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ON ORTHOGONAL POLYNOMIALS ON A HALF LINE AND THE ASSOCIATED KERNEL POLYNOMIALS

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On orthogonal polynomials on a half line and the associated kernel polynomials *)

by

E.A. van Doorn

ABSTRACT

We consider a sequence of polynomials $\mathcal{P}$ which is orthogonal with respect to a distribution whose support is contained in $[0,\infty)$. Our main concern is the derivation of limit theorems for $\mathcal{P}$. In the course of this we establish some links between the sets of limit points of zeros of $\mathcal{P}$ and $\mathcal{P}^*$, the set of kernel polynomials associated with $\mathcal{P}$, and the parameters in the three term recurrence relation for $\mathcal{P}$.

KEY WORDS & PHRASES: *orthogonal polynomials, kernel polynomials, Hamburger moment problem*

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*) This report has been submitted for publication elsewhere.

I Introduction

Let $\mathcal{P} = \{P_n(x)\}_{n=0}^{\infty}$ be a sequence of monic polynomials satisfying the recurrence relation

$$\begin{aligned} P_n(x) &= (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \qquad (n > 1) \\ P_0(x) &= 1 \ , \quad P_1(x) = x - c_1 \ , \end{aligned} \tag{1.1}$$

where c_n is real and $\lambda_{n+1} > 0$ $(n > 0)$. A recurrence of this type is a necessary and sufficient condition for $\mathcal{P}$ to constitute an orthogonal sequence. That is, there is a mass distribution $d\psi$ on the real line (with total mass 1 and infinite support) such that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\psi(x) = \delta_{nm} \prod_{j=1}^{n} \lambda_{j+1} \ , \tag{1.2}$$

where the empty product is interpreted as unity.

$P_n(x)$ has n real, simple zeros $x_{n1}(\mathcal{P}) < x_{n2}(\mathcal{P}) < \ldots < x_{nn}(\mathcal{P})$. Moreover, the zeros of $P_n(x)$ and $P_{n+1}(x)$ separate each other, that is,

$$x_{n+1,i}(\mathcal{P}) < x_{ni}(\mathcal{P}) < x_{n+1,i+1}(\mathcal{P}) \qquad (i = 1,2,\ldots,n), \tag{1.3}$$

so that the limits

$$\xi_i(\mathcal{P}) = \lim_{n\to\infty} x_{ni}(\mathcal{P}) \qquad (i \geq 1)$$

exist, and

$$-\infty \leqq \xi_1(\mathcal{P}) \leqq \xi_2(\mathcal{P}) \leqq \cdots .$$

Throughout this paper we will assume that the zeros of $P_n(x)$ are positive, that is, we assume $\xi_1(\mathcal{P}) \geqq 0$.

We let

$$\sigma(\mathcal{P}) = \lim_{i\to\infty} \xi_i(\mathcal{P}) ,$$

and note that for $i \geqq 1$

$$\xi_i(\mathcal{P}) = \xi_{i+1}(\mathcal{P}) \implies \xi_i(\mathcal{P}) = \sigma(\mathcal{P}) . \tag{1.4}$$

The distribution $d\psi$ of (1.2) is uniquely determined by $\{c_n, \lambda_{n+1}\}$ if and only if the Hamburger moment problem (Hmp) associated with $\mathcal{P}$ is determined. In this case we have

$$\operatorname{supp}(d\psi) \cap (-\infty, \sigma(\mathcal{P})] = \overline{\Xi(\mathcal{P})} \quad \text{if} \quad \sigma(\mathcal{P}) < \infty \tag{1.5}$$

and

$$\operatorname{supp}(d\psi) = \Xi(\mathcal{P}) \qquad \text{if} \quad \sigma(\mathcal{P}) = \infty , \tag{1.6}$$

where $\Xi(\mathcal{P}) = \{\xi_1(\mathcal{P}), \xi_2(\mathcal{P}), \ldots\}$ and a bar denotes closure. If the Hmp for $\mathcal{P}$ is indeterminate, then $\sigma(\mathcal{P}) = \infty$ and there is exactly one distribution $d\psi$ satisfying (1.2) and (1.6); any other distribution satisfying (1.2) has at least one supporting point smaller than $\xi_1(\mathcal{P})$. (See Chihara [1-4] for the above results).

By $\mathcal{P}^* = \{P_n^*(x)\}_{n=0}^{\infty}$ we denote the sequence of <u>kernel polynomials</u>

(with parameter 0) associated with $\mathcal{P}$, that is,

$$xP_n^*(x) = P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)} P_n(x) \; . \tag{1.7}$$

$\mathcal{P}^*$ constitutes a sequence of monic, orthogonal polynomials [4, Theorem I.7.1], and therefore there exist real numbers c_n^* and positive numbers λ_{n+1}^* $(n > 0)$ determining a recurrence of the type (1.1) for $\mathcal{P}^*$.

In what follows we write

$$x_{ni} = x_{ni}(\mathcal{P}) \quad \text{and} \quad x_{ni}^* = x_{ni}(\mathcal{P}^*) .$$

The numbers ξ_i, ξ_i^*, σ and σ^* and the sets Ξ and Ξ^* are defined similarly. As a preliminary result we note that as a consequence of the separation theorem

$$x_{ni} < x_{ni}^* < x_{n+1,i+1} \qquad (i = 1,2,\ldots,n) \tag{1.8}$$

[4, Theorem I.7.2] we have

$$\xi_i \leqq \xi_i^* \leqq \xi_{i+1} \qquad (i \geqq 1) \; . \tag{1.9}$$

After these introductory remarks we are prepared to give the plan of the paper. In Section II we will establish some results relating Ξ to the parameters $\{c_n, \lambda_{n+1}\}$. The kernel polynomials (1.7) will be studied in Section III, our main result being the precise conditions for which an equality holds in (1.9). Using our findings of Sections II and III we will prove some limit theorems for $\mathcal{P}$ in Section IV.

II Orthogonal polynomials on a half line

The assumption $\xi_1 \geq 0$ makes that we can invoke a result of Chihara [4, Theorem I.9.1] stating that $\xi_1 \geq 0$ is a necessary and sufficient condition for the existence of a unique sequence $\{\gamma_n\}_{n=2}^{\infty}$ of positive numbers such that

$$c_n = \gamma_{2n-1} + \gamma_{2n}, \quad \lambda_{n+1} = \gamma_{2n}\gamma_{2n+1} \quad (n > 0), \tag{2.1}$$

where $\gamma_1 = 0$. Clearly, $\{\gamma_n\}$ can be determined recursively from $\{c_n, \lambda_{n+1}\}$.

It will be convenient to introduce the quantities

$$G_n = \prod_{i=1}^{n} \gamma_{2i}/\gamma_{2i+1}, \quad H_n = \prod_{i=1}^{n} \gamma_{2i+1}/\gamma_{2i+2} \quad (n \geq 0) \tag{2.2}$$

(where we use Chihara's [5] notation), maintaining the convention that the empty product denotes unity. Also, following [3], we will use the convention that $\sum' a_i^{-1}$ denotes the series obtained after deleting from $\{a_i\}$ any terms that are equal to zero $(\sum' a_i^{-1} = \infty$ if $a_i = 0$ for all i).

<u>Theorem 1</u>. The following statements are equivalent:

(i) $\xi_1 > 0$ and $\sum' \xi_i^{-1} < \infty$,

(ii) $\sum_{n=0}^{\infty} H_n \sum_{i=0}^{n} G_i < \infty$,

(iii) $\{P_n(x)/P_n(0)\}_n$ converges uniformly on bounded sets to an entire function whose zeros are simple and are precisely the points ξ_i $(i \geq 1)$,

(iv) $\{P_n(x)/P_n(0)\}_n$ is bounded as $n \to \infty$ for at least one $x < 0$.

The equivalence of (i) and (iii) above was proven by Chihara [3, Theorem 2]. For a proof of the equivalence of (ii), (iii) and (iv) (essentially due to Stieltjes) we refer to Karlin and McGregor [8, Lemma 4]. The results of the latter paper are stated in the context of birth-death processes and can be translated to our present notation in the manner indicated by Chihara [5, pp. 335-336]. Karlin and McGregor do not mention the fact that Ξ is the set of zeros of the entire function in (iii), but the proof of Chihara's result shows that this is actually a consequence of the uniform convergence.

Before we can say something about the case $\xi_1 = 0$ and $\sum' \xi_i^{-1} < \infty$, we must mention a result noted in [9, p. 229] (without proof) and [7, Theorem A.3] in the context of birth-death processes. In the present notation it may be stated as follows.

<u>Theorem 2</u>. The series $\sum' \xi_i^{-1}$ converges if and only if

$$\sum_{n=0}^{\infty} G_{n+1} \sum_{i=0}^{n} H_i < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} H_n \sum_{i=0}^{n} G_i < \infty . \tag{2.3}$$

Theorems 1 and 2 are easily seen to imply the following.

<u>Corollary 2.1</u>. One has $\xi_1 = 0$ and $\sum' \xi_i^{-1} < \infty$ if and only if

$$\sum_{n=0}^{\infty} G_{n+1} \sum_{i=0}^{n} H_i < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} H_n \sum_{i=0}^{n} G_i = \infty .$$

Another consequence of Theorem 2, for which we will have later use, is the following.

Corollary 2.2. If $\sum' \xi_i^{-1} = \infty$, then the Hmp for $\mathcal{P}$ is determined.

Proof. If $\sum' \xi_i^{-1} = \infty$, then, by Theorem 2,

$$\sum_{n=0}^{\infty} G_{n+1} \sum_{i=0}^{n} H_i = \infty ,$$

whence

$$\sum_{n=0}^{\infty} G_{n+1} \left(\sum_{i=0}^{n} H_i \right)^2 = \infty .$$

But the latter is a necessary and sufficient condition for the Hmp associated with $\mathcal{P}$ to be determined [5, Theorem 3]. □

III Kernel polynomials

By [4, Theorem I.9.1] the parameters c_n^* and λ_{n+1}^* $(n > 0)$ in the recurrence for P^*, the set of kernel polynomials associated with P, satisfy

$$c_n^* = \gamma_{2n} + \gamma_{2n+1}\,, \quad \lambda_{n+1}^* = \gamma_{2n+1}\gamma_{2n+2} \quad (n > 0), \tag{3.1}$$

where $\{\gamma_n\}_{n=2}^{\infty}$ is the sequence of positive numbers which is uniquely determined by $\{c_n,\lambda_{n+1}\}$ through (2.1). Note that c_1^* is written in (3.1) as the sum of two positive numbers, which makes the representation (3.1) essentially different from (2.1). However, in view of (1.9) our assumption $\xi_1 \geqq 0$ implies that $\xi_1^* \geqq 0$, so that results analogous to those of the previous section are valid for P^*. It can easily be verified that the appropriate quantities γ_n^*, G_n^* and H_n^* satisfy

$$\left.\begin{aligned} \gamma_{2n}^* &= \gamma_{2n+1} \sum_{j=0}^{n} G_j \Big/ \sum_{j=0}^{n-1} G_j \\ \gamma_{2n+1}^* &= \gamma_{2n+2} \sum_{j=0}^{n-1} G_j \Big/ \sum_{j=0}^{n} G_j \end{aligned}\right\} \quad (n > 0) \tag{3.2}$$

and

$$\left.\begin{aligned} G_n^* &= H_n\left(\sum_{j=0}^{n} G_j\right)^2 \\ H_n^* &= G_1^{-1}(1+G_1)G_{n+1}\left(\sum_{j=0}^{n} G_j\right)^{-1}\left(\sum_{j=0}^{n+1} G_j\right)^{-1} \end{aligned}\right\} \quad (n \geqq 0), \tag{3.3}$$

so that these results for P^* may be formulated in terms of G_n and H_n.

But we can say more about $\mathcal{P}^*$. First, note that by (1.9)

$$\sum{}' \xi_i^{-1} < \infty \iff \sum{}' (\xi_i^*)^{-1} < \infty , \tag{3.4}$$

so that (2.3) is a necessary and sufficient condition for $\sum{}'(\xi_i^*)^{-1}$ to converge. Then, by interpreting Lemma 4 of Karlin and McGregor [8] in terms of $\mathcal{P}^*$, the next result emerges.

Theorem 3. The following statements are equivalent:

(i) $\displaystyle\sum_{n=0}^{\infty} G_{n+1} \sum_{i=0}^{n} H_n < \infty ,$

(ii) $\{(\sum_{j=0}^{n} G_j)P_n^*(x)/P_n^*(0)\}_n$ converges uniformly on bounded sets to an entire function whose zeros are simple and precisely the points ξ_i^* $(i \geq 1)$,

(iii) $\{(\sum_{j=0}^{n} G_j)P_n^*(x)/P_n^*(0)\}_n$ is bounded as $n \to \infty$ for at least one $x < 0$.

Again, Karlin and McGregor do not mention the fact that Ξ^* is the set of zeros of the entire function in (ii), but, as in Theorem 1, this extension is a direct consequence of the uniform convergence.

Some relations between the polynomials P_n and P_n^* $(n \geq 0)$ will now be derived for future use. To begin with it is easy to see that

$$P_n(0) = (-1)^n \prod_{i=1}^{n} \gamma_{2i} \tag{3.5}$$

and

$$P_n^*(0) = (-1)^n \prod_{i=1}^{n} \gamma_{2i+1} \sum_{j=0}^{n} G_j \ . \qquad (3.6)$$

From (1.7) and (3.5) we then obtain

$$xP_n^*(x) = P_{n+1}(x) + \gamma_{2n+2}P_n(x) \ . \qquad (3.7)$$

Combining this result with (2.1) and (1.1) gives us

$$P_n(x) = P_n^*(x) + \gamma_{2n+1}P_{n-1}^*(x) \ . \qquad (3.8)$$

The orthonormalized polynomials corresponding to P_n and P_n^* will be denoted by p_n and p_n^*, respectively. From (1.2), (2.1) and (3.1) we get

$$p_n(x) = \left(\prod_{i=2}^{2n+1} \gamma_i\right)^{-\frac{1}{2}} P_n(x) \qquad (3.9)$$

and

$$p_n^*(x) = \left(\prod_{i=3}^{2n+2} \gamma_i\right)^{-\frac{1}{2}} P_n^*(x) \ . \qquad (3.10)$$

Hence, by (3.7) and (3.8),

$$xp_n^*(x) = (\gamma_2\gamma_{2n+2})^{\frac{1}{2}}p_n(x) + (\gamma_2\gamma_{2n+3})^{\frac{1}{2}}p_{n+1}(x) \qquad (3.11)$$

and

$$p_n(x) = (\gamma_{2n+1}/\gamma_2)^{\frac{1}{2}}p_{n-1}^*(x) + (\gamma_{2n+2}/\gamma_2)^{\frac{1}{2}}p_n^*(x) \ . \qquad (3.12)$$

Using (3.11) and (3.12) we can write

$$\sum_{j=0}^{n} p_j^2(x) = \sum_{j=0}^{n} \{(\gamma_{2j+1}/\gamma_2)^{\frac{1}{2}} p_{j-1}^*(x) + (\gamma_{2j+2}/\gamma_2)^{\frac{1}{2}} p_j^*(x)\} p_j(x) =$$

$$\sum_{j=0}^{n} (\gamma_{2j+1}/\gamma_2)^{\frac{1}{2}} p_{j-1}^*(x) p_j(x) +$$

$$+ \sum_{j=0}^{n} (\gamma_{2j+2}/\gamma_2)^{\frac{1}{2}} p_j^*(x) \{x(\gamma_2\gamma_{2j+2})^{-\frac{1}{2}} p_j^*(x) - (\gamma_{2j+2}/\gamma_{2j+3})^{-\frac{1}{2}} p_{j+1}(x)\} =$$

$$\gamma_2^{-1} x \sum_{j=0}^{n} (p_j^*(x))^2 - (\gamma_{2n+3}/\gamma_2)^{\frac{1}{2}} p_{n+1}(x) p_n^*(x) .$$

With (3.5), (3.6), (3.9) and (3.10) this result can be formulated as follows.

<u>Lemma 1</u>. For $n \geqq 0$ one has

$$\sum_{j=0}^{n} p_j^2(x) = \gamma_2^{-1} x \sum_{j=0}^{n} (p_j^*(x))^2 + \frac{P_{n+1}(x)}{P_{n+1}(0)} \frac{P_n^*(x)}{P_n^*(0)} \sum_{j=0}^{n} G_j .$$

The relevance of this lemma resides in the fact that we will make use of the following well-known result from the theory of moments [10, Corollary 2.6 and Theorem 2.13].

<u>Theorem 4</u>. Let $R = \{r_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials which is <u>orthonormal</u> with respect to a distribution $d\phi$ (with infinite support). Then

$$d\phi(x) \leqq \left(\sum_{n=0}^{\infty} r_n^2(x)\right)^{-1} \qquad (x \text{ real}) \qquad (3.13)$$

Moreover, if the Hmp for $\mathcal{R}$ is determined (so that $d\phi$ is unique), then equality holds in (3.13); if the Hmp for $\mathcal{R}$ is indeterminate, then the right hand side of (3.13) is positive and for every x there is exactly one distribution $d\phi$ for which equality holds in (3.13).

We are now in a position to state the main result of this section.

Theorem 5. (i) If $\xi_1 = 0$, then $\xi_i^* = \xi_{i+1}$ $(i \geqq 1)$.
(ii) If $\xi_1 > 0$ and the Hmp for $\mathcal{P}$ is determined, then $\xi_i^* = \xi_i$ $(i \geqq 1)$.
(iii) If $\xi_1 > 0$ and the Hmp for $\mathcal{P}$ is indeterminate, then $\xi_i < \xi_i^* < \xi_{i+1}$ $(i \geqq 1)$.

Proof. Part (i) has been established in [6, Lemma A.1].

As for (ii) we denote by $d\psi$ the unique distribution with respect to which the P_n are orthogonal. Then $\mathcal{P}^*$ is orthogonal with respect to $d\phi(x) = \gamma_2^{-1} x d\psi(x)$ (see [1, 4]). Note that

$$\mathrm{supp}(d\phi) = \mathrm{supp}(d\psi) \ , \qquad (3.14)$$

since $\xi_1 = \min(\mathrm{supp}(d\psi)) > 0$, and

$$\int_{-\infty}^{\infty} d\phi(x) = \gamma_2^{-1} \int_{-\infty}^{\infty} (P_1(x) + \gamma_2) d\psi(x) = 1,$$

so that $d\phi$ is properly normalized. If the Hmp for $\mathcal{P}^*$ is determined, then $d\phi$ is the unique distribution for $\mathcal{P}^*$, so that $\xi_i^* = \xi_i$ $(i \geqq 1)$ by (1.5) and (1.6) (in terms of $\mathcal{P}^*$) and (3.14). Therefore, let us assume that the Hmp for P^* is indeterminate. From [5, Theorem 3] we then have

$$\sum_{n=1}^{\infty} H_n \left(\sum_{i=1}^{n} G_i \right)^2 < \infty ,$$

whence

$$\sum_{n=0}^{\infty} H_n \sum_{i=0}^{n} G_i < \infty .$$

Invoking Theorem 1 we conclude that $\{P_n(x)/P_n(0)\}_n$ converges to an entire function $Q_\infty(x)$, say, whose zeros are precisely the points ξ_i $(i \geq 1)$. Also from Theorem 1 we see that $\sum' \xi_i^{-1} < \infty$, so that, by (3.4), $\sum' (\xi_i^*)^{-1} < \infty$. Since $\xi_1^* \geq \xi_1 > 0$ we can apply Theorem 1 to P^* yielding that $\{P_n^*(x)/P_n^*(0)\}_n$ converges to an entire function $Q_\infty^*(x)$, say, whose zeros are precisely the points ξ_i^* $(i \geq 1)$.

Evidently, we have $\sigma = \infty$. Now let x be an arbitrary number which is not in Ξ. Then, by (1.6) and Theorem 4, $\sum_{j=0}^{\infty} p_j^2(x) = \infty$, whereas, also by Theorem4, $\sum_{j=0}^{\infty} (p_j^*(x))^2 < \infty$, since the Hmp for P^* is indeterminate. From Lemma 1 we therefore obtain

$$\frac{P_{n+1}(x)}{P_{n+1}(0)} \frac{P_n^*(x)}{P_n^*(0)} \sum_{j=0}^{n} G_j \to +\infty \quad (n \to \infty) ,$$

implying that $P_n(x)/P_n(0)$ and $P_n^*(x)/P_n^*(0)$ must have the same sign for n sufficiently large. A simple argument shows that this will happen for every $x \notin \Xi$ only if $Q_\infty(x)$ and $Q_\infty^*(x)$ have identical zeros, i.e., $\xi_i^* = \xi_i$ $(i \geq 1)$.

Finally turning to part (iii) we note that if $\xi_1 > 0$ and the Hmp for $\mathcal{P}$ is indeterminate, then also the Stieltjes moment problem for $\mathcal{P}$ is indeterminate. By [5, Theorem 2] this is equivalent to $\sum(G_j + H_j) < \infty$. Under the latter condition, however, the validity of (iii) was established in [5, p. 340]. □

IV Further limit theorems

For any sequence $\{a_n\}_{n=0}^{\infty}$ we denote by $S(\{a_n\})$ the number of sign changes in the sequence $\{a_n\}$ after deleting all zero terms. By convention, $S(\{0\}) = -1$. Now let $\mathcal{R} = \{R_n(x)\}_{n=0}^{\infty}$ be any sequence of monic orthogonal polynomials. From [6, Theorem 3] we then have

$$S(\{(-1)^n R_n(x)\}) = \begin{cases} 0 & \text{if} \quad x \leq \xi_1(\mathcal{R}) \\ k & \text{if} \quad \xi_k(\mathcal{R}) < x \leq \xi_{k+1}(\mathcal{R}) \quad (k \geq 1) \end{cases} \tag{4.1}$$

Assuming $\xi_1(\mathcal{R}) > -\infty$ we subsequently define

$$I(\mathcal{R}) = \bigcup_{i=1}^{\infty} (-\infty, \xi_i(\mathcal{R})] .$$

Note that $I(\mathcal{R}) = (-\infty, \sigma(\mathcal{R})]$ or $I(\mathcal{R}) = (-\infty, \sigma(\mathcal{R}))$, depending on the occurrence of the event $\xi_i(\mathcal{R}) = \xi_{i+1}(\mathcal{R})$ for some i (cf. (1.5)).

Returning to the context of the previous sections we note that $I(\mathcal{P}) = I(\mathcal{P}^*)$ in view of (1.9). Now applying (4.1) to both $\mathcal{P}$ and $\mathcal{P}^*$ one readily sees that for each $x \in I(\mathcal{P})$ there exists an integer $N = N(x)$ such that the sequence $\{P_n(x)/P_n(0)\}_{n=N}^{\infty}$ is monotone and without sign changes. In particular for $x < 0$ ($\leq \xi_1$) it is easily shown that the sequence $\{P_n(x)/P_n(0)\}_{n=0}^{\infty}$ is positive and increasing. Whether $P_n(x)/P_n(0)$ tends to infinity or not as $n \to \infty$ must be decided from Theorem 1. In what follows we restrict our attention to positive x. Theorem 5 enables us to relate the behaviour of $\{P_n(x)/P_n(0)\}$ to the points ξ_i $(i \geq 1)$. Indeed, from Theorem 5 (ii) and (4.1), applied to $\mathcal{P}$ and $\mathcal{P}^*$, we easily obtain the following lemma.

Lemma 2. If the Hmp for P is determined, $\xi_1 > 0$ and $\xi_k < x \leq \xi_{k+1}$ $(k \geq 0,\ \xi_0 = 0)$, then there exists an integer $N = N(x)$ such that the sequence $\{(-1)^k P_n(x)/P_n(0)\}_{n=N}^{\infty}$ is positive and decreasing ($N(x) = 0$ if $0 < x \leq \xi_1$).

Under the conditions of this lemma the sequence $\{P_n(x)/P_n(0)\}_n$ tends to a finite limit. The next theorem, which is a generalization of [3, Lemma 2] gives a criterion for this limit to be zero when $x < \sigma$ (compare Theorem 1).

Theorem 6. If $\xi_1 > 0$, $\sum' \xi_i^{-1} = \infty$ and $0 < x < \sigma$, then $\{P_n(x)/P_n(0)\}_n$ converges to zero.

Proof. By Corollary 2.2 the Hmp for P is determined when $\sum' \xi_i^{-1} = \infty$, so that Lemma 2 applies. Let a be any positive number smaller than σ, and

$$R = \max_{0 \leq x \leq a} \max_{n \geq 0} |P_n(x)/P_n(0)| .$$

By Lemma 2, $R < \infty$. Moreover, by [3, Lemma 2], $\{P_n(x)/P_n(0)\}_n$ tends to zero for $0 < x < \xi_1$. The result follows by the Stieltjes-Vitali theorem. □

Remark: We conjecture that Theorem 6 remains valid when σ is replaced by $\lim_{n\to\infty} x_{nn}$.

Let us now turn to the case $\xi_1 = 0$. Theorem 5 (i) and (4.1), applied to $\mathcal{P}$ and $\mathcal{P}^*$, readily yield the next lemma.

Lemma 3. If $\xi_1 \geqq 0$ and $\xi_k < x \leqq \xi_{k+1}$ $(k \geqq 1)$, then there is an integer $N = N(x)$ such that the sequence $\{(-1)^k P_n(x)/P_n(0)\}_{n=N}^{\infty}$ is positive and increasing.

Our final result gives the limit of the sequence $\{(-1)^k P_n(x)/P_n(0)\}_n$ for $x \notin \Xi$.

Theorem 7. If $\xi_1 = 0$ and $\xi_k < x < \xi_{k+1}$ $(k \geqq 1)$, then $\{(-1)^k P_n(x)/P_n(0)\}_n$ tends to infinity.

Proof. First suppose that the Hmp for $\mathcal{P}$ is determined. From (3.5) and (3.9) we see that

$$\sum_{n=0}^{\infty} p_n^2(x) = \sum_{n=0}^{\infty} G_n P_n^2(x)/P_n^2(0) .$$

By (1.5), (1.6) and Theorem 4 we have on the one hand $\sum_{n=0}^{\infty} p_n^2(0) < \infty$, whence $\sum G_n < \infty$, and on the other hand $\sum_{n=0}^{\infty} p_n^2(x) = \infty$ for $\xi_k < x < \xi_{k+1}$. It follows that in the latter case $\{P_n(x)/P_n(0)\}$ must be unbounded. The required result follows by Lemma 3.

Next assume that the Hmp for $\mathcal{P}$ is indeterminate, so that, by Corollary 2.2, $\sum' \xi_i^{-1} < \infty$. Let $a > 0$ and $\mathcal{R} = \{R_n(x)\}_{n=0}^{\infty}$, where $R_n(x) = P_n(x-a)$. Clearly, $\xi_i(\mathcal{R}) = \xi_i + a$ (so that $\xi_1(\mathcal{R}) > 0$) and $\sum' (\xi_i(\mathcal{R}))^{-1} < \infty$. Applying Theorem 1 to $\mathcal{R}$ yields that $R_n(a)/R_n(0) \to 0$ $(n \to \infty)$, whereas for $\xi_k < x < \xi_{k+1}$ we have that $R_n(x+a)/R_n(0)$ tends to a non-zero limit. Since

$$\frac{P_n(x)}{P_n(0)} = \frac{R_n(x+a)}{R_n(0)} \Big/ \frac{R_n(a)}{R_n(0)} \quad ,$$

it follows that $|P_n(x)/P_n(0)| \to \infty$ as $n \to \infty$. Lemma 3 now gives the required result. □

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