* $|I_L \cap I_{L'}| \ge 2$, *then* $I_L = I_{L'}$.

<u>PROOF</u>. If L and L' meet, they coincide and the conclusion is true. Since Z_{i1} is connected we may assume that there is a line $M \subset Z_{i1}$ meeting both L and L'. Since Y_i is isomorphic to Y we can find a line $\bar{M}$ not contained in any Z_{ij} such that $\pi_X(\bar{M}) = M$. Let a and a' be the points of $\bar{M}$ projecting to the points $L \cap M$ and $L' \cap M$ of M. Now $a \in X_j$ and $a' \in X_{j'}$, say. Let $\bar{L}$ be the line on a in X_j such that $\pi_X(\bar{L}) = L$. Let $\bar{\bar{L}}$ be the projection of $\bar{L}$ into

$X_{j'}$. Let $L'' = \pi_X(\bar{\bar{L}})$. Then $I_L = I_{\bar{L}} = I_{\bar{\bar{L}}} = I_{L''}$ and L" meets L', so L" = L'. □

This proposition shows that our product construction cannot yield a near polygon in all cases - there are special restrictions on the factors. Let us give a simple sufficient condition for Z to be a near polygon.

PROPOSITION 3. *Suppose that all projections* $\pi_{jk}: X_j \to X_k$ *and* $\pi'_{ik}: Y_i \to Y_k$ *are isomorphisms. Then* Z *is a near polygon and the* Y_i, X_j *are big subspaces of* Z.

PROOF. (Note that π_{jk} is well defined since $\pi_{jk}|_{Z_{ij}} : Z_{ij} \to Z_{ik}$ is well defined. These restrictions clearly are isomorphisms.)
Let L be a line in X_j and a_0 a point of X_k, $k \neq j$. If $a_0a_1 \ldots a_r b_1 \ldots b_s$ is a shortest path from a_0 to some point $b_s \in L$, where $a_i \in X_k$ $(0 \le i \le r)$ and $b_i \in X_j$ $(1 \le i \le s)$ then so is $a_0a_1 \ldots a_{r-1}b_0b_1 \ldots b_s$, where $b_0 = \pi_{kj}(a_{r-1})$. Thus, we may assume that a shortest path from a_0 to a point of L starts with $a_0\ \pi_{kj}(a_0)$, but X_j is a near polygon (and geodetically closed, again since the π_{jk} are isomorphisms) so there is a unique point on L closest to a_0. □

A converse of this proposition holds: when Z is a near polygon satisfying the hypotheses of Proposition 1, then clearly the π_{jk} are isomorphisms.

THEOREM. *There is a unique hexagon with quads, with lines of size* 3 *and at least one big quad and* $t+1 = 9$. *It has* $v = 243$ *and is a product* $Q \times_L Q'$ *of two big quads meeting in a line. The corresponding Fischer space has* 18 *points and is the disjoint union of two affine planes* AG(2,3).

PROOF. We already saw that any such near hexagon necessarily has the form $Q \times_L Q'$ where Q and Q' are 27-point quads and L is a 3-line. Now since Q' is not the direct product of two lines, the hypothesis of Proposition 2 is satisfied and it follows that the Z_{i1} form a spread of lines in Q with the property that if three lines of the spread have a transversal, then they have three transversals. For the dual generalized quadrangle GQ(4,2) (i.e., $U(4,2^2)$) this means that we have an ovoid with the property that $|x^\perp \cap y^\perp \cap z^\perp| \in \{0,3\}$ for any three points x,y,z of the ovoid.
Now Brouwer & Wilbrink [3] show that there are precisely two isomorphism classes of ovoids, those lying on a nontangent plane ("plane ovoids") and

those lying on three concurrent lines ("tripods"). It is a simple exercise to show that only the plane ovoids satisfy our restriction.
This means that the spreads in Q and Q' in terms of which our product is defined can be chosen in a unique way up to isomorphism. Remains to show that if we choose the spreads in this way, we actually do get a near hexagon. To this end, let $L = \{a,b,c\}$ be a line in X_j and $a' = \pi_{jk}(a)$, $b' = \pi_{jk}(b)$. We want to show that $a' \sim b'$. (By Proposition 3 this suffices.)
If $L \subset Y_i$ for some i or if $j = 1$ or $k = 1$ then this is clear. Let $a \in Z_{hj}$ and $b \in Z_{ij}$. Suppose $L \cap Y_1 \neq \emptyset$. If a or b is in $L \cap Y_1$ then $a' \sim b'$ by definition of the lines aa' and bb'. Thus we may assume that $c \in Y_1$. Then $\pi_Y(aa') = \pi_Y(bb') = cc'$ where $c' = \pi_{jk}(c)$. If the third point c" of the line cc' lies in Z_{11} then also the third points a" and b" of the lines aa' and bb' lie in X and clearly $\pi_X(L) = \{a'',b'',c''\}$. Now the line $\pi_{1k}(\{a'',b'',c''\})$ contains a',b',c', so that $a' \sim b'$. If $c'' \notin Z_{11}$ then the points $z = \pi_X(c)$ and $z' = \pi_X(c')$ are distinct. By our hypothesis on the spread we have $\{1,h,i\} = I_L = I_{a'c'}$, so that the third point of the line a'c' lies in Z_{ik}. But since $c' \sim b' \in Z_{ik}$, this third point must be b', as was to be proved. Now suppose $L \cap Y_1 = \emptyset$. If the line aa' meets X_1 then by what we already proved $\pi'_{hi}(aa') = bb'$ and $a' \sim b'$. Thus we may assume that the lines aa' and bb' do not meet X_1. Now our argument uses the hypothesis on the spread and the uniqueness of the 3×3 Latin square: if $Z_{11} = \{A,B,C\}$ then each point of Z has as first coordinate one of A,B and C, and if M is a line not meeting $X_1 \cup Y_1$ then the first coordinates of the three points of M are (a permutation of) A,B,C. If we assume that the first coordinates of a, b and c are A, B and C and that $a' = (B,h,k)$ then both the neighbour of a' in Z_{ik} and b' have coordinates (c,i,k) so that $a' \sim b'$. (For: the projections of the three lines aa'a", bb'b" and cc'c" on Y are three mutually disjoint lines exhausting $Z_{1j} \cup Z_{1k} \cup Z_{1\ell}$, if $c \in Z_{\ell j}$, etc.) This ends the proof, and also shows that we cannot expect a general theorem in the same vein. □

5. THE ASCHBACHER NEAR HEXAGON

Remains to discuss the case $t+1 = 15$, $v = 567$. From the fact that equality holds in the inequality $t \leq 14$ derived in section 3C, we see that each point outside a given 15-pt quad Q_1 lies in 2 big quads and in one 15-pt quad meeting Q_1. It follows that all local spaces are isomorphic and have the following structure: $|L_x| = 15$, there are two 5-lines on each point, 6 5-lines altogether; there are three 3-lines on each point, 15 3-lines altogether; given a point u outside a 3-line L there is a unique 3-line on u meeting L, i.e., the 3-lines are the lines of a generalized quadrangle GQ(2,2) and the 5-lines are the ovoids in that generalized quadrangle.

Thus, our near hexagon has 6.567/27 = 126 big quads. The corresponding Fischer space (E,L) has 126 points, 45 2-lines on each point and 40 3-lines on each point. Since any component has size at least 81 it follows that E is connected.

Given any two points $a,b \in E$, it never happens that $E_a = E_b$ (where E_x is the set of all points joined to x by a 2-line) and by Beukenhout [4, Prop. 13 and 12] it follows that any block of imprimitivity B of E contains 2-lines only and has size a power of two. Since $4 \nmid 126$, if B is not a singleton then $|B| = 2$. But now the quotient Fischer space would be primitive, with 63 points, 20 3-lines on each point and 22 2-lines on each point, yielding a strongly regular graph with parameters $(v,k,\lambda) = (63,22,?)$ but such a graph does not exist. (The only parameter set satisfying the divisibility conditions has $\mu = 11$, $\lambda = 1$, but this graph dies, e.g. because of the absolute bound.)

This shows that we have a primitive Fischer space, and by Fischer [7] the corresponding Fischer group is one of Sym(n), Sp(2n,2), $O^{\pm}(2n,2)$, $PSU(n,2^2)$, $O^{\pm,\pm}(n,3)$, Fi_j (j=22,23,24). But 126 is not of the form $\binom{n}{2}$ or $2^{2n}-1$ or $2^{2n-1} \pm 2^{n-1}-1$ or $\frac{1}{3}(2^{n+1}-(-1)^{n+1})(2^n-(-1)^n)$ and less than 3510 so that the Fischer group necessarily is $O_6^-(3)$.

Consequently, we may identify the big quads with the points x in PG(5,3) for which $Q(x) = 1$, for a fixed quadratic form Q with Witt index 2; the 3-lines correspond to tangent lines, the 2-lines to hyperbolic lines.

Each point of the near hexagon lies in 6 big quads (and these big quads meet each other so are joined by 2-lines), so points of the near hexagon correspond to orthonormal bases in the geometry (note that the elliptic lines have two points with $Q(x) = 1$ and two with $Q(x) = -1$; the first pair and the second pair are both orthogonal), but since there are exactly 567 orthonormal bases, these are all the points.

This proves:

<u>THEOREM</u>. *There exists a unique near hexagon with* s+1 = 3, t+1 = 15 *with quads, having at least one 27-point quad. It has* v=567 *and arises from the* $O_6^-(3)$ *geometry as described before. It has no 9-point quads, and in each point the local structure is a linear space* B[{3,5};15], *the 3-lines forming the* GQ(2,2) *generalized quadrangle with the 5-lines as ovoids.* □

6. CONCLUSION

Summing up, we have:

<u>THEOREM</u>. *Let* (X,*L*) *be a near hexagon with lines of size* 3 *having quads (i.e., such that any two points at distance two have at least two common neighbours), and such that at least one 27-point quad occurs. Then we have one of the following cases, and each is unique:*

	v	t+1	t_2+1	group
(i)	891	21	5	$P\Gamma U(6,2^2)$.
(ii)	81	6	2,5	$S_3 \times P\Gamma U(4,2^2)$.
(iii)	243	9	2,5	$(\mathbb{Z}_3 \times AGL(2,3) \times AGL(2,3)).2$
(iv)	567	15	3,5	$PO_6^-(3)$. □

REFERENCES

[1] ASCHBACHER, M., *Flag structures on Tits Geometries*, preprint, 1982.

[2] BROUWER, A.E. & H.A. WILBRINK, *The structure of near polygons with quads*, report ZW 177/82, Math. Centr., Amsterdam; to appear, Geom. Dedic.

[3] BROUWER, A.E. & H.A. WILBRINK, *Ovoids and fans in the generalized quadrangle* GQ(4,2), report ZN 102/81, Math. Centr., Amsterdam.

[4] BEUKENHOUT, F., *La gêometrie des groupes de Fischer*, preprint, Univ. Libre de Bruxelles.

[5] CAMERON, P.J., *Dual polar spaces*, Geometriae Dedicata 12 (1982) 75-85.

[6] COHEN, A.M., *Geometries originating from certain distance-regular graphs*, in: Cameron, Hirschfeld and Hughes (eds.), *Finite Geometries and Designs*, LMS Lecture Note 49, London, 1981, pp. 81-87.

[7] FISCHER, B., *Finite groups generated by 3-transpositions* I, Inventiones Math. 13 (1971) 232-246.

[8] KANTOR, W.M., *Some geometries that are almost buildings*, Europ. J. Combinatorics 2 (1981) 239-247.

[9] SHULT, E.E. & A. YANUSHKA, *Near n-gons and line systems*, Geometriae Dedicata 9 (1980) 1-72.

MC, 830218