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# HEREDITARY SUBDIRECT IRREDUCIBILITY IN GRAPHS

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Hereditary subdirect irreducibility in graphs*)

by

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ABSTRACT

We give a characterization of productive hereditary classes of anti-reflexive graphs in which every full subgraph of a subdirectly irreducible graph is subdirectly irreducible as well. This generalizes also previous results for the symmetric case.

KEY WORDS & PHRASES: *subdirectly irreducible, meet-irreducible, hereditary productive class of graphs*

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## 0. INTRODUCTION

The concept of the subdirect irreducibility was introduced for algebras by G. Birkhoff. It can be defined more generally for categories, in particular for graphs: Let $\underline{C}$ be a class of (some) graphs. Then a $\underline{C}$-graph A (i.e. a graph $A \in \underline{C}$) is said to be *subdirectly irreducible* (SI) if, whenever an isomorphic copy A' of A is contained as a full subgraph in a product $\mathop{\mathsf{X}}_{i \in \alpha} B_i$ with $B_i \in \underline{C}$ and $p_j(A') = B_j$ for all the projections, there is a j such that the restriction of $p_j$ to A' is an isomorphism onto $B_j$. (This formulation is due to A. Pultr - see [P]).

Importance of investigation this topic is following: having a list of subdirectly irreducible $\underline{C}$-graphs, one can construct any $\underline{C}$-graph from subdirectly irredicibles using such simple operations as product and restrictions to full subgraphs. If subdirectly irreducibles are, in some sense, "simple" then this procedure may be useful for recording of graphs to the machine memory.

Characterization theorem for the subdirect irreducibility is given in [PV]. It enables us to find a list of subdirectly irreducibles in various categories. This theorem, however, does not solve the problem when the list of subdirectly irreducibles is closed to subobjects. This question is particularly interesting for the case of systems of antireflexive graphs where the list of subdirectly irreducibles is often infinite and so such a hereditary (with respect to full subgraphs) can be useful for its description.

In [V1], a characterization of systems of symmetric antireflexive graphs in which any full subgraph of a subdirectly irreducible one is again SI, is given. In the present note, we are going to generalize this characterization to the case of antireflexive graphs.

## 1. NOTATIONS AND DEFINITIONS

1.1. <u>NOTATIONS</u> . Denote $\underline{G}$ the class of all antireflexive graphs. For any ordinal n denote $K_n = (n, \{(i,j) \mid i,j \in n,\ i \neq j\})$ (i.e. the complete antireflexive graph with n vertices), $K_n' = (n, \{(i,j) \mid i,j \in n, \mid i \neq j,\ (i,j) \neq (0,1)\})$ (i.e. the antireflexive graph on n vertices with just one edge missing),

$L_n = (n, \{(i,j) \mid i,j \in n,\ i < j\})$.

$L_n^+ = (n, \{(i,j) \mid i,j \in n,\ i < j\} \cup \{(1,0)\})$,

$L_n^- = (n, \{(i,j) \mid i,j \in n,\ i > j\} \cup \{(0,1)\})$,

$A_3 = (3, \{(0,1),(1,0),(0,2),(2,1)\})$,

$C_3 = (3, \{(0,1),(1,2),(2,0)\})$,

$A_4 = (4, \{(0,1),(1,0),(0,2),(1,2),(2,1),(1,3),(2,3),(3,2),(3,0),(0,3)\})$.

Further, put $K = \{K_n \mid n \in \mathrm{Ord}\}$, $K' = \{K_n' \mid n \in \mathrm{Ord}\}$, $L^+ = \{L_n^+ \mid n \in \mathrm{Ord}\}$, $L^- = \{L_n^- \mid n \in \mathrm{Ord}\}$, $S = \{(x,\emptyset) \mid X \text{ is a set}\}$ (the class of sets = discrete graphs),

$T = \{(X,R) \mid \forall x,y \in X,\ x \neq y \Rightarrow |\{(x,y),\ (y,x)\} \cap R| = 1\}$ (the class of all tournaments),

$U = \{(n,R) \mid n \leq 6,\ |R| = n + [\frac{n}{2}],\ x \neq y \Rightarrow |\{(x,y),\ (y,x)\} \cap R| \geq 1$ and $(n,R)$ contains neither $K_3'$ nor $A_3$ as a full subgraph$\}$,

$V = \{(n,R) \mid n \leq 4,\ x \neq y \Rightarrow |\{(x,y),\ (y,x)\} \cap R| \geq 1,\ R \supseteq \{(0,1),\ (1,0),\ (2,3),\ (3,2)\} \cap n \times n$ and $(n,R)$ does not contain $K_3'$ as a full subgraph$\}$,

$W = \{A \in \underline{G} \mid$ any full subgraph of $A$ with 3 vertices is isomorphic either to $A_3$ or to $L_3\}$.

Let $\underline{D}$ be a collection of graphs. Then $SP(\underline{D})$ denotes (similarly as in [NP]) the class of all the graphs which can be embedded as full subgraphs into products of graphs from $\underline{D}$.

1.2. DEFINITION. A class $\underline{C}$ of graphs closed to categorical products $\times$ ($\times_{i \in I}(X_i, R_i) = (\times_{i \in I} X_i, R)$ where $((x_i)_I, (y_i)_I) \in R \Leftrightarrow (x_i, y_i) \in R_i$ for any $i \in I$) and to full subgraphs is said to be *hereditary with respect to subdirect irreducibility* (HSI) if any full subgraph of a SI graph is again SI.

2. MAIN THEOREM.

We are going to prove the following:

2.1. THEOREM. *Let* $\underline{C} \subset \underline{G}$ *be a productive hereditary class of graphs (i.e. a* *class closed to categorical products* X *and to full subgraphs). Then* $\underline{C}$ *is* HS1 *iff either* $\underline{C} = S$*, or* $\underline{C} = SP(\underline{D})$ *where* $\underline{D}$ *satisfies one of the following conditions:*

(i) $\underline{D} \subset K \cup K'$

(ii) $\underline{D} \subset K \cup \{K_3', A_4\}$

(iii) $\underline{D} \subset K \cup L^{+} \cup T$

(iv) $\underline{D} \subset K \cup L^{-} \cup T$

(v) $\underline{D} \subset K \cup U$

(vi) $\underline{D} \subset K \cup V$

(vii) $\underline{D} \subset K \cup W$

## 3. SUBDIRECT IRREDUCIBILITY

Before proving Main Theorem, recall the characterization of subdirectly irreducibles:

3.1. DEFINITION. a) A graph (X,R) is said to be *meet-irreducible* (in $\underline{C}$) iff, whenever $R = \bigcap_{i\in I} R_i, (X,R_i) \in \underline{C}$ then there exist $i_0 \in I$ such that $R_{i_0} = R$.
b) A graph (X,R) is said to be *maximal* (in $\underline{C}$) iff $R' \supset R$, $(X,R') \in \underline{C}$ implies that $R' = R$.
c) A *monomorphic system* is a system $(u_i : (X,R) \to (Y_i, R_i))_{i\in I}$ of homomorphisms such that if $u_i\alpha = u_i\beta$ for all $i \in I$ then $\alpha = \beta$.

3.2. THEOREM. *A* $\underline{C}$*-graph* $A = (X,R)$ *is* SI *iff either* A *is maximal in* $\underline{C}$ *and for any monomorphic system* $(u_i : A \to B_i)_{i\in I}$ *there exists an* $i_0 \in I$ *such that* $u_{i_0}$ *is one-to-one, or* A *is not maximal, it is meet-irreducible in* $\underline{C}$ *and for any* $\phi\ A \to B$ *not one-to-one there exist* $R' \supsetneqq R$ *such that* $\phi$ *can be extended to a homomorphism* $\bar{\phi} : (X,R') \to B$.

3.3. REMARK. The previous theorem is just a reformulation of Theorem 3.6 from [V2].

Using Theorem 3.2 one can characterize subdirectly irreducible $\underline{G}$-graphs:

3.4. PROPOSITION. *A* $\underline{G}$*-graph* A *is* SI *in* $\underline{G}$ *iff* $A \in K \cup K'$.

PROOF. One can easily see that meet-irreducible G-graphs are just elements of $K \cup K'$. Since any S1 graph must be meet-irreducible, it has to be an element of $K \cup K'$.

Any element of $K$ is maximal G-graph which cannot be mapped to a G-graph of a smaller cardinality, hence it S1. Any element A of $K'$ is non-maximal meet-irreducible graph. Moreover, every mapping $\phi: A \to B$ is one-to-one. Hence, A is S1. □

4. PROOF OF THE MAIN THEOREM

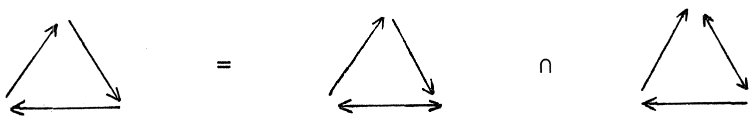
We are going to prove Theorem 2.1 by a series of lemmas:

4.1. LEMMA. *Let* C $\neq S$ *be a productive hereditary class of* G*-graphs. If* C *is* HS1 *then for every* (X,R) *a* S1 C*-graph and for any* $x,y \in X$, $x \neq y$, *there is* $\{(x,y), (y,x)\} \cap R \neq \emptyset$.

PROOF. Since C $\neq S$, C must contain a non-discrete graph. Since C is hereditary, it must contain a non-discrete graph A with two vertices. Hence, $(2,\emptyset) \cong A \times (1,\emptyset)$ is not S1 and HS1 of C implies the assertion of lemma. □

4.2. LEMMA. *Let* C *be a productive hereditary class of* G*-graphs. If* $A_3 \in$ C *then* $C_3$ *is not* S1 *in* C.

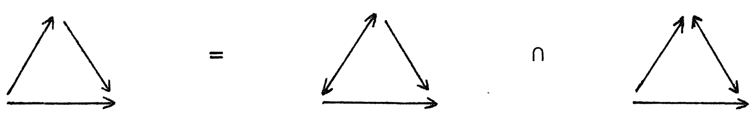
PROOF.



Hence, $C_3$ is not meet-irreducible and according to 3.2 it is not S1. □

4.3. LEMMA. *Let* C *be a productive hereditary class of* G*-graphs. If* $L_3^+ \in$ C, $L_3^- \in$ C, *then* $L_3$ *is not* S1 *in* C.

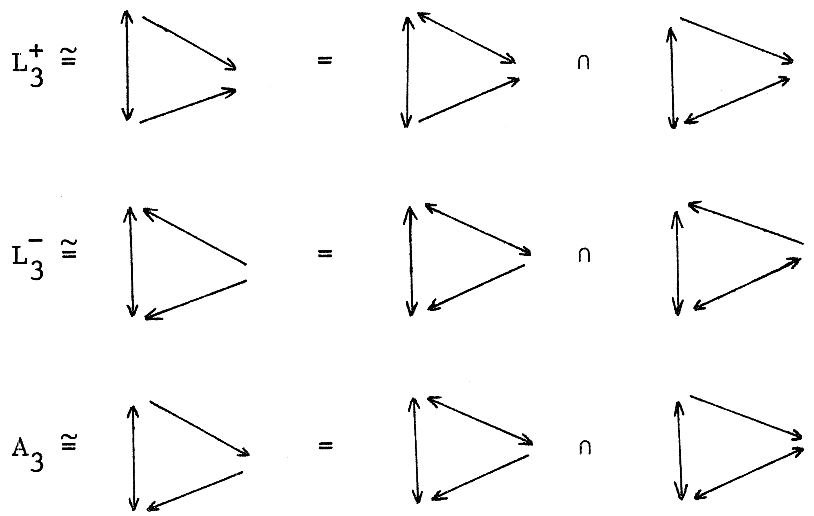
PROOF.



Hence, $L_3$ is not meet-irreducible and according to 3.2 it is not S1. □

4.4. LEMMA. *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $K_3' \in \underline{C}$ *then* $L_3^+$, $L_3^-$, $A_3$ *are not* S1 *in* $\underline{C}$.

PROOF.




Proposition 3.2 finishes the proof. □

4.5. LEMMA. *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $\underline{C}$ *is* HS1, $K_3' \in \underline{C}$, $K_4' \notin \underline{C}$, *then every* S1 $\underline{C}$*-graph with 4 vertices is either isomorphic to* $K_4$ *or isomorphic to* $A_4$.

PROOF. Let A be a S1 $\underline{C}$-graph with 4 vertices. Since $\underline{C}$ is HS1, A cannot contain a two-point discrete graph as a full subgraph. Lemmas 4.2 - 4.4 imply that any 3-point full subgraph of A has to be isomorphic either to $K_3$, or to $K_3'$. Since $K_4' \notin \underline{C}$, there is either $A \cong K_4$ or $A \cong A_4$. □

4.6. LEMMA. *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $\underline{C}$ *is* HS1, $K_3' \in \underline{C}$, $K_4' \notin \underline{C}$, *then every* S1 $\underline{C}$*-graph with* $n \geq 5$ *vertices is isomorphic to* $K_n$.

PROOF. Suppose that there exists S1 $\underline{C}$-graph with more than 4 vertices which is isomorphic to no $K_n$. Hereditary of $\underline{C}$ implies that there exist $A = (5,R) \ncong K_5$ which is S1 in $\underline{C}$, Lemma 4.5 and the assumptions $K_3' \in \underline{C}$,

$K_4' \notin C$ imply that

$$A \restriction 4 = (4, R \cap 4 \times 4) \cong A_4.$$

Suppose that $R \cap 4 \times 4 = \{(0,1), (1,0), (0,2), (1,2), (2,1), (1,3), (2,3), (3,2), (3,0), (0,3)\}$. Similarly, $A \restriction \{1,2,3,4\} \cong A_4$. Hence, $R \cap \{1,2,3,4\} \times \{1,2,3,4\} = \{(1,2), (2,1), (1,3), (2,3), (3,2), (1,4), (4,1), (3,4), (4,3)\} \cup \{(i,j)\}$ where $(i,j) = (2,4)$ or $(i,j) = (4,2)$. Therefore, $B = A \restriction \{0,2,4\} \ncong K_3$, $B \ncong K_3'$. Hence, B is not S1 which contradicts the HS1 property of $\underline{C}$. □

4.7. LEMMA. *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{C}$*-graphs. If* $\underline{C}$ *is* HS1, $K_3' \notin \underline{C}$, $L_3^- \notin \underline{C}$, $A_3 \notin \underline{C}$, $L_3^+ \in \underline{C}$, *then every* S1 $\underline{C}$*-graph is an element of* $K \cup L^+ \cup T$.

PROOF. Let A be a S1 $\underline{C}$-graph. If $A \notin K \cup T$ then A must contain three-point full subgraph B which is neither a tournament, nor an isomorphic copy of $K_3$. Since $K_3' \notin \underline{C}$, $L_3^- \notin \underline{C}$, $A_3 \notin \underline{C}$, there is $B \cong L_3^+$ and $A \cong L_n^+$. □

4.8. LEMMA. *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $\underline{C}$ *is* HS1, $K_3' \notin \underline{C}$, $L_3^+ \notin \underline{C}$, $A_3 \notin \underline{C}$, $L_3^- \in \underline{C}$, *then every* S1 $\underline{C}$*-graph is an element of* $K \cup L^- \cup T$.

PROOF is similar to the proof of Lemma 4.7. □

4.9. LEMMA. *Any tournament on 4 vertices contains* $L_3$ *as a full subgraph.*

PROOF is obvious. □

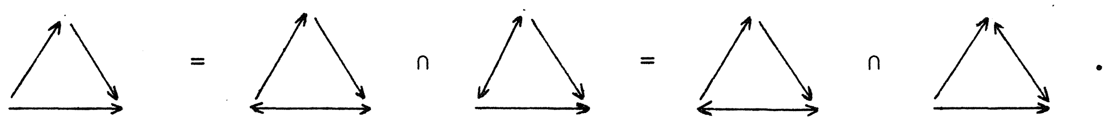
4.10. LEMMA. *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $\underline{C}$ *is* HS1, $L_3^+$, $L_3^-$, *are* S1 $\underline{C}$*-graph, then every* S1 $\underline{C}$*-graph is an element of* $K \cup U$.

PROOF. By Lemma 4.4, $K_3' \notin C$. Hence, $K_n' \notin C$ for any $n \geq 3$. By Lemma 4.3, $L_3$ is not S1 in $\underline{C}$. Suppose there exists a S1 $\underline{C}$-graph A with $n \geq 7$ vertices which is not isomorphic to $K_n$. Since $K_3' \notin \underline{C}$, A contains a tournament on 4 vertices as a full subgraph. By Lemma 4.9 and HS1 property of $\underline{C}$, $L_3$ is S1 is $\underline{C}$ which is a contradiction.

If B is a S1 $\underline{C}$-graph with $n \leq 6$ vertices, $B \ncong K_n$, then B does not contain $K_3'$ as a full subgraph, on the other hand, B does not contain $L_3$ as a full subgraph as well. Hence, $B \in \mathcal{U}$. □

4.11. <u>LEMMA.</u> *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs which is* HS1. *If* $A_3 \in \underline{C}$, $L_3^+ \in \underline{C}$ ($L_3^- \in \underline{C}$, *resp.*) *then no tournament with at least 3 vertices is* S1.

<u>PROOF.</u>



Hence, $L_3$ is not S1. By 4.2, $C_3$ is not S1. HS1 property of $\underline{C}$ implies that no tournament with at least three verices is S1. □

4.12. <u>LEMMA.</u> *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $K_3' \notin \underline{C}$, $A_3 \in \underline{C}$, $L_3^+ \in \underline{C}$, ($L_3^- \in \underline{C}$, *resp.*) *and* $\underline{C}$ *is* HS1, *then every* S1 $\underline{C}$*-graph is an element of* $\mathcal{K} \cup \mathcal{V}$.

<u>PROOF.</u> By Lemma 4.11, any S1 $\underline{C}$-graph A contains no tournament with at least 3 vertices. Hence, A is either complete, or an element of $\mathcal{V}$. □

4.13. <u>LEMMA.</u> *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $K_3' \notin \underline{C}$, $L_3^+ \notin \underline{C}$, $L_3^- \notin \underline{C}$, $A_3 \in \underline{C}$ *and* $\underline{C}$ *is* HS1 *then every* S1 $\underline{C}$*-graph is an element of* $\mathcal{K} \cup \mathcal{W}$.

<u>PROOF.</u> Let A be a S1 $\underline{C}$-graph, $A \notin \mathcal{K}$. By 4.2, A does not contain $C_3$ as a full subgraph. Hence, $A \in \mathcal{W}$. □

4.14. <u>LEMMA.</u> *Let* $\underline{C}$ *be a productive hereditary class of* $\underline{G}$*-graphs. If* $K_4' \in \underline{C}$ *and* $\underline{C}$ *is* HS1, *then every* S1 $\underline{C}$*-graph is an element of* $\mathcal{K} \cup \mathcal{K}'$.

<u>PROOF.</u> Suppose there is a S1 $\underline{C}$-graph $A \notin \mathcal{K} \cup \mathcal{K}'$. Then either, A has at most 3 vertices, or A contains a full subgraph B with four vertices such that $B \ncong K_4$, $B \ncong K_4'$.

In the first case, either there is $A \cong (2,\emptyset)$ which is a contradiction, or $A = (3,R)$, $A \ncong K_3$, $A \ncong K_3'$, Lemma 4.4 implies that A is not S1.

In the second case, by the similar argument as in 4.4, B is an intersection of isomorphic copies of $K_4'$, hence not S1. □

4.15. <u>PROOF OF MAIN THEOREM.</u>

A. Suppose that $\underline{C}$ is HS1, $\underline{C} \neq \mathcal{S}$, $\underline{D} = \{A \in \underline{C};\ A \text{ is S1}\}$.

If $K_4' \in \underline{C}$ then $\underline{D} \subset \mathcal{K} \cup \mathcal{K}'$ according to 4.14.

If $K_4' \notin \underline{C}$, $K_3' \in \underline{C}$, then $\underline{D} \subset \mathcal{K} \cup \{K_3', A_4\}$ according to 4.6 and 4.4.

If $K_3' \notin \underline{C}$, $A_3 \notin \underline{C}$, then either $\underline{D} \subset \mathcal{K} \cup \mathcal{L}^+ \cup \mathcal{T}$ ($\underline{D} \subset \mathcal{K} \cup \mathcal{L}^- \cup \mathcal{T}$, resp.) by Lemmas 4.7 and 4.8, or $\underline{D} \subset \mathcal{K} \cup \mathcal{U}$ by Lemma 4.10.

If $K_3' \notin \underline{C}$, $A_3 \in \underline{C}$, $L_3^+ \in \underline{C}$ ($L_3^- \in \underline{C}$, resp.) then $\underline{D} \subset \mathcal{K} \cup \mathcal{V}$ according to Lemma 4.12.

If $K_3' \notin \underline{C}$, $L_3^+ \notin \underline{C}$, $L_3^- \notin \underline{C}$, $A_3 \in \underline{C}$, then $D \subset \mathcal{K} \cup \mathcal{W}$ according to Lemma 4.13.

If $\underline{C} \cap \{K_3', A_3, L_3^+, L_3^-\} = \emptyset$ then $\underline{D} \subset \mathcal{K} \cup \mathcal{T}$.

B. One can check that each of systems $\mathcal{K} \cup \mathcal{K}'$, $\mathcal{K} \cup \{K_3', A_4\}$, $\mathcal{K} \cup \mathcal{L}^+ \cup \mathcal{T}$, $\mathcal{K} \cup \mathcal{L}^- \cup \mathcal{T}$ $\mathcal{K} \cup \mathcal{U}$, $\mathcal{K} \cup \mathcal{V}$, $\mathcal{K} \cup \mathcal{W}$ is hereditary. If $\underline{D}$ is its subsystem closed to full subgraphs, then $\underline{D}$ is a system of subdirectly irreducibles of SP(D) and $\underline{C}$ = SP(D) is HS1. □

4.16. <u>CONCLUDING REMARK.</u> In [V1], types of dimensions of graphs are studied. Recall that a *product dimension* of a graph A in $\underline{C}$ is $\text{p-dim}_{\underline{C}} A = \min\{\alpha \mid A$ is a full subgraph of $\prod_{i \in \alpha} A_i$ with $A_i$ S1 in $\underline{C}\}$, a *subdirect dimension* $\text{s-dim}_{\underline{C}} A = \min\{\alpha \mid A$ is a full subgraph of $\prod_{i\in\alpha}$ with $A_i$ S1 and $p_i m$ onto$\}$ ($p_i$ are projections, m is an embedding).

Theorem 2.1 implies that, if $\underline{C} \subset \underline{G}$ is a productive hereditary class of graphs then $\text{p-dim}_{\underline{C}} \equiv \text{s-dim}_{\underline{C}}$ iff $\underline{C}$ = SP(D) where $\underline{D}$ satisfies one of the conditions (i) - (vii) from 2.1.

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