stichting
mathematisch
centrum

AFDELING TOEGEPASTE WISKUNDE TW 244/83 AUGUSTUS
(DEPARTMENT OF APPLIED MATHEMATICS)

J.J.E. VAN DER MEER

CLINES INDUCED BY A GEOGRAPHICAL BARRIER

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 35Bxx, 35R05, 34C35, 92A10

Clines induced by a geographical barrier

by

J.J.E. van der Meer

ABSTRACT

The consequences of a geographical barrier in the habitat are studied in the context of a one-dimensional reaction-diffusion model.

By the symmetry of the problem, each steady state solution generates three more - not necessarily different - solutions. It is proved that only monotone steady state solutions can be stable.

We consider a special type of steady state solutions which occur in pairs of two by their symmetry. Necessary and sufficient conditions for these steady states to be stable are derived.

A cline is a nonconstant *stable* steady state solution. It is proved that two is the maximum number of clines of the special type. Moreover, for large values of the parameter, i.e., for small penetrability of the barrier, it is proved that only steady state solutions of the special type can be stable.

Finally, it is shown that the ω-limit set of any initial condition is a steady state solution.

KEY WORDS & PHRASES: *nonlinear diffusion equations, transmission condition*

CONTENTS

0. Introduction
1. The L-curve
2. The steady state solutions
3. The stability of steady state solutions
4. The initial value problem, a Lyapunov functional and the asymptotic behaviour

Appendix

References

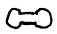
0. INTRODUCTION

The joint effect of selection and migration on the genetic composition of a population is frequently described by a one-dimensional reaction-diffusion equation

0.1
$$u_t = u_{xx} + f(u) \quad \text{in } [-L,L]$$
$$u_x(-L) = u_x(L) = 0.$$

For some background on this one gene - two alleles problem we refer to Nagylaki [13], Fife [5], and Fife and Peletier [6].

It has been proved by Chafee [2] that (0.1) does not admit *stable nonconstant* steady states (clines). This result has been extended to higher dimensional but *convex* domains by Casten and Holland [1] and Matano [11].

A complementary result of Matano, also in [11], shows that for a class of dumb-bell shaped domains (i.e. ) clines do exist. The work of Hale [7] and Hale and Vegas [8] is concerned with the bifurcation of these nonconstant solutions from constants as the domain is perturbed.

In Fife and Peletier [6], one can find a one dimensional reaction diffusion equation on an interval, with homogeneous Neumann-boundary conditions, which *has* stable nonconstant steady state solutions, due to nonhomogeneous diffusion and/or space-dependent selection.

In 1976 Nagylaki [12] adapted the model (0.1) to the situation in which the habitat is intersected by a geographical barrier. Under the assumption that the habitat is homogeneous, except for one geographical barrier which is situated at exactly the middle of the habitat, this adaptation takes the form of a transmission condition in 0:

$$u_x(0+,t) = u_x(0-,t) = \frac{1}{\mu}(u(0+,t)-u(0-,t)),$$

for some $\mu \in \mathbb{R}^+$, which measures the penetrability of the barrier.

In 1979 Ten Eikelder [4] analysed the effect of this transmission condition in an unbounded domain, in which selection is space-dependent. He proved the existence of a cline under some restrictions on the reaction-function f.

Motivated by these observations we shall now analyse the following evolution problem:

$$\text{E.P.}\begin{cases} u_t = u_{xx} + f(u) & x \in [-L,0) \cup (0,L] \\ u_x(-L,t) = 0 \\ u_x(L,t) = 0 \\ u_x(0-,t) = u_x(0+,t) = \frac{1}{\mu}(u(0+,t)-u(0-,t)) \\ u(x,0) = \psi(x) & x \in [-L,0) \cup (0,L]. \end{cases}$$

In the present paper f will be the rather special cubic

$$f(u) = u(1-u)(u-\tfrac{1}{2}).$$

However, it should be clear that our results can serve as the starting point of a perturbation analysis for "nearby" functions f, for instance

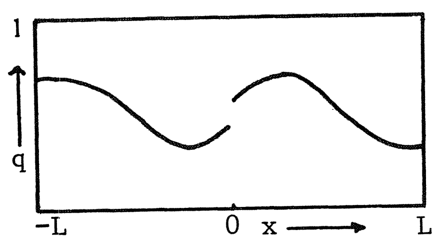
$$f(u) = u(1-u)(u-a)$$

with

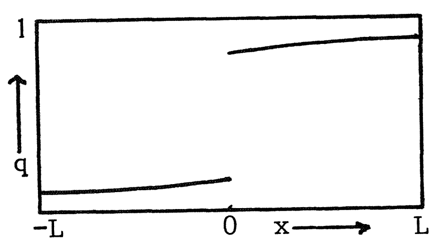
$$|a-\tfrac{1}{2}| \ll 1 \quad [10].$$

With respect to the special type of steady state solutions mentioned in the abstract, which have the following shape:

e.g.



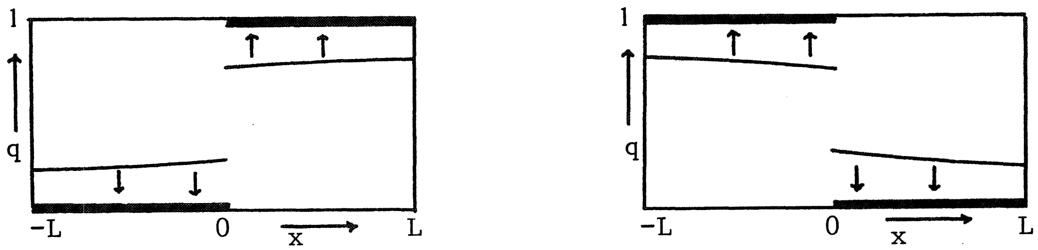
or



we prove that for any L there exists a value $\hat{\mu}(L)$ such that

- for $\mu < \hat{\mu}(L)$ there are no clines of this type
- for $\mu > \hat{\mu}(L)$ there exist exactly two clines of this type.

We shall show that, when μ tends to infinity, those clines converge uniformly on $[-L,L]$ to q_{01} and q_{10}:



q_{01} q_{10}

These results read in section 2 and section 3, and in these two sections, we use the L-curve, introduced in section 1. The L-curve is a curve in the phase-plane, indicating which parts of the trajectories represent a function q, with domain an interval of length L - which can be taken the interval $[0,L]$ - and $q_x(L) = 0$. In section 1, we shall derive some important properties of the L-curve, which can be used in both section 2, where we analyse the steady state equation, and section 3, where we analyse the stability of steady state solutions.

In section 4 we shall analyse (E.P.) and a Lyapunov functional will be put on the stage. We shall conclude that the ω-limit set of any initial condition is a steady state solution.

Although the biological interpretation clearly requires $\mu > 0$ and $0 \leq u(x,t) \leq 1$, we will not make these restrictions in all of our calculations, since this would give rise to a great loss of insight.

The symmetry of the domain and of the reaction-function f play an important role in our analysis and one can view the origination of the clines as the result of two symmetry breaking bifurcations. Nevertheless we think that our main technical tool, the L-curve introduced in section 1, is useful in the study of similar problems without symmetry. Moreover, we think it will be useful in the study of reaction-diffusion equations with

Neumann boundary conditions and a nonconstant, though locally constant, diffusion coefficient. Work in this direction is presently in progress.

To end this introduction, note that this paper asserts the possiblity of the existence of a *stable* situation, in which on each side of the geographical barrier, which is difficult to penetrate, lives a part of one population, though with a completely different genetical composition. Moreover, when the barrier is sufficiently hard to pass, the influence of one part of the population on the other will decrease with the distance to the barrier, e.g. within the population part which is predominantly "white", there exist no subcolonies of "black" (or even "grey") animals.

1. THE L-CURVE

This section will deal with a curve in the phase plane of

$$\zeta_{xx} + f(\zeta) = 0.$$

The phase-portrait of this equation is most easily obtained by introducing formally $P(\zeta) = \zeta_x(\zeta)$ to find $PP_\zeta + f(\zeta) = 0$ and subsequently, by integration,

$$P^2(w) - P^2(v) + 2\int_v^w f(\xi)d\xi = 0.$$

For our special function f the phase portrait is

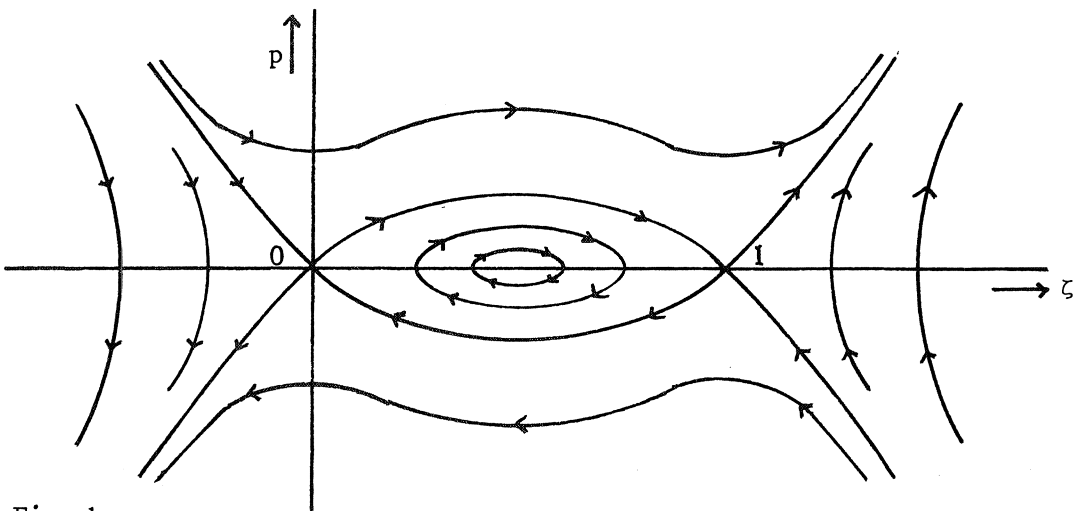


Fig. 1

At the risk of confusion we shall call the length of the *domain* of the function ζ corresponding to a certain part of one of the trajectories *the length of that part of that trajectory* (so that this is *not* the length of the curve in the phase plane).

Clearly, steady state solutions of (E.P.) consist of two parts of length L starting or ending at the line P = 0. This observation motivates the introduction of L-points which are obtained by pacing a length L along such trajectories. Exploiting the symmetry we restrict our attention to trajectories ending at the interval $[\frac{1}{2},1]$ on the ζ-axis.

1.1. DEFINITION. Let $\rho: (-\infty,\infty) \times [\frac{1}{2},1] \to [0,1]$ be the solution of the ω-L-problem:

$$\omega.\text{L.P.} \begin{cases} \dfrac{\partial^2\rho}{\partial x^2}(x,\omega) + f(\rho(x,\omega)) = 0 \quad x \in (-\infty,\infty) \\ \dfrac{\partial\rho}{\partial x}(L,\omega) = 0 \\ \rho(L,\omega) = \omega. \end{cases}$$

The L-curve is the set $\Gamma_L = \{(\rho(0,\omega), \frac{\partial\rho}{\partial x}(0,\omega)) \mid \omega \in [\frac{1}{2},1]\}$.

Note that Γ_L is a *smooth* curve connecting (1,0) to $(\frac{1}{2},0)$.

A sketch of $\Gamma_{\frac{9\pi}{2}}$ is presented in figure 2. Note that one can arrive at an L-point by first passing through a periodic orbit a number of times, and that, as a consequence, the L-curve may wind around $(\frac{1}{2},0)$ (a detailed analysis of this behaviour is presented later on).

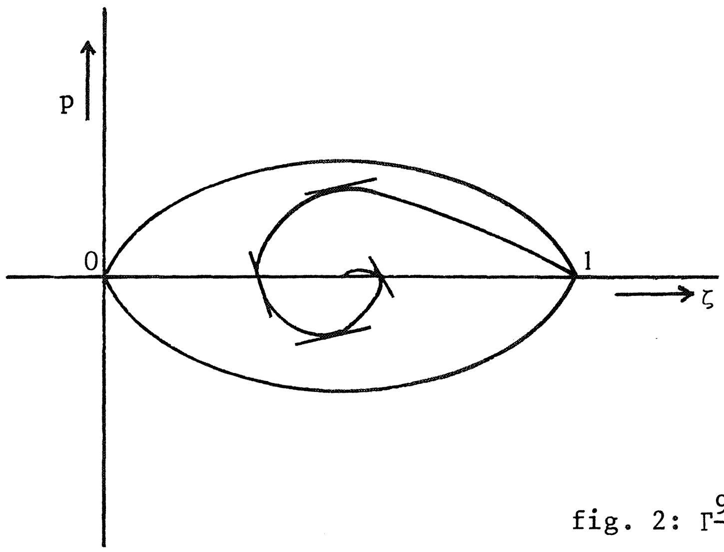


fig. 2: $\Gamma_{\frac{9\pi}{2}}$.

In the following, we shall pay special attention to the length of the parts of periodic orbits between a line $P(\zeta) = \frac{2}{\mu}(\zeta-\frac{1}{2})$, and $P = 0$ (including possibly one or more half periods!).

The first reason to do this is: Let $\rho(x,\omega)$ satisfy

$$\rho_x(0,\omega) = \frac{2}{\mu}(\rho(0,\omega)-\tfrac{1}{2}),$$

$$\rho_x(L,\omega) = 0,$$

then, defining $\rho(-x,\omega) = 1 - \rho(x,\omega)$ $x \in [0,L]$, $\rho(x,\omega)$ satisfies the steady state problem corresponding to (E.P.). The second reason is somewhat more intricate. In short, a tangential intersection of Γ_L and a line through $(\frac{1}{2},0)$ would lead to bifurcation of steady state solutions. However, corollary (1.5) will exclude such intersections for $\omega \neq \frac{1}{2}$.

The family of periodic orbits in the (ζ,P)-plane is given by

$$P^2(\zeta) = 2\int_\zeta^\omega f(\xi)d\xi$$

where the intersection point $(\omega,0)$ with $\frac{1}{2} < \omega < 1$ is used to parametrize the family.

The collection of straight lines through $(\frac{1}{2},0)$ can be parametrized by their slope λ as follows:

$$P(\zeta) = \lambda(\zeta-\tfrac{1}{2}) \quad -\infty < \lambda < \infty.$$

Each line intersects each periodic orbit in exactly two points which are found by solving

$$\lambda^2(\zeta-\tfrac{1}{2})^2 = 2\int_\zeta^\omega f(\xi)d\xi.$$

Let $\zeta(\lambda,\omega)$ denote the solution of this equation which satisfies $\zeta(\lambda,\omega) \geq \frac{1}{2}$ (note that one can fix ω and obtain $\zeta(\lambda,\omega)$ by the implicit function theorem and $\zeta(0,\omega)=\omega$).

We observe that

$$\frac{\partial\zeta}{\partial\lambda}(\lambda,\omega) = \frac{-\lambda(\zeta(\lambda,\omega)-\frac{1}{2})^2}{\lambda^2(\zeta(\lambda,\omega)-\frac{1}{2})+f(\zeta(\lambda,\omega))}$$

$$\frac{\partial\zeta}{\partial\omega}(\lambda,\omega) = \frac{f(\omega)}{\lambda^2(\zeta(\lambda,\omega)-\frac{1}{2})+f(\zeta(\lambda,\omega))} > 0$$

The first of these relations implies sign $\frac{\partial\zeta}{\partial\lambda}$ = sign $-\lambda$ and consequently

$$\zeta(\pm\infty,\omega) = \lim_{\lambda\to\pm\infty} \zeta(\lambda,\omega) = \tfrac{1}{2}.$$

Next, consider two lines through $(\frac{1}{2},0)$ with slopes λ_1, λ_2 such that

$$0 \le \lambda_2 < \lambda_1 \le \infty$$

and define

$$\zeta_i(\omega) = \zeta(\lambda_i,\omega), \quad i = 1,2.$$

The length of the piece of the periodic orbit with parameter ω between $\zeta_1(\omega)$ and $\zeta_2(\omega)$ is given by

$$\mathcal{D}(\omega,\lambda_1,\lambda_2) = \int_{\zeta_1(\omega)}^{\zeta_2(\omega)} \frac{du}{P(u)} = \int_{\zeta_1(\omega)}^{\zeta_2(\omega)} \frac{du}{\sqrt{2\int_u^\omega f(\xi)d\xi}}$$

<u>1.4. THEOREM</u>. *For* $0 \le \lambda_2 < \lambda_1 \le \infty$ *and* $\frac{1}{2} < \omega < 1$

$$\frac{\partial\mathcal{D}}{\partial\omega}(\omega,\lambda_1,\lambda_2) > 0.$$

Further, a straightforward computation shows that

$$\mathcal{D}(\omega,\lambda_1,\lambda_2) \to 2\,(\arctan 2\lambda_1 - \arctan 2\lambda_2) \text{ as } \omega\downarrow\tfrac{1}{2},$$

and

$$\mathcal{D}(1,\lambda_1,\lambda_2) = \sqrt{2}\left(\ell n \frac{\zeta_2}{1-\zeta_2} - \ell n \frac{\zeta_1}{1-\zeta_1}\right)$$

where

$$\zeta_i = \tfrac{1}{2} - \frac{\lambda_i}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2\lambda_i^2+1}.$$

We prove theorem (1.4) in the appendix.

Using the symmetry of the phase portrait we obtain a similar result for lines which are allowed to have a negative slope. So we can state the following: Take any pair of different lines ℓ_1 and ℓ_2 containing $(\frac{1}{2},0)$. Then starting in $(\frac{1}{2},0)$ and going outward, the length of trajectory parts between ℓ_1 and ℓ_2 is strictly increasing, as long as we remain on the periodic orbits inside the heteroclinic orbit connecting (0,0) and (1,0). See figure 3.

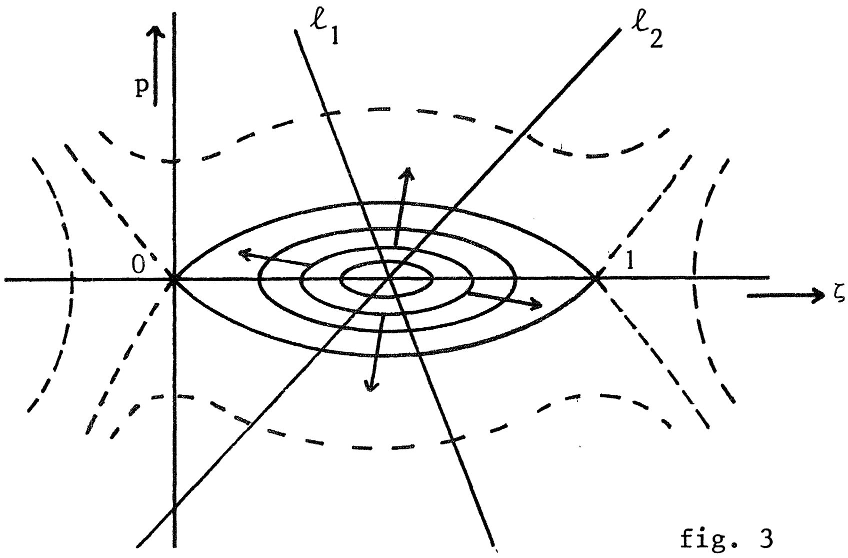


fig. 3

<u>1.5. COROLLARY</u>. *An intersection of* Γ_L *and a line* $P(\zeta) = \lambda(\zeta-\frac{1}{2})$, *not in* $(\frac{1}{2},0)$, *is nontangential.*

<u>PROOF</u>. For $\omega = 1$, this will be proved in lemma (1.9). Suppose, on the contrary, $P(\zeta) = \lambda(\zeta-\frac{1}{2})$ is the *tangent* to Γ_L at $(\rho(0,\omega), \rho_x(0,\omega))$, for some $\omega \in (\frac{1}{2},1)$. Then, one finds, for some $n \in \mathbb{N}_0$ (cf. fig. 4!)

$$\text{for } \lambda \geq 0 \quad \frac{\partial \mathcal{D}}{\partial \omega}(\omega,\lambda,0) + n\,\frac{\partial \mathcal{D}}{\partial \omega}(\omega,\infty,0) = 0,$$

$$\text{for } \lambda < 0 \quad \frac{\partial \mathcal{D}}{\partial \omega}(\omega,\infty,|\lambda|) + n\,\frac{\partial \mathcal{D}}{\partial \omega}(\omega,\infty,0) = 0.$$

But this clearly contradicts theorem (1.4). □

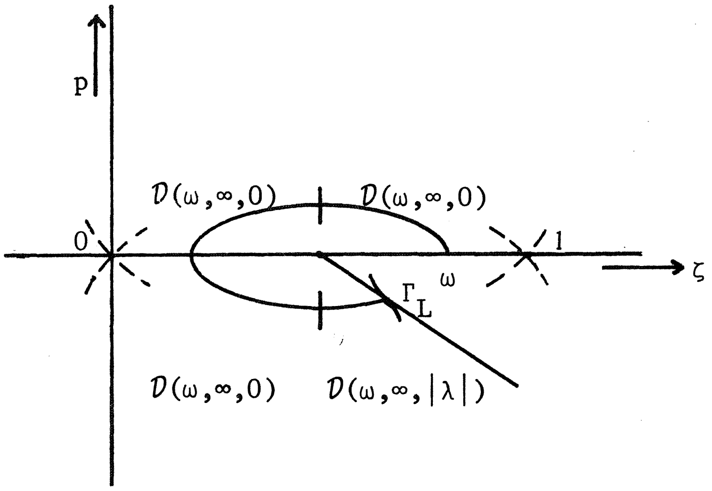


fig. 4: $\lambda < 0$, n=3

We need some more information about the intersection points of Γ_L and the lines $P = 0$ and $\zeta = \frac{1}{2}$. In order to give a precise formulation we introduce some notation.

1.6 DEFINITION. For $\frac{1}{2} < \omega < 1$, $\ell(\omega) = 4\,\mathcal{D}(\omega,\infty,0)$. So $\ell(\omega)$ is the *period* of the orbit through $(\omega,0)$.

By theorem (1.4), $\ell'(\omega) > 0$. In the sequel we shall use the subscript ω to indicate a derivative with respect to ω.

1.7 DEFINITION. For $\omega \in [\frac{1}{2},1]$, we define $S(\omega)$ to be the slope of Γ_L in the point $(\rho(0,\omega), \frac{\partial\rho}{\partial x}(0,\omega))$, i.e.

$$S(\omega) = \frac{\frac{\partial\rho_\omega}{\partial x}(0,\omega)}{\rho_\omega(0,\omega)} .$$

1.8 THEOREM. *For* $\omega \in (\frac{1}{2},1]$, $S(\omega) < 0$ *when* Γ_L *intersects the line* $P = 0$ *in* $\zeta = \rho(0,\omega)$. $S(\omega) > 0$ *when* Γ_L *intersects the line* $\zeta = \frac{1}{2}$ *in* $P = \frac{\partial\rho}{\partial x}(0,\omega)$.

PROOF. This result follows from the symmetry and periodicity of the functions $\rho(x,\omega)$, $\frac{1}{2} < \omega < 1$. For $\omega = 1$, we refer to lemma (1.9). Since for all four half-axes the proof is rather similar, we prove $S(\omega) < 0$, when Γ_L intersects $P = 0$ in $\zeta = \rho(0,\omega)$ with $0 < \rho(0,\omega) < \frac{1}{2}$, only.

In this case, for some $i \in \mathbb{N}_0$, $L = (i+\frac{1}{2})\ell(\omega)$. Hence

$$\rho(0,\omega) = \rho(L - (i+\tfrac{1}{2})\ell(\omega), \omega) = 1 - \omega,$$

$$\frac{\partial\rho}{\partial x}(0,\omega) = \frac{\partial\rho}{\partial x}(L - (i+\tfrac{1}{2})\ell(\omega), \omega) = 0,$$

and

$$\rho_\omega(0,\omega) - \frac{\partial\rho}{\partial x}(0,\omega).(i+\tfrac{1}{2})\ell'(\omega) = -1,$$

$$\frac{\partial\rho_\omega}{\partial x}(0,\omega) - \frac{\partial^2\rho}{\partial x^2}(0,\omega).(i+\tfrac{1}{2})\ell'(\omega) = 0.$$

Using $\frac{\partial\rho}{\partial x}(0,\omega) = 0$, $\frac{\partial^2\rho}{\partial x^2}(0,\omega) = -f(1-\omega) > 0$, and $\ell'(\omega) > 0$, we find

$$S(\omega) = \frac{-(i+\frac{1}{2})\ell'(\omega)f(1-\omega)}{-1} < 0. \quad \square$$

For $\omega = \frac{1}{2}$ or $\omega = 1$ we can solve (ω.L.P) explicitly; $\rho_\omega(x,\frac{1}{2}) = \cos\frac{1}{2}(x-L)$ and $\rho_\omega(x,1)=\cosh\frac{1}{2}\sqrt{2}(x-L)$. As a consequence it is an easy matter to calculate $S(\frac{1}{2})$ and $S(1)$:

<u>1.9 LEMMA</u>. $S(\frac{1}{2}) = \frac{1}{2}\tan\frac{1}{2}L$, $S(1) = -\frac{1}{2}\sqrt{2}\tanh\frac{1}{2}L\sqrt{2}$.

By the smoothness of Γ_L, $S(\omega)$ is a smooth function of $\omega \in [\frac{1}{2},1]$. Therefore a smooth anglefunction $\theta(\omega)$ is *uniquely* defined by $S(\omega) = \tan\theta(\omega)$, continuity and the normalization $\theta(1) = \operatorname{Arctan} S(1)$. See fig. 5.

<u>1.10 THEOREM</u>. $\theta'(\omega) < 0$ *for* $\omega \in (\frac{1}{2},1)$.

A proof of theorem (1.10), based on the Sturmian oscillation theorem and theorem (1.8) is given in the appendix.

We define a second anglefunction $\hat{\theta}(\omega)$ in the same way, using $\hat{S}(\omega) = \frac{\partial\rho/\partial x\ (0,\omega)}{\rho(0,\omega)-\frac{1}{2}}$ for $\omega = (\frac{1}{2},1]$, $\hat{S}(\frac{1}{2}) = \lim_{\omega\downarrow\frac{1}{2}}\hat{S}(\omega)$, and the normalization $\hat{S}(1) = 0$. So, in a sense, $\hat{\theta}(\omega)$ is the angle between the line through $(\frac{1}{2},0)$ and the point $(\rho(0,\omega), \frac{\partial\rho}{\partial x}(0,\omega))$ on the L-curve, and the ζ-axis.

By lemma (1.9) clearly $\hat{\theta}'(1) < 0$ and by corollary (1.5) we find $\hat{\theta}'(\omega) \neq 0$ for $\omega \in (\frac{1}{2},1]$. Therefore we can state the following

<u>1.11 THEOREM</u>. $\hat{\theta}'(\omega) < 0$ *for* $\omega \in (\frac{1}{2},1]$.

Again, see figure 5.

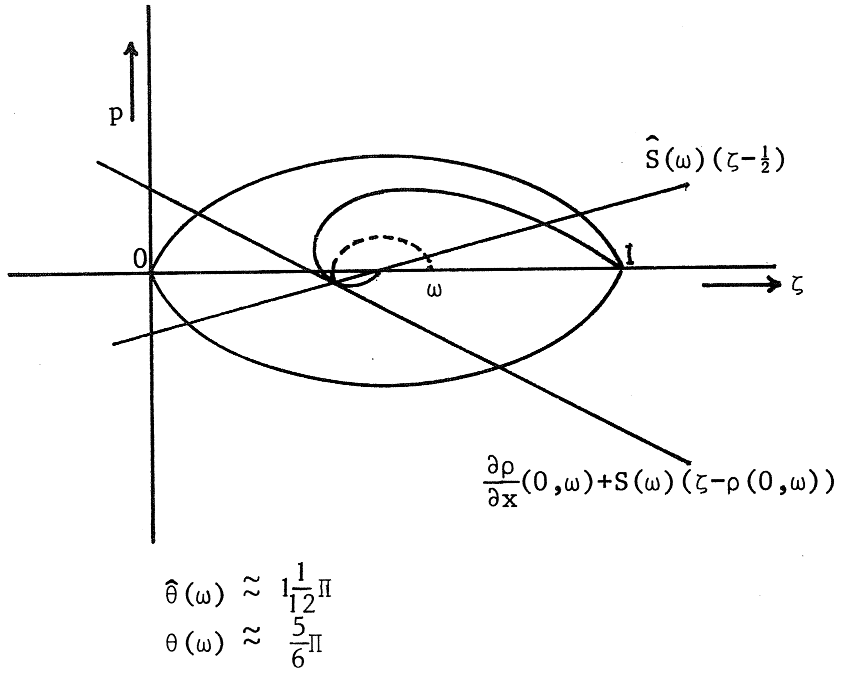


fig. 5: $\Gamma_{\frac{5\pi}{2}}$

The lines P = 0 and $\zeta = \frac{1}{2}$ devide the phase plane in four quadrants which we shall number counterclockwise as usual.

A critical point of Γ_L is a point $(\rho(0,\omega), \frac{\partial\rho}{\partial x}(0,\omega))$ where either $\rho_\omega(0,\omega)$ or $\frac{\partial\rho_\omega}{\partial x}(0,\omega)$ equals zero.

A quadrant component of Γ_L is a component of the intersection of Γ_L and a quadrant, without $(\frac{1}{2},0)$.

A combination of the theorems (1.8), (1.10) and (1.11) yields

1.12 COROLLARY. *Each quadrant component of* Γ_L *contains exactly one critical point. For the first quadrant, this is a maximum of* P, *for the second, a minimum of* ζ, *for the third a minimum of* P *and for the fourth a maximum of* ζ.

We end section 1 with a few simple observations.

1.13 COROLLARY. (i) *A trajectory piece reaching* (0,0) *or* (1,0) *has infinite length.* (ii) $\ell(\omega)$ *converges to* 4π *for* ω *tending to* $\frac{1}{2}$.

PROOF. This corollary follows immediately from lemma (1.4). Note that the first statement reflects the fact that (0,0) and (1,0) are equilibria of the system

$$\dot{y} = z$$
$$\dot{z} = -f(y). \quad \square$$

Using our knowledge of the value of $\ell(\frac{1}{2})$, the theorems (1.10) and (1.11) and corollary (1.12), we find the following

1.14 COROLLARY. $\theta(\frac{1}{2}) = \hat{\theta}(\frac{1}{2}) \in [n\pi, (n+1)\pi]$ *for* $L \in [2n\pi, 2(n+1)\pi]$ *or, in words, the* L-*curve winds* $[\frac{L}{4\pi}]$-*times around* $(\frac{1}{2},0)$.

Note that for any two periodic orbits and $i \in \mathbb{N}_0$, for L sufficiently large Γ_L spirals at least i times between these two periodic orbits.

1.15 REMARK. Note that for $\hat{\theta}(\omega) \in (n\pi, (n+1)\pi)$, $n \in \mathbb{N}_0$, $\frac{\partial \rho}{\partial x}(x,\omega)$ has exactly n zeros on (0,L), and that - within each quadrant - n follows at once from a numbering of the components, starting with the one at the outside. So the number of zeros of $\frac{\partial \rho}{\partial x}(x,\omega)$ on (0,L) is nonincreasing in ω, and it changes by one each time $\frac{\partial \rho}{\partial x}(0,\omega)$ passes the line P = 0.

To end section 1, note that in figure 2 we already used the knowledge about Γ_L derived so far.

2. THE STEADY STATE SOLUTIONS

Using the results of section 1, we shall obtain a fairly detailed picture of the steady state solutions of (E.P.). For these solutions we shall use the character q. In this section and in section 3, we shall often distinguish steady states with range in the interval [0,1] from other ones. The main reason is that, in fact, the reaction function f does not have a biological interpretation outside the interval [0,1] of its domain. Another reason can be found in proposition (2.2); note that only for $\mu > 0$ the transmission condition of (E.P.) models a geographical barrier.

By the symmetry of the steady state problem

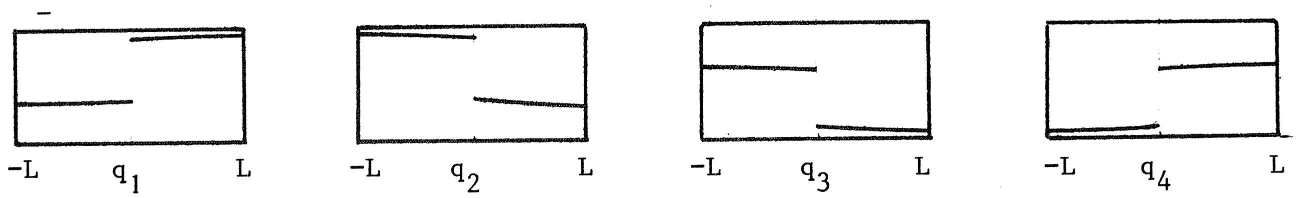
$$\text{S.S.P.}\begin{cases} q_{xx} + f(q) = 0 \quad x \in [-L,0) \cup (0,L] \\ q_x(-L) = 0 \\ q_x(L) = 0 \\ q_x(0-) = q_x(0+) = \frac{1}{\mu}(q(0+)-q(0-)) \end{cases}$$

each solution q_1 generates three more, not necessarily different, solutions q_i, $i = 2,3,4$ as follows

$$q_2(x) = q_1(-x)$$

$$q_3(x) = 1 - q_1(x) \quad \text{for } x \in [-L,0) \cup (0,L]$$

$$q_4(x) = 1 - q_1(-x):$$



For those solutions which have their range in [0,1] at least one of these four satisfies $(q(0+), q_x(0+)) \in \Gamma_L$. Therefore we can take this property as a normalization of *such* solutions. Note that, for q a normalized steady state solution,

$$(1-q(0-),\ q_x(0-)) \in \Gamma_L, \text{ when } q(-L) \le \tfrac{1}{2},$$

and

$$(q(0-),\ -q_x(0-)) \in \Gamma_L, \text{ when } q(-L) \ge \tfrac{1}{2}.$$

However, the normalization is *not* unique! E.g., in the preceding example, both q_1 and q_4 are normalized steady state solutions. Since this will cause no trouble (cf. theorem (3.4)), we shall put no further restrictions on normalized steady state solutions.

2.1 DEFINITION. A steady state solution q is called trivial when $q(x) = q(-L)$ for $x \in [-L,L]$. A *nontrivial* solution is called symmetric if $q(x) = q(-x)$ for $x \in [0,L]$, anti-symmetric if $q(x) = 1-q(-x)$ for $x \in (0,L]$, and a-symmetric when it is neither symmetric nor anti-symmetric.

2.2 PROPOSITION. *Let q be a steady state solution. When* $\mu > 0$, *the range of q is in the interval* $[0,1]$. *This statement continues to be true when* $\mu < 0$, *when q is symmetric. When q is trivial, it is one of the constant functions* 0, $\frac{1}{2}$ *or* 1.

PROOF. These properties of q follow immediately from the phase portrait of $\zeta_{xx} + f(\zeta) = 0$. Note that for a symmetric solution q, $q_x(0) = 0$. □

By proposition 2.2. for symmetric solutions of (S.S.P.), and for other solutions of (S.S.P.) when $\mu > 0$, we can restrict our analysis to normalized steady states (i.e. having range in $[0,1]$ and $(q(0+), q_x(0+)) \in \Gamma_L$).

2.3 PROPOSITION. *The normalized symmetric solutions of (S.S.P.) are uniquely determined by the points of intersection of the L-curve and the* ζ*-axis. The normalized anti-symmetric solutions of (S.S.P.) are uniquely determined by the intersections of the line* $P(\zeta) = \frac{2}{\mu}(\zeta-\frac{1}{2})$ *and the L-curve.*

PROOF. Since symmetric steady states satisfy and are determined by

$$
\begin{aligned}
q_{xx} + f(q) &= 0 \quad x \in [0,L] \\
q_x(0) &= 0 \\
q_x(L) &= 0
\end{aligned}
$$

the first statement is a direct consequence of the definition of Γ_L and the normalization. Since anti-symmetric solutions satisfy both (S.S.P.) and $q(x) = 1-q(-x)$, $x \in (0,L]$, they are found by solving

$$
\begin{aligned}
q_{xx} + f(q) &= 0 \quad x \in [0,L] \\
q_x(0) &= \frac{1}{\mu}(2q(0)-1) \\
q_x(L) &= 0
\end{aligned}
$$

Hence, the second assertion follows. □

In fact, in section 1, we have proved the properties of the L-curve which are relevant for finding all symmetric and anti-symmetric normalized steady state solutions. E.g., we have already proved the following

2.4 PROPOSITION. *For* $L \in (n.2\pi, (n+1)2\pi)$, $n \in \mathbb{N}_0$, *there exist exactly* n *normalized symmetric steady state solutions. For* $i \in \{0,1,\dots,n-1\}$ *there exists exactly one such solution with* i *zeroes of its derivative for* $x \in (0,L)$.
Of course, this is proved by the observation that Γ_L intersects $P = 0$ exactly $n + 2$ times (note the trivial steady states!) and $\hat{S}(\frac{1}{2}) \in (n\pi, (n+1)\pi)$.

With respect to anti-symmetric solutions, recall $\hat{\theta}'(\omega) < 0$ and note that there exists a normalized anti-symmetric steady state q with $q(L) = \omega$ iff $\hat{\theta}(\omega) = \arctan \frac{2}{\mu} + m\pi$ for some $m \in \mathbb{N}_0$. So using this, and the results of section 1 - more specifically, theorem (1.11) and remark (1.15) -, we can state the following

2.5 THEOREM. *For* $i \in \mathbb{N}_0$ *there is exactly one normalized anti-symmetric steady state solution* q *with* i *zeroes of* q_x *on* $(0,L)$ $(i \in \mathbb{N}_0)$ *iff*

$$\begin{cases} \mathrm{Arctan}\, \frac{2}{\mu} + i\pi < \hat{\theta}(\frac{1}{2}) & \text{when } \mu > 0 \\ \mathrm{Arctan}\, \frac{2}{\mu} + (i+1)\pi < \hat{\theta}(\frac{1}{2}) & \text{when } \mu < 0, \end{cases}$$

when this condition is not satisfied, there is no such solution.

Although we could give more results in terms of L, μ, $q(L)$ and the number of zeros of q_x, we prefer to end this discussion about normalized symmetric and anti-symmetric steady states with the figures 6 and 7, which will illuminate the main ideas.

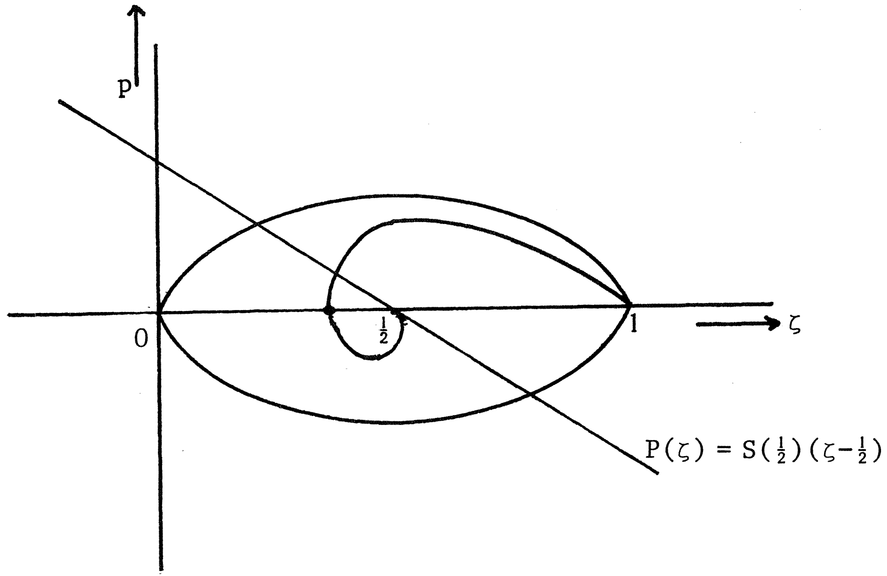


Fig. 6: $\Gamma_{\frac{7\pi}{2}}$

In figure 7 one finds a diagram showing the symmetric and anti-symmetric normalized solutions of (S.S.P.). We used $\omega = q(L)$ to represent such steady state solutions. Recall that $S(\frac{1}{2}) = \frac{1}{2}\tan\frac{1}{2}L$ (lemma (1.9)).

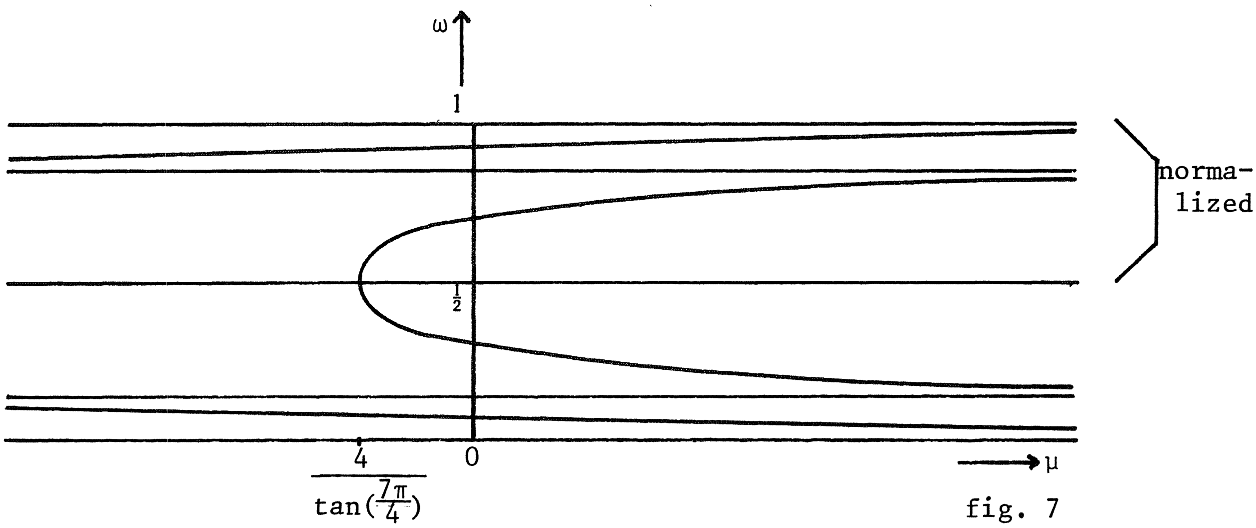


fig. 7

In this section, we have not yet paid attention to a-symmetric steady state solutions. However, it will turn out to be sufficient to obtain only a minimum of information about these solutions, in order to prove that most of the *normalized* a-symmetric steady states are unstable, (which we shall do in section 3). Note that such solutions are numerous for large values of L and μ, which most easily can be seen from Γ_L and its mirror images in $\zeta = \frac{1}{2}$ and $P = 0$; however,

2.6 THEOREM. *The set of steady state solutions is discrete.*

The proof of this theorem, which is based on f being analytic, is given in the appendix.

2.7 PROPOSITION. *Two branches of normalized a-symmetric steady state solutions can not intersect. Therefore, the only bifurcations which can occur along a branch of normalized a-symmetric steady state solutions are turning point bifurcations and intersections with branches of symmetric or anti-symmetric steady states.*

A proof of this theorem, based on our knowledge of Γ_L, is given in the appendix. The idea of the proof is the fact that at every a-*symmetric* steady state q on a branch, in a neighbourhood of q, μ is function of ω.

In section 3, a special class of a-symmetric normalized steady state solutions (in fact, a special *branch* of such solutions) has to be handled separately. We shall now introduce that class ad hoc, and derive some of its properties.

The following notations will be used frequently in the appendix; in the sequel of this section we only use ω_0.

Let

$$\tfrac{1}{2} < \omega_m < \omega_{m-2} < \dots < \omega_1$$

$$\tfrac{1}{2} < \omega_n < \omega_{n-2} < \dots < \omega_0$$

such that

$$\rho_\omega(0,\omega) = 0 \quad \text{iff } \omega \in \{\omega_m, \omega_{m-2} \quad \dots, \omega_1\}$$

$$\frac{\partial \rho_\omega}{\partial x}(0,\omega) = 0 \quad \text{iff } \omega \in \{\omega_n, \omega_{n-2}, \dots, \omega_0\}$$

and

$$\omega_1 := \tfrac{1}{2} \text{ for } L \leq \pi.$$

2.8 THEOREM. *Suppose* q *is a normalized* a-*symmetric steady state solution, for which the function* η *defined by*

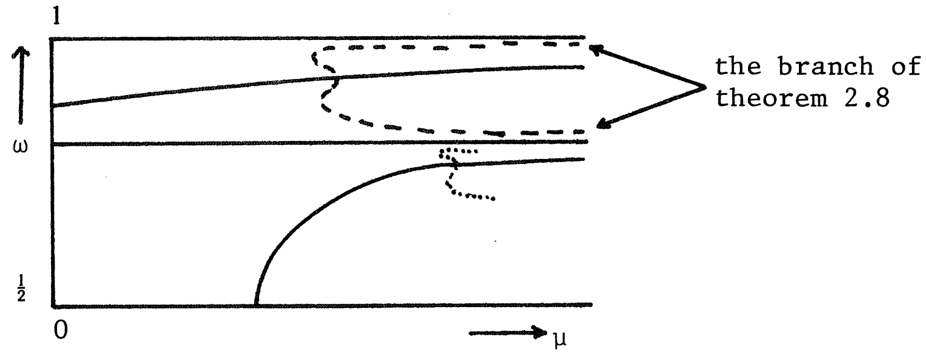
$$\eta_{xx} + f'(q)\eta = 0 \quad \text{on } [-L,0],\ [0,L]$$

$$\eta_x(-L) = \eta_x(L) = 0$$

$$\eta(-L) = \eta(L) = 1$$

does not change sign on either [-L,0] *or* [0,L]. *Then* q *is situated on exactly one branch in the* ω-μ-*diagram, which exists only for* $\mu > 0$.

We prove theorem 2.8 in the appendix: for its meaning, compare the following picture:



2.9 THEOREM (a sufficient condition). *When*

$$\mu > \frac{2\rho(0,\omega_0)}{\rho_x(0,\omega_0)}$$

there exist at most four (not all normalized!) a-*symmetric steady state solutions which satisfy the hypothesis of theorem (2.8). More specifically, the branch mentioned in theorem (2.8) does not contain turning points for such values of* μ.

We prove theorem 2.9 in the appendix. In fact, we would like to prove the following

2.10 Conjecture. The branch mentioned in theorem (2.8) does not contain any turning points at all.

This (unfortunately unproved) property would imply the stable nonconstant normalized steady states to be anti-symmetric, by the results of section 3. However, this property is proved valid for $\mu > \frac{2\rho(0,\omega_0)}{\rho_x(0,\omega_0)}$, and for $\mu < 0$ there do not exist such solutions at all, as will be seen in section 3. Moreover, it will be proved that for $\mu > \frac{\frac{1}{2}(\rho(0,\omega_0)-\frac{1}{2})}{\rho_x(0,\omega_0)}$ (which is larger than zero!) there does exist *exactly* one stable normalized anti-symmetric solution, for other values of μ such solutions do not exist.

3. THE STABILITY OF STEADY STATE SOLUTIONS

Formally, we can state (E.P.) as

$$\frac{du}{dt} + A(u) = f(u) \quad t > 0$$

$$u(0) = \psi(x)$$

where

$$A: \mathcal{D}(A) \to X$$

is defined by

$$Au = -u_{xx},$$

with

$$X = C[-L,0] \times C[0,L], \ \|u\| = \sup\{|u(x)| \mid x \in [-L,0], [0,L]\},$$

and

$$\mathcal{D}(A) = \{u \in C^2[-L,0] \times C^2[0,L] \mid u_x(-L) = u_x(L) = 0$$
$$u_x(0-) = u_x(0+) = \frac{1}{\mu}(u(0+)-u(0-))\}.$$

In section 4 we shall prove this operator A to be sectorial in the Banach space X. By this result, it is justified to infer the stability of a steady state solution q from an analysis of the spectrum of $-A + f'(q)$.

Note that for $c \in X$ we also use c for the bounded operator on X defined by

$$u \to c.u: \qquad (c.u)(x) = c(x)u(x)$$
$$x \in [-L,0], [0,L].$$

In fact, the steady state solution q is asymptotically stable when $\sigma(-A+f'(q))$ contains elements with real part strictly less than zero only, and is unstable when $\sigma(-A+f'(q))$ contains an element with real part larger than zero.

For more background on these techniques we refer to [9]. Also in section 4 we shall prove, for q a steady state solution,

$$\sigma(-A+f'(q)) = P\sigma(-A+f'(q)) \subset \mathbb{R}.$$

Let Y = C[0,L] and let A_s and A_a be the (unbounded) operators on Y, with

$$\mathcal{D}(A_s) = \{\eta \in C^2[0,L] \mid \eta_x(0) = 0,\ \eta_x(L) = 0\}$$
$$\mathcal{D}(A_a) = \{\eta \in C^2[0,L] \mid \eta_x(0) = \frac{2}{\mu}\eta(0),\ \eta_x(L) = 0\},$$

and $A_s\eta = -\eta_{xx}$, $A_a\eta = -\eta_{xx}$.

Again, for $c \in Y$ (or X!) we also use c for the bounded operator on Y defined by

$$\eta \to c.\eta\colon\ (c.\eta)(x) = c(x)\eta(x)\quad x \in [0,L].$$

3.1 LEMMA. *For* q *a trivial, symmetric or anti-symmetric steady state solution*

$$P\sigma(-A+f'(q)) = P\sigma(-A_s+f'(q)) \cup P\sigma(-A_a+f'(q)).$$

PROOF. By the symmetry of f and q, we have

$$f'(q(x)) = f'(q(-x)) \text{ for } x \in [0,L].$$

Now, one can easily see that $P\sigma(-A_s+f'(q))$ corresponds to elements of $P\sigma(-A+f'(q))$ with symmetric eigenfunctions, $P\sigma(-A_a+f'(q))$ to elements with anti-symmetric eigenfunctions. Moreover, assume $\lambda \in P\sigma(-A+f'(q))$, $\eta \in \mathcal{D}(A)$ an eigenfunction. Define $\hat{\eta}(x) = \eta(-x)$ for $x \in [-L,0]$, $[0,L]$. Then $\eta(x) + \hat{\eta}(x)$ is either nonzero symmetric and an eigenfunction for λ, or $\eta(x) = -\eta(-x)$ for $x \in [0,L]$. But this proves

$$P\sigma(-A+f'(q)) \subset P\sigma(-A_s+f'(q)) \cup P\sigma(-A_a+f'(q)). \qquad \square$$

3.2 LEMMA. *Let* c *be a not necessarily symmetric (or anti-symmetric) element of* X.
Then:

1) $P\sigma(-A+c)$, $P\sigma(-A_s+c)$ *and* $P\sigma(-A_a+c)$ *are bounded above,*

2) *when* $P\sigma(-A_a+c) = \ldots \gamma_3 < \gamma_2 < \gamma_1 < \gamma_0$

   $P\sigma(-A_s+c) = \ldots \lambda_3 < \lambda_2 < \lambda_1 < \lambda_0$

   for $\mu > 0 \ldots \gamma_1 < \lambda_1 < \gamma_0 < \lambda_0$

   for $\mu < 0 \ldots \lambda_1 < \gamma_1 < \lambda_0 < \gamma_0$

3) *the eigenfunction of* $-A_s+c$ *corresponding to* λ_n *has exactly* n *zeros. moreover* $\lambda_n \to -\infty$ *for* $n \to \infty$.

4) *the eigenfunction of* $-A+c$ *corresponding to its dominant (largest) eigenvalue does neither change sign on* $[-L,0]$ *nor on* $[0,L]$.

Using the property $P\sigma(-A+c)$, $P\sigma(-A_s+c)$, $P\sigma(-A_a+c) \subset \mathbb{R}$, which we shall prove in section 4, we prove this lemma in the appendix. Note that by normalizing *all* eigenfunctions η by demanding

$$\eta(L) = 1,$$

one simply proves the geometric multiplicity of an eigenvalue to be 1.

<u>3.3 LEMMA</u>. *For* $c \in Y$, *the <u>dominant</u> eigenvalue of* $(-A_s+c)$ *belongs to the range of* c.

<u>PROOF</u>. Let $\mathcal{R}(c) = [\alpha,\beta]$, then for $\lambda > \beta$, we apply the maximum principle. For $\lambda < \alpha$, and η an eigenfunction for λ, note that η_{xx} has at least one zero on $(0,L)$. But since $\eta_{xx} = (\lambda-c)\eta$ and $\lambda-c < 0$ on $[0,L]$, η has at least one zero on $(0,L)$. Using lemma (3.2)3), this proves lemma (3.3). □

<u>3.4 THEOREM</u>. *For* q_1 *a steady state solution, and* q_2, q_3 *and* q_4 *as defined in section 2,*

$$P\sigma(-A+f'(q_1)) = P\sigma(-A+f'(q_2)) = P\sigma(-A+f'(q_3)) = P\sigma(-A+f'(q_4)).$$

<u>PROOF</u>. Since

$$f'(1-\xi) = f'(\xi) \text{ for all } \xi,$$

$$P\sigma(-A+f'(q_1)) = P\sigma(-A+f'(q_3))$$

and

$$P\sigma(-A+f'(q_2)) = P\sigma(-A+f'(q_4)).$$

Suppose $\eta_1 \in \mathcal{D}(A)$ satisfies

$$\eta_{xx} + f'(q_1)\eta = \lambda\eta$$

then $\eta_2 \in \mathcal{D}(A)$ defined by $\eta_2(x) = \eta_1(-x)$ for $x \in [-L,0]$, $[0,L]$, clearly satisfies

$$\eta_{xx} + f'(q_1(-x))\eta = \lambda\eta$$

i.e.,

$$P\sigma(-A+f'(q_1)) = P\sigma(-A+f'(q_2)). \quad \square$$

3.5 THEOREM. *Let* q *be a normalized steady state solution. Then* 0 *is an eigenvalue of* $-A + f'(q)$ *if and only if*

$$\mu = \frac{\rho_\omega(0,\omega)}{\frac{\partial\rho_\omega}{\partial x}(0,\omega)} + \frac{\rho_\omega(0,\alpha)}{\frac{\partial\rho_\omega}{\partial x}(0,\alpha)} \quad \textit{or} \quad \frac{\partial\rho_\omega}{\partial x}(0,\omega) = \frac{\partial\rho_\omega}{\partial x}(0,\alpha) = 0$$

where

$$\alpha = q(-L) \quad \textit{when } q(-L) \geq \tfrac{1}{2},$$

$$\alpha = 1-q(-L) \textit{ when } q(-L) < \tfrac{1}{2}, \textit{ and } \omega = q(L).$$

Note that the subscript ω *denotes a derivative with respect to the second argument of* ρ *(recall definition (1.1)).*

We prove theorem (3.5), using the proof of theorem (2.8), in the appendix.

3.6 COROLLARY. *Let* q *be a normalized steady state solution and* $\omega = q(L)$, *then*

$$0 \in P\sigma(-A_s+f'(q)) \textit{ iff } \frac{\partial\rho_\omega}{\partial x}(0,\omega) = 0$$

$$0 \in P\sigma(-A_a+f'(q)) \textit{ iff } \frac{2}{\mu} = \frac{\frac{\partial\rho_\omega}{\partial x}(0,\omega)}{\rho_\omega(0,\omega)} .$$

However, note that knowledge of $P\sigma(-A_s+f'(q))$ *or* $P\sigma(-A_s+f'(q))$ *is useful, only for* q *a trivial, symmetric or anti-symmetric steady state solution.*

The trivial steady state solutions.

3.7 LEMMA.

$$P\sigma(-A_s+f'(1)) \subset (-\infty,-\tfrac{1}{2}].$$

$$P\sigma(-A_a+f'(1)) \subset (-\infty,0) \text{ for } \mu \in \mathbb{R}\backslash[-\frac{2\sqrt{2}}{\tanh\frac{1}{2}L\sqrt{2}}, 0],$$

and it contains exactly one positive element for

$$\mu \in (-\frac{2\sqrt{2}}{\tanh\frac{1}{2}L\sqrt{2}}, 0).$$

Moreover, $P\sigma(-A_a+f'(1))$ *changes discontinuously for* $\mu = 0$.

PROOF. Since $f'(1) = -\frac{1}{2}$, the first statement is proved by lemma (3.3). For

$$\lambda > -\tfrac{1}{2}, \quad \lambda \in P\sigma(-A_a-\tfrac{1}{2}) \quad \text{iff} \quad \eta(x) = \cosh(\sqrt{\tfrac{1}{2}+\lambda}(x-L))$$

satisfies

$$\eta_x(0) = \frac{2}{\mu}\eta(0), \quad \text{i.e. } \mu = -\frac{2}{\sqrt{\frac{1}{2}+\lambda}\tanh\sqrt{\frac{1}{2}+\lambda}L}.$$

This also proves the last statement, which is a consequence of using μ instead of $\frac{1}{\mu}$ as parameter. □

3.8 REMARK. For $\mu = -\frac{2\sqrt{2}}{\tanh\frac{1}{2}L\sqrt{2}}$ a branch of a-symmetric steady states bifurcates from the constant steady state 1. However, the range of these steady states lies partially outside the interval [0,1], and for that reason we shall pay no attention to them.

3.9 LEMMA. *For* $L \in (n2\pi, (n+1)2\pi)$, $n \in \mathbb{N}_0$, $P\sigma(-A_s+f'(\frac{1}{2}))$, *which is contained in* $(-\infty, \frac{1}{4}]$, *has exactly* n + 1 *elements larger than zero. Further* $0 \in P\sigma(-A_a + f'(\frac{1}{2}))$ iff $\mu = \frac{4}{\tan\frac{1}{2}L}$.

PROOF. By lemma (3.3), since $f'(\frac{1}{2}) = \frac{1}{4}$, $P\sigma(-A_s+\frac{1}{4}) \subset (-\infty\frac{1}{4}]$. Further,

$$\lambda \in P\sigma(-A_s+\tfrac{1}{4}) \text{ iff } \eta(x) = \cos(\sqrt{\tfrac{1}{4}-\lambda}(x-L)) \text{ satisfies}$$

$$\eta_x(0) = 0, \quad \text{i.e.}$$

$$\sqrt{\tfrac{1}{4}-\lambda}\sin\sqrt{\tfrac{1}{4}-\lambda}\,L = 0, \text{ i.e.}$$

$$\sqrt{1-4\lambda}\,L = m.2\pi \text{ for some } m \in \mathbb{N}_0.$$

To prove the last assertion, $0 \in P\sigma(-A_a + \frac{1}{4})$ iff

$\eta(x) = \cos \frac{1}{2}(x-L)$ satisfies

$$\eta_x(0) = \frac{2}{\mu}\eta(0), \text{ i.e., } \mu = \frac{4}{\tan \frac{1}{2} L}. \quad \square$$

We summarize some of the results obtained so far in

3.10 THEOREM. $q \equiv \frac{1}{2}$ *is unstable for all* μ,

$q \equiv 1$ *and* $q \equiv 0$ *are stable for* $\mu \in \mathbb{R} \setminus [-\frac{2\sqrt{2}}{\tanh \frac{1}{2}L\sqrt{2}}, 0]$

and unstable for $\mu \in (-\frac{2\sqrt{2}}{\tanh \frac{1}{2}L\sqrt{2}}, 0]$.

In this theorem, we used theorem (3.4) to obtain results for $q \equiv 0$ from those for $q \equiv 1$.

The symmetric steady state solutions.

3.11 THEOREM. *Every symmetric steady state solution* q *is unstable.*

PROOF. By proposition (2.2) and theorem (3.4) we can assume q to be normalized, without losing generality. Let q be represented by $(q(0), q_x(0)) = (q(0), 0) \in \Gamma_L$. Using – only in this proof – L as bifurcation parameter, one finds q on a *branch* of symmetric steady states bifurcation from $\frac{1}{2}$.
Lemma (3.3) implies that symmetric steady states, in a neighbourhood of $\frac{1}{2}$ are unstable, since the associated operator $(-A_s+f'(q))$ will have a positive eigenvalue. We prove the theorem by showing that 0 can not be an element of $P\sigma(-A_s+f'(\tilde{q}))$, for any symmetric steady state $\tilde{q}$ corresponding to some $\tilde{L} > 0$. Let $\omega = \tilde{q}(\tilde{L})$, then

$$\tilde{q}(x) \mid_{[0,\tilde{L}]} = \tilde{\rho}(x,\omega),$$

where $\tilde{\rho}$ is as in definition (1.1), using $\tilde{L}$ instead of L. By corollary (3.6)

$$0 \in P\sigma(-A_s+f'(q)) \quad \text{iff} \quad \frac{\partial \tilde{\rho}_\omega}{\partial x}(0,\omega) = 0.$$

However, by theorem (1.8), this cannot occur. $\square$

As a corollary of theorem (1.8) and corollary (3.6) we have the following

3.12 PROPOSITION. *Let* q *be a normalized symmetric steady state and* $\omega = q(L)$.

Then $0 \in P\sigma(-A_a+f'(q))$ *iff* $\frac{2}{\mu} = S(\omega)$. *For this value of* μ, *which is less than zero since* $S(\omega) < 0$ *(see theorem (1.8)), two a-symmetric steady state solutions bifurcate from* q.

In figure 8 and figure 9 we sketch the way a-symmetric steady states bifurcate from a symmetric steady state q, for μ as in proposition (3.12).

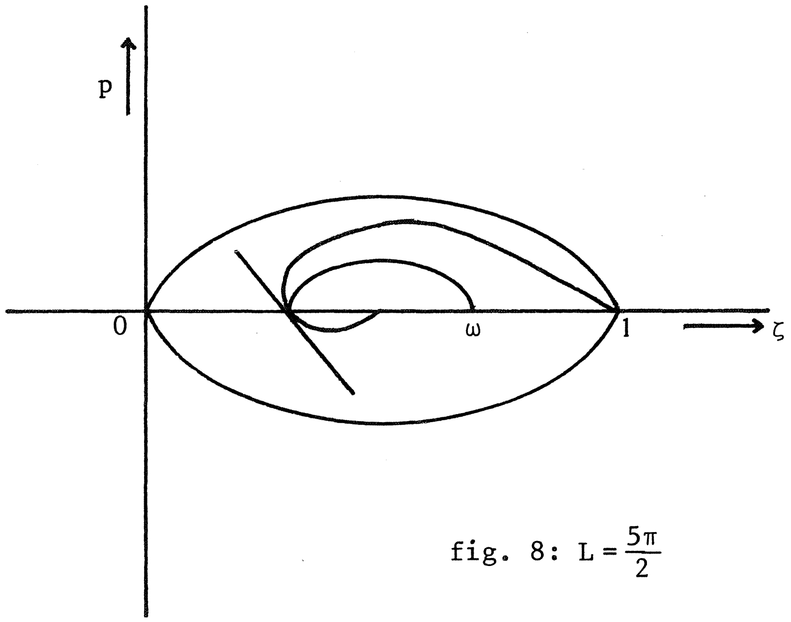


fig. 8: $L = \frac{5\pi}{2}$

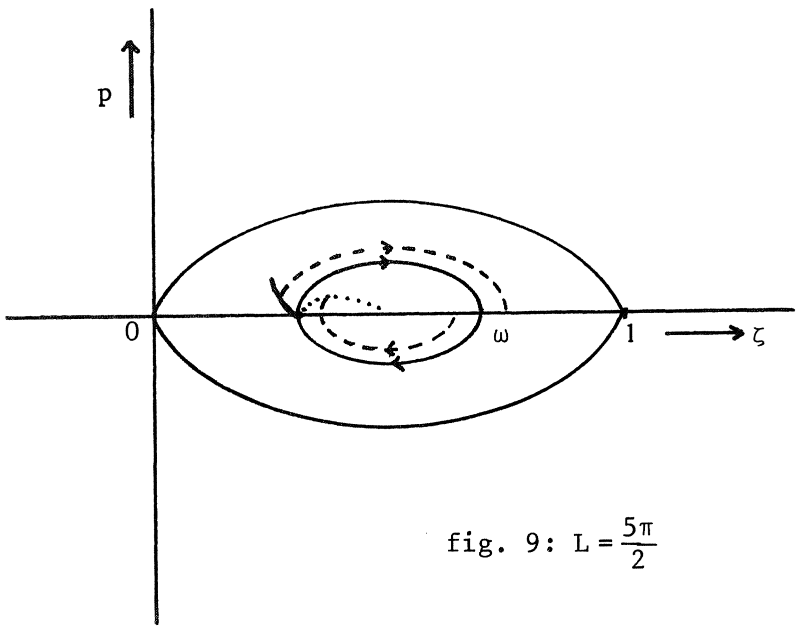


fig. 9: $L = \frac{5\pi}{2}$

In figure 9, we sketched both Γ_L and part of its image with respect to the line P = 0. Since $\frac{2}{\mu} = S(\omega)$, two points (ζ_1,p_1), (ζ_2,p_2), on $P(\zeta) = S(\omega)(\zeta-q(0))$ and $P(\zeta) = -S(\omega)(\zeta-q(0))$ respectively, with $p_1 = p_2$ satisfy

$$p_1 = p_2 = \frac{1}{\mu}(\zeta_1-\zeta_2).$$

Using $P(\zeta) = S(\omega)(\zeta-q(0))$, $P(\zeta) = -S(\omega)(\zeta-q(0))$ being tangent on Γ_L, respectively its mirror image in P = 0, at (q(0),0), this should explain figure 9.

The anti-symmetric steady state solutions.

In this subsection, we shall consider normalized anti-symmetric steady state solutions q (so note in particular that q has its range in the interval [0,1]).

Some of our results will be stated in terms of $\rho(x,\omega)$, but these can easily be translated in terms of anti-symmetric steady states by means of the results of section 2.

As a first straightforward corollary of lemma (3.3) and lemma (3.9), we have the following

3.13 PROPOSITION. $P\sigma(-A_s+f'(\rho(.,\omega)))$ *depends continuously on* $\omega \in [\frac{1}{2},1]$. *For* $L \in (n.2\pi, (n+1).2\pi)$, *it contains exactly* n+1 *elements larger than zero for* $\omega-\frac{1}{2} \ll 1$, *and no such elements for* $1-\omega \ll 1$.

The following proposition is in fact, due to our - more or less - detailed knowledge of Γ_L.

3.14 PROPOSITION. *Let* $\lambda_i(\omega)$ *denote the* i-*th element of* $P\sigma(-A_s+f'(\rho(.,\omega)))$. *Then, when*

$$\lambda_i(\omega) = 0,$$
$$\lambda_i'(\omega) < 0,$$

i.e., the number of positive elements of $P\sigma(-A_s+f'(\rho(.,\omega)))$ *is non-increasing in* ω.

PROOF. For $L \in (n.2\pi, (n+1).2\pi)$, $n \in \mathbb{N}_0$, there exist values

$$\omega_{2n} < \omega_{2n-2} < \ldots < \omega_2 < \omega_0$$

such that $\frac{\partial\rho_\omega}{\partial x}(0,\omega) = 0$ iff $\omega = \omega_i$ for some $i \in \{0,\ldots,2n\}$, by corollary (1.12) and corollary (1.14). By corollary (3.6) and proposition (3.13), proposition (3.14) follows. □

<u>3.15 PROPOSITION</u>. *For* $\omega \in (\frac{1}{2},1)$, $\frac{2}{\mu} = \hat{S}(\omega)$, 0 *is not contained in* $P\sigma(-A_a+f'(\rho(.,\omega)))$.

<u>PROOF</u>. By lemma (3.6), $0 \in P\sigma(-A_a+f'(\rho(.,\omega)))$ should imply $S(\omega) = \frac{2}{\mu}$. However, this clearly contradicts corollary (1.5). □

<u>3.16 THEOREM</u>. *As in proposition (3.14), let*

$$\omega_0 = \max\{\omega \mid \frac{\partial\rho_\omega}{\partial x}(0,\omega) = 0\},$$

and suppose q *is a normalized anti-symmetric steady state solution. Then, when* $q(L) > \omega_0$, q *is asymptotically stable, when* $q(L) < \omega_0$, q *is unstable.*

<u>PROOF</u>. Since $P\sigma(-A_s+f'(q)) = P\sigma(-A_s+f'(\rho(.,\omega)))$ for $\omega = q(L)$, the last assertion is proved by proposition (3.14). Suppose $q(L) > \omega_0$, then, by proposition (3.13) and proposition (3.14), using $\omega = q(L)$,

$$P\sigma(-A_s+f'(q)) \subset (-\infty,0).$$

By corollary (1.12), $q(L) > \omega_0$ implies q to be a solution of (S.S.P.) for some $\mu > 0$. Apply lemma (3.2) to conclude

$$P\sigma(-A_a+f'(q)) \subset (-\infty,0),$$

and in combination with lemma (3.1), this proves theorem (3.16). □

Of course, a normalized anti-symmetric steady state solution is determined by its part on [0,L]. Note that theorem (3.16) states that such a solution is stable when *that part* starts on Γ_L between the point $(\rho(0,\omega_0), \rho_x(0,\omega_0))$ and the point (1,0), otherwise, it is unstable! Note that the *stable* anti-symmetric steady states are monotone!
It may also give some clarification, to sketch the way a branch of a-symmetric steady states bifurcates from a branch of normalized anti-symmetric steady states, at an anti-symmetric solution q, such that $\frac{\partial\rho_\omega}{\partial x}(0,q(L)) = 0$, i.e. the point $(q(0+), q_x(0))$ is a maximum (or minimum) of P along Γ_L.
See figure 10.

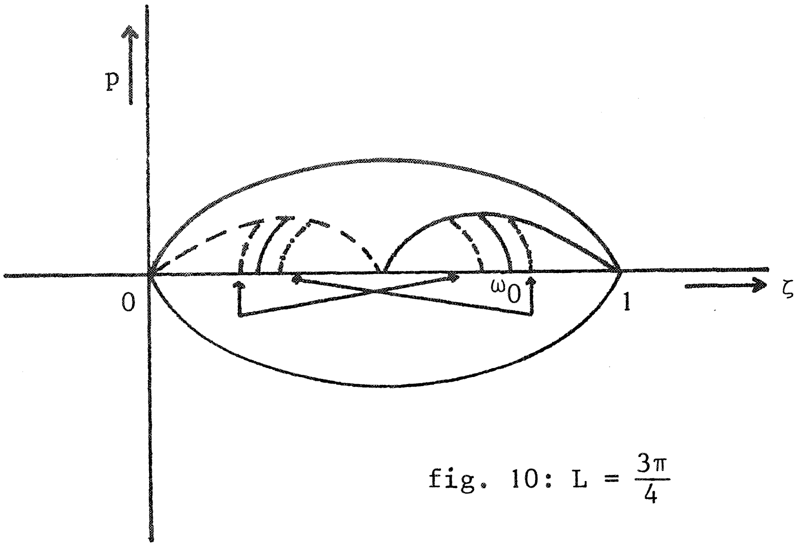


fig. 10: $L = \frac{3\pi}{4}$

When we choose $\omega \sim \omega_0$ such that $\omega - \omega_0 \ll 1$, then there exists exactly one *a-symmetric* solution $\tilde{q}$ with

$$\|\tilde{q} - q\| \text{ small and } \tilde{q}(L) = \omega.$$

Moreover $\left|\frac{\tilde{q}(0+)-\tilde{q}(0-)}{\tilde{q}_x(0)} - \frac{2}{\hat{S}(\omega_0)}\right|$ will be small and tends to zero, for $\omega \downarrow \omega_0$. This should explain the sketch figure 10.

Note that for $\omega = \omega_0$, the branch of a-symmetric solutions bifurcating from the anti-symmetric solution with $q(L) = \omega_0$ is the branch mentioned in theorem (2.8), theorem (2.9) and conjecture (2.10)!

<u>3.17 THEOREM</u>. *For* $\mu > \frac{2(\rho(0,\omega_0)-\frac{1}{2})}{\rho_x(0,\omega_0)}$ *exactly one stable normalized anti-symmetric steady state solution exists, for other values of* μ *such solutions do not exist.*

Moreover, this solution is strictly increasing on $(-L,0]$, $[0,L)$ *and converges, uniformly on* $[-L,0]$, $[0,L]$ *to* q_{01} *- which is shown in the introduction - when* μ *tends to infinity.*

<u>PROOF</u>. For the proof of this theorem, consider theorem (3.16), and recall the discussion about the steady state solutions in section 2, which includes the last assertion. □

As can easily be seen in figure 1, for some $\mu < 0$, there do exist anti-symmetric steady state solutions with range completely outside the interval $[0,1]$.

The a-symmetric steady state solutions.

3.18 LEMMA. *Let* q *be a normalized a-symmetric steady state solution, and define*

$$\omega = q(L)$$

$$\alpha = \begin{cases} q(-L) \text{ when } q(-L) > \frac{1}{2} \\ 1-q(-L) \text{ when } q(-L) < \frac{1}{2}. \end{cases}$$

When either $\rho_\omega(x,\omega)$ *or* $\rho_\omega(x,\alpha)$ *has at least one zero on* $[0,L]$, *zero can be an eigenvalue of* $(-A+f'(q))$ *but* not *the dominant one.*

PROOF. As in the proof of theorem (2.8) and theorem (3.5), when $0 \in P\sigma(-A+f'(q))$, the eigenfunction η, normalized by $\eta(L) = 1$, satisfies

$$\eta(x) = \rho_\omega(0,\omega) \qquad x \in [0,L],$$

$$\eta(x) = -\rho_\omega(-x,\alpha) \cdot \frac{\frac{\partial\rho_\omega}{\partial x}(0,\omega)}{\frac{\partial\rho_\omega}{\partial x}(0,\alpha)} \qquad x \in [-L,0].$$

Now we can apply lemma (3.2) 4) which proves the lemma. □

Note that all normalised a-symmetric steady state solutions, *except (partly) those on the branch mentioned in theorem (2.8),* do satisfy the hypothesis of lemma (3.18).

3.19 THEOREM. *All normalized a-symmetric steady state solutions, not on the branch mentioned in theorem (2.8), are unstable.*

PROOF. Suppose such an a-symmetric solution is on a branch, bifurcating from either a symmetric steady state or an anti-symmetric steady state which is *not* monotone. Then, since such a branch will contain *unstable* a-symmetric solutions close to the bifurcation point, it will contain *only* unstable a-symmetric solutions by lemma 3.18.
(Note that the branch mentioned in theorem (2.8) is excluded in this proof!)
Otherwise, suppose such an a-symmetric solution on a branch which exists *merely* of a-symmetric solutions (which necessarily all will be normalized!).

Then that branch can *not* cross the line $\mu = 0$ (only branches consisting of symmetric *or* a-symmetric solutions can). One can easily prove this by the observation that Γ_L and its mirror images in either $\zeta = \frac{1}{2}$ or $P = 0$ intersect *only* on the line $\zeta = \frac{1}{2}$ and $P = 0$ respectively, which can be proved by corollary (1.12), theorem (1.8) and the simple fact that Γ_L intersects each periodic orbit exactly once.
Hence, such a branch does contain at least one turning point bifurcation for an a-symmetric solution q on it, satisfying the hypothesis of lemma (3.18), and hence q is unstable. But then, again lemma (3.18) proves all solutions on that branch to be unstable. □

3.20 REMARK. There do exist branches of a-symmetric solutions only, for L sufficiently large. A simple way to prove this is to show the existence of a solution q such that the function q_x has at least two more zeros on $[-L,0]$ than on $[0,L]$. Such a solution q cannot exist on a branch bifurcating from either a symmetric or an a-symmetric solution, as can easily be seen.

3.21 THEOREM (a sufficient condition). *For* $\mu > \dfrac{2\rho(0,\omega_0)}{\rho_x(0,\omega_0)}$, *all normalized a-symmetric steady state solutions are unstable.*

PROOF. A simple observation of the proofs of theorem (3.2) 4) and theorem (3.5) shows that - using $\omega = q(L)$, $\alpha = 1-q(-L)$, since we are dealing only with the special branch mentioned in theorem (2.8) -, when

$$\mu > \frac{\rho_\omega(0,\omega)}{\frac{\partial\rho_\omega}{\partial x}(0,\omega)} + \frac{\rho_\omega(0,\alpha)}{\frac{\partial\rho_\omega}{\partial x}(0,\alpha)},$$

then q is unstable.
We can use this for μ sufficiently large, since $\frac{1}{S(\alpha)} + \frac{1}{S(\omega)}$ will be bounded above, for such values of μ. So in fact we proved for *all* L, and μ suffisiently large, the solutions on the special branch of theorem (2.8) to be unstable.
But since stability does not change on this branch for $\mu > \dfrac{2\rho(0,\omega_0)}{\rho_x(0,\omega_0)}$ by absence of bifurcations, this proves the theorem. □

3.22 CONJECTURE (and a corollary of conjecture (2.10)). All normalized a-symmetric solutions are unstable by absence of bifurcations of the special branch of theorem (2.8) for $\mu \neq \dfrac{2(\rho(0,\omega_0)-\frac{1}{2})}{\rho_x(0,\omega_0)}$.

The discussion about the special branch in theorem (2.8), theorem (3.17) and theorem (3.19), proves the following

3.23 REMARK. For a normalized steady state solution q a *necessary* condition for it to be stable is q is strictly monotone on (-L,0], [0,L).

We end this section with a bifurcation-diagram for $L = \frac{7\pi}{2}$ (compare with figure 6 and figure 7).
In this diagram (see figure 11)

- -steady states, whose ranges are not contained in [0,1],
- -the (unstable!) branches, merely existing of normalized a-symmetric solutions,
- -the branches of a-symmetric solutions bifurcating from either symmetric or anti-symmetric solutions

are (partly) deleted.
Further, this diagram is made *under the assumption of conjecture (3.22)*. However, this only effects the figure with respect to the branch of a-symmetric solutions, which appears from the change of stability of the anti-symmetric solutions, and moreover, only for $0 < \mu < \dfrac{2\rho(0,\omega_0)}{\rho_x(0,\omega_0)}$.

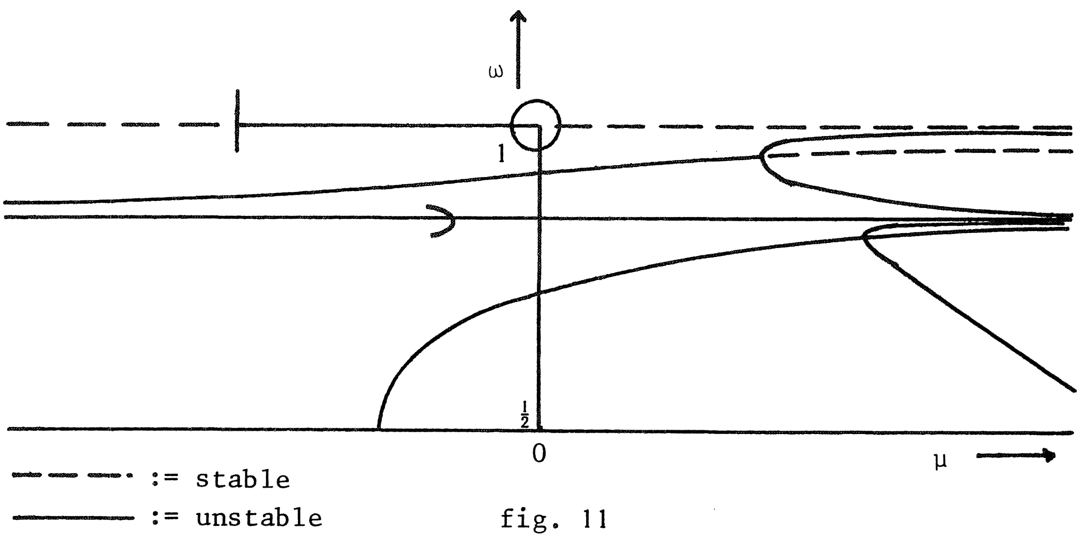


fig. 11

4. THE INITIAL VALUE PROBLEM, A LYAPUNOV FUNCTIONAL AND THE ASYMPTOTIC BEHAVIOUR

To prove some spectral properties of the operators $(-A+c)$ on X, $(-A_a+c)$, $(-A_s+c)$ on Y, we extend those operators to the complexification of X and Y, which we denote by X_c and Y_c. For

$$v \in X_c, \quad \|v\| = \max_{x \in [-L,0],[L,0]} \{|v(x)|\}.$$

We provide X_c with the inner-product

$$\langle v,w\rangle = \int_{-L}^{0} v(x)\,\overline{w(x)}\,dx + \int_{0}^{L} v(x)\,\overline{w(x)}\,dx, \quad v,w \in X.$$

Note however that

$$\|v\| \neq \sqrt{\langle v,v\rangle}.$$

<u>4.1 PROPOSITION</u>. *For* $c \in X$, $\mu \in \mathbb{R}$, $(-A+c)$ *is a symmetric linear operator on* X_c. *Hence* $P\sigma(-A+c) \subset \mathbb{R}$.

<u>PROOF</u>. For $v,w \in \mathcal{D}(A)$ (extended to X_c)

$$\begin{aligned}
\langle(-A+c)v,w) &= \int_{-L}^{0} (v_{xx}+cv)\bar{w}\,dx + \int_{0}^{L} (v_{xx}+cv)\bar{w}\,dx \\
&= v_x\,\bar{w}\,\Big|_{-L}^{0} - v\bar{w}_x\Big|_{-L}^{0} + \int_{-L}^{0} v\,\overline{(w_{xx}+cw)}\,dx \\
&\quad + v_x\,\bar{w}\,\Big|_{0}^{L} - v\bar{w}_x\Big|_{0}^{L} + \int_{0}^{L} v\,\overline{(w_{xx}+cw)}\,dx \\
&= v_x(0)\,(\overline{w(0-)} - \overline{w(0+)}) \\
&\quad + \overline{w_x(0)}\,(v(0+) - v(0-)) + \langle v,(-A+c)w\rangle \\
&= -v_x\,\overline{\mu w_x(0)} + \overline{w_x(0)}\,\mu v_x(0) \\
&\qquad + \langle v, (-A+c)w\rangle \\
&= \langle v, (-A+c)w\rangle. \quad \square
\end{aligned}$$

For c an element of Y, we can extend c to a symmetric element of X, and hence, by lemma (3.1) and proposition (4.1), when $\mu \in \mathbb{R}$, also $P\sigma(-A_s+c)$, $P\sigma(-A_s+c) \subset \mathbb{R}$.

4.2 PROPOSITION. *For* c *an element of* X, (-A+c) *has compact resolvent.*

The proof of proposition (4.2), based on the existence of Green's functions for both [-L,0], and [0,L] (see [3]), can be found in the appendix.

4.3 COROLLARY. *For* c *an element of* X, $\mu \in \mathbb{R}$, $P\sigma(-A+c) = \sigma(-A+c) \subset \mathbb{R}$.

And this corollary is proved in [15], chapter VI, theorem (5.1). In the following lemmas we shall prove A to be a sectorial operator on X - see [9], chapter I, definition 1.3.1.

4.4 LEMMA. *When* $\mu \in \mathbb{R}$, A *is a linear, closed, symmetric, densely defined operator on* X. *Moreover* $\sigma(A) = P\sigma(A) \subset \mathbb{R}$.

The proof of this theorem is partly straightforward, partly trivial by earlier results.
Note that by lemma (3.2), $\sigma(A)$ is bounded *below*. Hence, trivially, one can choose $a \in \mathbb{R}$, $\phi \in (0,\frac{\pi}{2})$ such that

$$S_{a,\phi} = \{\lambda \mid \phi \le |\arg(\lambda-a)| \le \pi,\ \lambda \ne a\} \subset \mathbb{C}\backslash\sigma(A).$$

4.5 THEOREM. *There exist constants* a, ϕ, M, *such that*

$$\|(\lambda-A)^{-1}\| \le \frac{M}{|\lambda-a|} \quad \textit{for all } \lambda \in S_{a,\phi}.$$

The proof of this theorem, essentially based on the existence of Green's functions for both [-L,0], [0,L], is given in the appendix.
Having proved A to be sectorial, we apply theorem (3.3.3) and theorem (3.3.4) of [9]. Note that one can choose $\alpha = 0$ in the hypotheses of these theorems.

4.6 COROLLARY. *For any* $\psi \in X$, *there exist* $T = T(\psi) > 0$ *such that (E.P.) has a unique solution* u *on* (0,T) *with* $u(x,0) = \psi$.
Moreover, when $T(\psi)$ *is maximal, either* $T = \infty$ *or else there exists a sequence*

$t_n \uparrow T$ *as* $n \to \infty$, *such that* $\|u(t_n,\psi)\| \to \infty$.

By theorem (5.1.5) and theorem (5.1.3) of [9], we can state

4.7 COROLLARY. *The stability of a steady state solution* q *is determined by* $P\sigma(-A+f'(q))$, *in the way already used in section 3.*

A Lyapunov functional.

4.8 THEOREM. *Let* $F(u) = \int_0^u f(\xi)d\xi$, *and define for* $u \in \mathcal{D}(A)$

$$V(u) = \int_{-L}^{0} \{\tfrac{1}{2}u_x^2 - F(u)\}dx + \int_{0}^{L} \{\tfrac{1}{2}u_x^2 - F(u)\}dx + \frac{\mu}{2}\, u_x^2(0,t).$$

Then V *is a strict Lyapunov functional for (E.P.) and* $\dot{V}(u) = 0$ *if and only if* u *is a steady state solution.*

Theorem (4.8) is proved in the appendix.

4.9 LEMMA. *For fixed* $\mu > 0$, *the set*

$$\{u \in C^2[-L,0] \times C^2[0,L] \mid u_x(0-) = u_x(0+) \text{ and } V(u) < K\}$$

is bounded in X.

Lemma (4.9) is proved in the appendix.

4.10 COROLLARY. *For* $\mu > 0$, $u(x,t) \to q(\psi)$, *for* $t \to \infty$, *where* $q(\psi)$ *is a steady state solution, which depends on the initial function* $\psi \in X$.

PROOF. By theorem (4.8) and lemma (4.9), *the orbit* u(x,t) is bounded in X. By theorem (3.3.6) of [9], $u(x,t)|_{t>0}$ is in a compact set of X. Therefore we can apply theorem (4.3.4) of [9] and since the set of steady state solutions is discrete by theorem (2.6), this proves corollary (4.10). □

CONCLUDING REMARKS

The condition $\mu > 0$ is rather essential for proving these last results. However, this is in accordance with the results of section 2, in which a

striking difference occurred for $\mu < 0$ or $\mu > 0$ (e.g. the existence of non-normalizable solutions, the stability of the constant steady states 0 and 1). In fact, we think, for μ negative and $|\mu| \ll 1$, unbounded orbits of (E.P.) can exist.
Only for $\mu > 0$, the transmission condition in (E.P.) describes a geographical barrier and for μ sufficiently large, we know exactly both the stable steady state solutions and the asymptotic behaviour. (Under the assumption of conjecture (2.10) this is known even for all $\mu > 0$.)
We think that, although we regret the fact we can not yet prove conjecture (2.10), our results show that the reaction diffusion equation E.P. is both tractable and nontrivial, in the sense that it exhibits the existence of clines.

ACKNOWLEDGEMENT

I like to express my gratitude to Prof. L.A. Peletier, who suggested this problem to me and who made me share his enthusiasm for reaction-diffusion equations.
Whenever that enthusiasm was in danger of perishing, I could count on the support of Odo Diekmann. For his encouragements and for the way in which he guided me, I would like to thank him more than a few words can say.

APPENDIX

SECTION 1.

A1. <u>Proof of theorem 1.4</u>. In this proof we transform $\hat{\zeta} = \zeta-\frac{1}{2}$, $\hat{\omega} = \omega-\frac{1}{2}$, etc., to simplify the calculations.

We *retain* the notations of section 1, now using $0 < \omega < \frac{1}{2}$ etc., and use $h(\xi) = f(\xi+\frac{1}{2}) = \xi(\frac{1}{4}-\xi^2)$.

Hence,

$$\mathcal{D}(\omega,\lambda_1,\lambda_2) = \int_{\zeta_1(\omega)}^{\zeta_2(\omega)} \frac{du}{\sqrt{2\int_u^{\omega} h(\xi)d\xi}},$$

$$= \int_{\frac{\zeta_1(\omega)}{\zeta_2(\omega)}}^{1} \frac{\zeta_2(\omega)du}{\sqrt{2\int_{\zeta_2(\omega)u}^{\zeta_2(\omega)} h(\xi)d\xi+2\int_{\zeta_2(\omega)}^{\omega} h(\xi)d\xi}}$$

$$= \int_{\frac{\zeta_1(\omega)}{\zeta_2(\omega)}}^{1} \frac{du}{\sqrt{2\int_{\zeta_2(\omega)u}^{\zeta_2(\omega)} \frac{h(\xi)}{\zeta_2(\omega)^2}d\xi+\lambda_2^2}}$$

$$= \int_{\frac{\zeta_1(\omega)}{\zeta_2(\omega)}}^{1} \frac{du}{\sqrt{2\int_u^1 \frac{h(\zeta_2(\omega)\xi)}{\zeta_2(\omega)}d\xi+\lambda_2^2}}$$

Hence,

$$\frac{\partial\mathcal{D}}{\partial\omega}(\omega,\lambda_1,\lambda_2) = -\frac{1}{\sqrt{2\int_{\frac{\zeta_1(\omega)}{\zeta_2(\omega)}}^{1} \frac{h(\zeta_2(\omega)\xi)}{\zeta_2(\omega)}d\xi+\lambda_2^2}}\ \frac{d}{d\omega}\left(\frac{\zeta_1(\omega)}{\zeta_2(\omega)}\right)$$

$$-\int_{\frac{\zeta_1(\omega)}{\zeta_2(\omega)}}^{1} \left(2\int_u^1 \frac{h(\zeta_2(\omega)\xi)}{\zeta_2(\omega)}d\xi+\lambda_2^2\right)^{-1\frac{1}{2}} \cdot$$

$$\left(\int_u^1 \frac{d}{d\omega}\left(\frac{h(\zeta_2(\omega)\xi)}{\zeta_2(\omega)}\right)d\xi\right)du$$

<u>Step 1</u>: $\frac{d}{d\omega}\left(\frac{\zeta_1(\omega)}{\zeta_2(\omega)}\right) < 0$

<u>Proof</u>: $\zeta_2(\omega)\frac{d\zeta_1}{d\omega}(\omega) - \zeta_1(\omega)\frac{d\zeta_2}{d\omega}(\omega) =$

$$\zeta_2(\omega)\frac{h(\omega)}{\lambda_1^2\zeta_1(\omega)+h(\zeta_1(\omega))} - \zeta_1(\omega)\frac{h(\omega)}{\lambda_2^2\zeta_2(\omega)+h(\zeta_2(\omega))} =$$

$$\frac{h(\omega)}{(\lambda_1^2\zeta_1(\omega)+h(\zeta_1(\omega)))(\lambda_2^2\zeta_2(\omega)+h(\zeta_2(\omega)))}\cdot(\lambda_2^2\zeta_2^2(\omega)+\zeta_2(\omega)h(\zeta_2(\omega)) - \lambda_1^2\zeta_1^2(\omega)-\zeta_1(\omega)h(\zeta_1(\omega))).$$

Since $\operatorname{sign}\frac{\partial\zeta}{\partial\lambda} = \operatorname{sign} -\lambda$, we obtain that $\operatorname{sign}\left(\frac{d}{d\omega}\left(\frac{\zeta_1(\omega)}{\zeta_2(\omega)}\right)\right)$ equals the sign of

$$2\int_{\zeta_2(\omega)}^{\omega} h(\xi)d\xi + \zeta_2(\omega)h(\zeta_2(\omega)) - 2\int_{\zeta_1(\omega)}^{\omega} h(\xi)d\xi - \zeta_1(\omega)h(\zeta_1(\omega)) = \tfrac{1}{2}(\zeta_1^4(\omega)-\zeta_2^4(\omega)),$$

which is clearly negative.

<u>Step 2</u>: $\frac{d}{d\omega}\left(\frac{h(\zeta_2(\omega)\xi)}{\zeta_2(\omega)}\right) < 0$ for $\xi > 0$

<u>Proof</u>: $\frac{d}{d\omega}\left(\frac{h(\zeta_2(\omega)\xi)}{\zeta_2(\omega)}\right) = \frac{\zeta_2(\omega)h'(\zeta_2(\omega)\xi)\xi-h(\zeta_2(\omega)\xi)}{\zeta_2^2(\omega)}\frac{d\zeta_2}{d\omega}(\omega)$

Since $\frac{d\zeta_2}{d\omega}(\omega) > 0$, and

$$\zeta_2(\omega)h'(\zeta_2(\omega)\xi)\xi - h(\zeta_2(\omega)\xi) = -2\zeta_2^3(\omega)\xi^3,$$

the result follows.

Thus we showed that both terms in the expression for $\frac{\partial\mathcal{D}}{\partial\omega}$ are positive. We end the proof of theorem 1.4 by an *outline* of the computations of $\mathcal{D}(\frac{1}{2},\lambda_1,\lambda_2)$ and $\mathcal{D}(1,\lambda_1,\lambda_2)$.

For $\hat{\omega}\downarrow 0$ (i.e. $\omega\downarrow\frac{1}{2}$):

Since (e.g.) $\cos\frac{1}{2}x$ satisfies

$$\eta_{xx} + f'(\tfrac{1}{2})\eta = 0,$$

and

$$\frac{-\frac{1}{2}\sin\frac{1}{2}x}{\cos\frac{1}{2}x} = \lambda_i \text{ iff } \tfrac{1}{2}x = m.\pi - \text{Arctan } 2\lambda_i \text{ for some } m \in \mathbb{Z}.$$

We find $\mathcal{D}(\frac{1}{2},\lambda_1,\lambda_2) = 2(-\text{Arctan } 2\lambda_2 + \text{Arctan } 2\lambda_1)$.

For $\hat{\omega} \uparrow \frac{1}{2}$ (i.e. $\omega \uparrow 1$);

$$\mathcal{D}(1,\lambda_1,\lambda_2) = \int_{\zeta_1-\frac{1}{2}}^{\zeta_2-\frac{1}{2}} \frac{\sqrt{2}}{(\frac{1}{4}-u^2)}\,du = \sqrt{2}\left(\ln\frac{\zeta_2}{1-\zeta_2} - \ln\frac{\zeta_1}{1-\zeta_1}\right)$$

and

$$\lambda_i\hat{\zeta}_i = \frac{(\frac{1}{4}-\hat{\zeta}_i^2)}{\sqrt{2}}, \quad \hat{\zeta}_i \geq 0, \ \hat{\zeta}_i = \zeta_i - \tfrac{1}{2},$$

hence

$$\zeta_i = \tfrac{1}{2} - \frac{\lambda_i}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2\lambda_i^2+1}. \quad \square$$

A2. <u>Proof of theorem 1.10.</u>

<u>Part 1.</u> For $\rho(x,\omega)$ defined by (ω.L.P.)

$$\left(\frac{\frac{\partial\rho_\omega}{\partial x}(0,\omega)}{\rho_\omega(0,\omega)}\right)_\omega = \frac{\int_0^L f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx}{\rho_\omega^2(0,\omega)}$$

<u>PROOF.</u> Since

$$1\quad\begin{cases} \dfrac{\partial^2\rho}{\partial x^2}(x,\omega) + f(\rho(x,\omega)) = 0 \quad \text{on } [0,L] \\ \rho(L,\omega) = \omega \\ \dfrac{\partial\rho}{\partial x}(L,\omega) = 0 \end{cases}$$

it follows

$$2\quad\begin{cases} \dfrac{\partial^2\rho_\omega}{\partial x^2}(x,\omega) + f'(\rho(x,\omega))\rho_\omega(x,\omega) = 0 \quad \text{on } [0,L] \\ \rho_\omega(L,\omega) = 1 \\ \dfrac{\partial\rho_\omega}{\partial x}(L,\omega) = 0 \end{cases}$$

and

$$3\quad\begin{cases} \dfrac{\partial^2\rho_{\omega\omega}}{\partial x^2}(x,\omega) + f'(\rho(x,\omega))\rho_{\omega\omega}(x,\omega) + f''(\rho(x,\omega))\rho_\omega^2(x,\omega) = 0 \text{ on } [0,L] \\ \rho_{\omega\omega}(L,\omega) = 0 \\ \dfrac{\partial\rho_{\omega\omega}}{\partial x}(L,\omega) = 0 \end{cases}$$

When we multiply 3 by $q_\omega(x,\omega)$ and use 2, then we find, when we integrate by parts

$$\frac{\partial\rho_{\omega\omega}}{\partial x}(x,\omega)\rho_\omega(x,\omega)\Big|_0^L - \int_0^L \frac{\partial\rho_{\omega\omega}}{\partial x}(x,\omega)\frac{\partial\rho_\omega}{\partial x}(x,\omega)\,dx$$

$$-\frac{\partial\rho_\omega}{\partial x}(x,\omega)\rho_{\omega\omega}(x,\omega)\Big|_0^L + \int_0^L \frac{\partial\rho_\omega}{\partial x}(x,\omega)\frac{\partial\rho_{\omega\omega}}{\partial x}(x,\omega)\,dx$$

$$+\int_0^L f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx = 0.$$

Using 3BC we find

$$\frac{\partial\rho_{\omega\omega}}{\partial x}(0,\omega)\rho_\omega(0,\omega) - \frac{\partial\rho_\omega}{\partial x}(0,\omega)\rho_{\omega\omega}(0,\omega) = \int_0^L f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx$$

and this proves part 1.

Note that we can prove theorem (1.10) by

<u>Part 2</u>.

$$\int_0^L f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx < 0.$$

Recall the definition (1.6) of $\ell(\omega)$ and note that $L = n\,\frac{\ell(\omega)}{2} + r$, $r \in [0, \frac{\ell(\omega)}{2})$, $n \in \mathbb{N}_0$ and $n+r \neq 0$.

We shall prove

$$\int_{L-i\frac{\ell(\omega)}{2}-b}^{L-i\frac{\ell(\omega)}{2}} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx < 0$$

for $0 < b \le \frac{\ell(\omega)}{2}$, and $0 \le i \le n$ *even,* i.e. working in the upper-halfplane. However, it will be easy to derive a similar result for i *odd,* and hence, since

$$\int_0^L f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx = \sum_{i=0}^{n-1}\int_{L-(i+1)\frac{\ell(\omega)}{2}}^{L-i\frac{\ell(\omega)}{2}} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx$$

$$+\int_0^{L-n\frac{\ell(\omega)}{2}} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)\,dx,$$

this proves part 2 and hence theorem (1.10).

For $[L-(i+1)\frac{\ell(\omega)}{2}, L-i\frac{\ell(\omega)}{2}]$, the map $x \to \rho(x,\omega)$ is 1-1 and $\rho(L-i\frac{\ell(\omega)}{2}, \omega) = \omega > \frac{1}{2}$.

Moreover, by the symmetry, mirror images of the trajectory have the same length, i.e.:

For

$$x \in [L-(i+1)\frac{\ell(\omega)}{2}, L-i\frac{\ell(\omega)}{2}], \quad \text{define}$$

$$M(x) = (L-(i+\tfrac{1}{2})\frac{\ell(\omega)}{2})-(x-(L-(i+\tfrac{1}{2})\frac{\ell(\omega)}{2})$$

$$= 2L - (i+\tfrac{1}{2})\ell(\omega) - x,$$

then

$$\rho(x,\omega) = 1 - \rho(M(x),\omega)$$

$$\frac{\partial\rho}{\partial x}(x,\omega) = \frac{\partial\rho}{\partial x}(M(x),\omega).$$

Note that the functions $\frac{\partial\rho}{\partial x}(x,\omega)$, $\rho_\omega(x,\omega)$ *and* $\rho_\omega(M(x),\omega)$ satisfy

$$\eta_{xx} + f'(\rho(x,\omega))\eta = 0 \quad \text{on } [L-(i+1)\frac{\ell(\omega)}{2}, L-i\frac{\ell(\omega)}{2}],$$

where we used $f'(\xi) = f'(1-\xi)$ to handle the case of $\rho_\omega(M(x),\omega)$.

Since

$$\frac{\partial\rho}{\partial x}(L-(i+1)\frac{\ell(\omega)}{2},\omega) = \frac{\partial\rho}{\partial x}(L-i\frac{\ell(\omega)}{2},\omega) = 0$$

and

$$\frac{\partial\rho}{\partial x}(x,\omega) \neq 0 \quad \text{for } x \in (L-(i+1)\frac{\ell(\omega)}{2},L-i\frac{\ell(\omega)}{2}),$$

the Sturmian oscillation theorem (see [14], Chapter 1, theorem 18), implies:

1) $\rho_\omega(x,\omega) - \rho_\omega(M(x),\omega)$ has exactly one zero on $[L-(i+1)\frac{\ell(\omega)}{2}, L-i\frac{\ell(\omega)}{2}]$, which occurs for $x = L - (i+\frac{1}{2})\frac{\ell(\omega)}{2}$.
By theorem (1.8) (take $x = L - i\frac{\ell(\omega)}{2}$) $\rho_\omega(x,\omega) - \rho_\omega(M(x),\omega) > 0$ for $x \in (L-(i+\frac{1}{2})\frac{\ell(\omega)}{2},L-i\frac{\ell(\omega)}{2}]$.

2) $\rho_\omega(x,\omega)$ has exactly one zero on $[L-(i+1)\frac{\ell(\omega)}{2}, L-i\frac{\ell(\omega)}{2}]$. By theorem (1.8), it occurs for $x \in (L-(i+1)\frac{\ell(\omega)}{2}, L-(i+\frac{1}{2})\frac{\ell(\omega)}{2})$ and $\rho_\omega(x,\omega) > 0$ for $x \in [L-(i+\frac{1}{2})\frac{\ell(\omega)}{2}, L-i\frac{\ell(\omega)}{2}]$.

For $b \le \frac{\ell(\omega)}{4}$, part 2 is proved by 2) and the observation

$$f''(\rho(x,\omega)) < 0 \quad \text{for } x \in (L-(i+\tfrac{1}{2})\tfrac{\ell(\omega)}{2}, L-i\tfrac{\ell(\omega)}{2}].$$

For $\frac{\ell(\omega)}{4} < b \le \frac{\ell(\omega)}{2}$,

$$\int_{L-i\frac{\ell(\omega)}{2}-b}^{L-i\frac{\ell(\omega)}{2}} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)dx =$$

$$\int_{L-i\frac{\ell(\omega)}{2}-b}^{M(L-i\frac{\ell(\omega)}{2}-b)} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)dx + \int_{M(L-i\frac{\ell(\omega)}{2}-b)}^{L-i\frac{\ell(\omega)}{2}} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)dx =$$

$$\int_{L-(i+\frac{1}{2})\frac{\ell(\omega)}{2}}^{M(L-i\frac{\ell(\omega)}{2}-b)} f''(\rho(x,\omega))(\rho_\omega^3(x,\omega)-\rho_\omega^3(M(x),\omega)dx$$

$$+ \int_{M(L-i\frac{\ell(\omega)}{2}-b)}^{L-i\frac{\ell(\omega)}{2}} f''(\rho(x,\omega))\rho_\omega^3(x,\omega)dx < 0$$

using $M(L-i\frac{\ell(\omega)}{2}-b) > L-(i+\frac{1}{2})\frac{\ell(\omega)}{2}$. □

SECTION 2.

A3 <u>Proof of theorem 2.6</u>. Let $\sigma(x,\alpha)$ be defined by

$$\sigma_{xx}(x,\alpha) + f(\sigma(x,\alpha)) = 0 \quad \text{on } [-L,0],\ [0,L]$$
$$\sigma_x(-L) = 0$$
$$\sigma(-L) = \alpha$$
$$\sigma_x(0+) = \sigma_x(0-)$$
$$\sigma(0+) = \mu\sigma_x(0-) + \sigma(0-).$$

Then $\sigma(x,\alpha)$ is a steady state solution iff $\sigma_x(L,\alpha) = 0$. Therefore, define the map

$$SH: \mathbb{R} \to \mathbb{R} \quad \text{by} \quad SH(\alpha) = \sigma_x(L,\alpha).$$

Since f is analytic, SH is an analytic function too. So the zeros of SH are isolated and consequently the set of steady state solutions is discrete. □

A4 Proof of proposition 2.7. Let $\frac{1}{2} < \omega_n < \omega_{n-2} < \dots < \omega_2 < \omega_0$ be such that

$$\frac{\partial\rho_\omega}{\partial x}(0,\omega) = 0 \quad \text{iff} \quad \omega \in \{\omega_n,\dots,\omega_2,\omega_0\}$$

then, by corollary (1.12) and theorem (1.8), and the simple fact that Γ_L intersects each periodic orbit exactly once,

$$\left|\frac{\partial\rho}{\partial x}(0,\omega_n)\right| < \left|\frac{\partial\rho}{\partial x}(0,\omega_{n-1})\right| < \dots < \frac{\partial\rho}{\partial x}(0,\omega_0).$$

So when q is a normalized a-symmetric steady state solution, and

$$\alpha = q(-L) \quad \text{when} \quad q(-L) > \tfrac{1}{2},$$
$$\alpha = 1-q(-L) \quad \text{when} \quad q(-L) < \tfrac{1}{2},$$

then, since

$$q|_{[-L,0]} = \rho(-x,\alpha) \quad \text{when} \quad q(-L) > \tfrac{1}{2}$$

$$= 1 - \rho(-x,\alpha) \text{ when } q(-L) < \tfrac{1}{2},$$

$$\frac{\partial \rho_\omega}{\partial x}(0,\omega) \neq 0,$$

or

$$\frac{\partial \rho_\omega}{\partial x}(0,\alpha) \neq 0.$$

Assume, without loss of generality $\frac{\partial \rho_\omega}{\partial x}(0,\alpha) \neq 0$. Then, one simply finds

$$\mu = \left(\frac{\tilde{q}(0+)-\tilde{q}(0-)}{\tilde{q}_x(0)}\right)$$

a function of $\tilde{q}(L)$ for $\tilde{q}$ a solution of

$$\tilde{q}_{xx} + f(\tilde{q}) = 0 \qquad \text{on } [-L,0],\ [0,L]$$

$$\tilde{q}_x(L) = \tilde{q}_x(-L) = 0$$

$$\tilde{q}_x(0-) = \tilde{q}_x(0+)$$

$$\sup_{\substack{[-L,0]\\ [0,L]}} \{|q(x)-\tilde{q}(x)|\} < \varepsilon$$

for ε sufficiently small. □

A5 Proof of theorem 2.8. Let

$$\alpha = \begin{cases} q(-L) \text{ when } q(-L) \geq \tfrac{1}{2} \\ 1-q(-L) \text{ when } q(-L) < \tfrac{1}{2}, \text{ and } \omega = q(L). \end{cases}$$

Then

$$q(x) = \rho(x,\omega) \text{ for } x \in [0,L]$$

$$q(x) = \begin{cases} \rho(-x,\alpha) \text{ when } q(L) \geq \tfrac{1}{2} \\ 1 - \rho(-x,\alpha) \text{ when } q(L) \leq \tfrac{1}{2} \end{cases} \quad \text{for } x \in [-L,0].$$

Note that

$$\eta(x) = \rho_\omega(x,\omega) \quad \text{for } x \in [0,L]$$

$$\eta(x) = \rho_\omega(-x,\alpha) \text{ for } x \in [-L,0],$$

since (e.g.) $\rho_\omega(-x,\alpha)$ satisfies both

$$\eta_{xx} + f'(q)\eta = 0 \quad \text{on} \quad [-L,0]$$

and

$$\eta(-L) = 1$$

$$\eta_x(-L) = 0,$$

and is therefore unique.
But now, under the assumptions of theorem (2.8), by corollary (1.12),

$$q(L) = \omega > \omega_1,$$

and

$$q(-L) < \tfrac{1}{2},$$

and hence

$$q(-L) = 1 - \alpha < 1 - \omega_1,$$

i.e.

$$\alpha > \omega_1. \qquad \text{(See figure a1)}$$

Let $q(x)$ be $\rho(x,\omega)$ on $[0,L]$, and *satisfy* the hypothesis of theorem (2.8). Now assume, $\omega > \omega_0$, then using corollary (1.12), $\alpha = 1-q(-L) > \omega_1$ is *uniquely* determined by q being *a-symmetric* and

$$\frac{\partial\rho}{\partial x}(0,\omega) = \frac{\partial\rho}{\partial x}(0,\alpha)$$

and it follows $\alpha < \omega_0$, or for ω too large, α cannot be found. When α is defined it follows, by a similar argument as used in the proof of theorem (2.6)

$$\rho(0,\omega) - (1-\rho(0,\alpha)) > 0$$

and

$$\frac{\partial\rho}{\partial x}(0,\omega) > 0.$$

So taking

$$\alpha = \alpha(\omega)$$

$$\mu = \mu(\omega),$$

one finds in this way

$$\mu(\omega) = \frac{\rho(0,\omega)-(1-\rho(0,\alpha(\omega))}{\frac{\partial\rho}{\partial x}(0,\omega)} > 0.$$

So for $\omega > \omega_0$ we found one branch-part satisfying theorem (2.8). Note that for $\omega = \omega_0$ this branch bifurcates from an anti-symmetric steady state solution.

Now using q_4, described in the beginning of section 2, this proves theorem (2.8), after the simple observation that this branch, in *total*, remains in the area $\mu > 0$ (note that - for $L > \pi$ - after this extension not all solutions q represented by this branch do satisfy the hypothesis of theorem (2.8); α can be in $(\frac{1}{2},\omega_1)$, for ω sufficiently large!). □

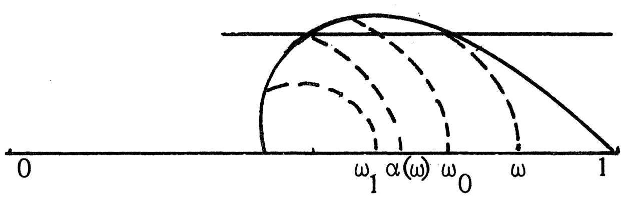


fig. a1

A6 <u>Proof of theorem 2.9</u>. Restrict ω to the interval $[\underline{\omega},1]$, $\hat{\theta}(\underline{\omega}) = \pi$. Define, for some $\mu > 0$

$$J(\zeta,P) = (\zeta-\frac{\mu}{2}P,P)$$

and consider the set

$$Q = \{J(\rho(0,\omega),\ \rho_x(0,\omega)) \mid \omega \in [\underline{\omega},1]\}$$

and its mirror-image in the line $\zeta = \frac{1}{2}$.

One simply shows an intersection of these two sets on $\zeta = \frac{1}{2}$ to represent an anti-symmetric solution, an intersection not on $\zeta = \frac{1}{2}$, to represent an a-symmetric solution.

The slope of the tangent at $(\rho(0,\omega)-\frac{\mu}{2}\rho_x(0,\omega),\rho_x(0,\omega)) \in Q$ is

$$JS(\omega) = \frac{S(\omega)}{1-\frac{\mu}{2}S(\omega)} .$$

Hence $JS'(\omega) = \dfrac{S'(\omega)}{(1-\frac{\mu}{2}S(\omega))^2}$, and Q has same convexity-properties as Γ_L.
Note that $JS(\omega) = 0$ iff $\omega = \omega_0$, under the restriction $\omega \in [\underline{\omega},1]$.
The idea of the lowerbound for μ is to take care that Q and its image intersect exactly once on both sides of $\zeta = \frac{1}{2}$, by placing this maximum of Q left of $\zeta = 0$. One can easily verify this to be a sufficient condition.
So

$$\rho(0,\omega_0) - \frac{\mu}{2}\rho_x(0,\omega_0) < 0,$$

i.e.

$$\mu > 2\frac{\rho(0,\omega_0)}{\rho_x(0,\omega_0)}. \quad \square$$

SECTION 3.

A7 <u>Proof of lemma 3.2</u>. Define

$$\eta(x,\lambda) \in C^2[-L,0] \times C^2[0,L]$$

$$\eta_{xx}(x,\lambda) + c(x)\eta(x) = \lambda\eta(x) \quad x \in [-L,0],\ [0,L]$$

$$\eta(-L) = \eta(L) = 1$$

$$\eta_x(-L) = \eta_x(L) = 0.$$

According to theorem 1.2, chapter 8 of [3], $\dfrac{\eta(0+,\lambda)}{\eta_x(0+,\lambda)}$ is piecewise strictly increasing in λ, jumping from infinity to minus infinity when $\eta_x(0+,\lambda) = 0$, and $\dfrac{\eta(0-,\lambda)}{\eta_x(0-,\lambda)}$ is piecewise strictly decreasing in λ, jumping from minus infinity to infinity when $\eta_x(0-,\lambda) = 0$.
Further, $\dfrac{\eta(0+,\lambda)}{\eta_x(o+,\lambda)} \uparrow 0$, $\dfrac{\eta(0-,\lambda)}{\eta_x(0-,\lambda)} \downarrow 0$, for $\lambda \to \infty$, and note that

$$\lambda \in P\sigma(-A_s+c) \text{ iff } \eta_x(0+,\lambda) = 0,$$

$$\lambda \in P\sigma(-A_a+c) \text{ iff } \frac{\mu}{2} = \frac{\eta(0+,\lambda)}{\eta_x(0+,\lambda)},$$

$$\lambda \in P\sigma(-A+c) \text{ iff } \mu = \frac{\eta(0+,\lambda)}{\eta_x(0+,\lambda)} - \frac{\eta(0-,\lambda)}{\eta_x(0-,\lambda)},$$

or

$$\eta_x(0+,\lambda) = \eta_x(0-,\lambda) = 0,$$

$$\eta(0+,\lambda) = \eta(0-,\lambda).$$

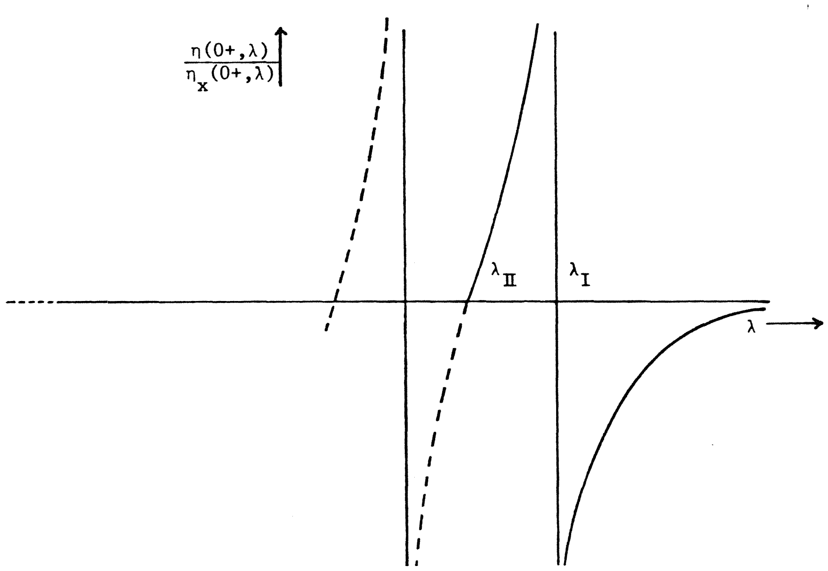
These observations prove (3.2) 1) and (3.2) 2). One can find the proof of (3.2) 3) in [3] (chapter 8, theorem 2.1).
Define

$$\lambda_I,\ \lambda_{II} \text{ by } \lambda_I = \max\{\lambda \mid \eta_x(0+,\lambda) = 0\}$$
$$\lambda_{II} = \max\{\lambda \mid \eta(0+,\lambda) = 0\}$$

and

$$\lambda_I',\ \lambda_{II}' \text{ by } \lambda_I' = \max\{\lambda \mid \eta_x(0-,\lambda) = 0\}$$
$$\lambda_{II}' = \max\{\lambda \mid \eta(0-,\lambda) = 0\}.$$ See fig. a2

We find $\lambda_{II} < \lambda_I < \infty$, $\lambda_{II}' < \lambda_I' < \infty$, and for λ larger than $\lambda_M = \max\{\lambda_{II},\lambda_{II}'\}$ $\eta(x,\lambda) \neq 0$ for $x \in [-L,0]$, $[0,L]$.



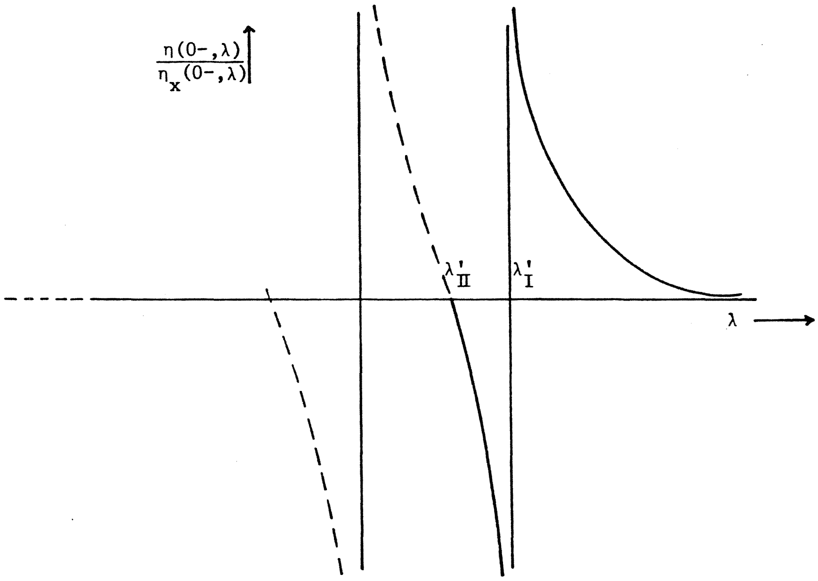


fig. a2

1) Suppose $\lambda_I = \lambda'_I$ then $\hat{\eta}(x,\lambda_I)$ defined by

$$\hat{\eta}(x,\lambda_1) = \eta(x,\lambda_1)\frac{\eta(0+,\lambda_1)}{\eta(0-,\lambda_1)} \quad \text{for} \quad x \in [-L,0)$$

$$\hat{\eta}(x,\lambda_1) = \eta(x,\lambda_1) \qquad\qquad \text{for} \quad x \in (0,L]$$

is an element of $\mathcal{D}(A)$, $\lambda_I \in P\sigma(-A+c)$, and in this case we have proved (3.3) 4), since $\lambda_{\text{dominant}} \geq \lambda_I > \lambda_M$.

2) Without loss of generality assume $\lambda_I > \lambda'_I$. When $\lambda'_I \geq \lambda_{II}$ it is easy to see that

$$\mathcal{R}\left(\left(\frac{\eta(0+,\lambda)}{\eta_x(0+,\lambda)} - \frac{\eta(0-,\lambda)}{\eta_x(0-,\lambda)}\right)\Big|_{(\lambda'_I,\infty)}\right) = (-\infty,\infty)$$

See fig. a3.

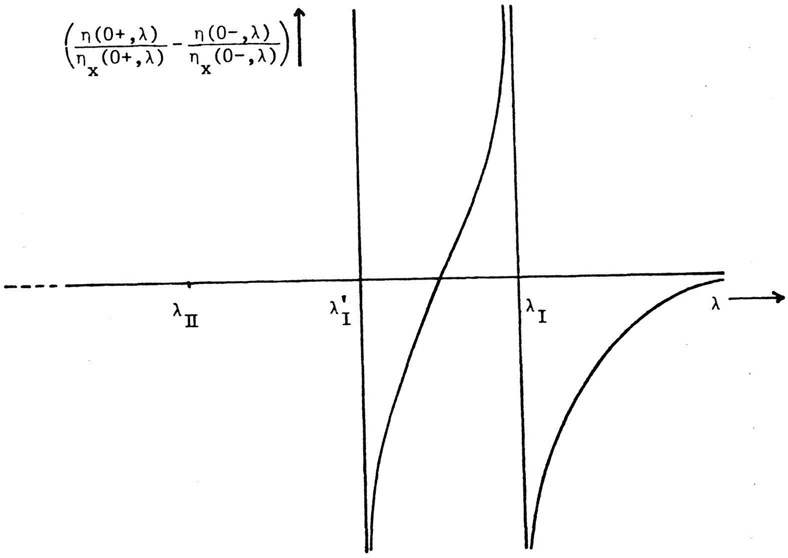


fig. a3

when $\lambda_I' < \lambda_{II}$ it is easy to see that

$$R\left(\left(\frac{\eta(0+,\lambda)}{\eta_x(0+,\lambda)} - \frac{\eta(0-,\lambda)}{\eta_x(0-,\lambda)}\right)\Big|_{(\lambda_{II},\infty)}\right) = (-\infty,\infty)$$

See fig. a4.

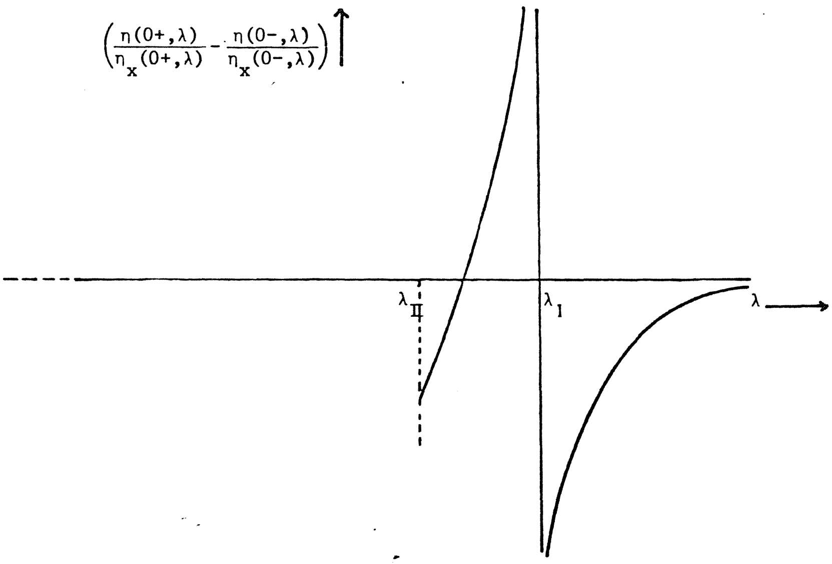


fig. a4

And also in this case we have proved (3.3) 4), since $\lambda_{dominant} > \lambda_M$. □

A8 Proof of theorem (3.5). Since $\rho_\omega(x,\omega)$ satisfies

$$\eta_{xx} + f'(q)\eta = 0 \quad \text{on} \quad [0,L]$$
$$\eta(L) = 1$$
$$\eta_x(L) = 0$$

and $\rho_\alpha(-x,\alpha)$ satisfies

$$\eta_{xx} + f'(q)\eta = 0 \quad \text{on} \quad [-L,0]$$
$$\eta(-L) = 1$$
$$\eta_x(-L) = 0$$

one finds, using the geometric multiplicity 1 of the eigenvalues, $0 \in P\sigma(-A+f'(q))$ if and only if

$$\frac{\partial\rho_\omega}{\partial x}(0,\omega) = \frac{1}{\mu}\left(\rho_\omega(0,\omega) - \frac{\rho_\omega(0,\alpha)}{-\frac{\partial\rho_\omega}{\partial x}(0,\alpha)} \; \frac{\partial\rho_\omega}{\partial x}(0,\omega)\right)$$

or

$$\frac{\partial\rho_\omega}{\partial x}(0,\omega) = \frac{\partial\rho_\omega}{\partial x}(0,\alpha) = 0,$$

in which case we take

$$\eta\big|_{[-L,0]} = \frac{\rho_\alpha(-x,\alpha)}{\rho_\alpha(0,\alpha)}\,\rho_\omega(0,\omega) \neq 0.$$

Hence, theorem (3.5) follows immediately. □

SECTION 4.

A9 Proof of proposition 4.2. In this proof we shall follow Chapter 12, Section 3 of [3].

Let $T(x,\xi,\lambda)$ denote the Green's function for the problem

$$(*) \qquad -\eta_{xx} = \lambda\eta \quad \text{on} \quad [0,L],$$
$$\eta_x(0) = \eta_x(L) = 0.$$

According to theorem (3.1) [3], for $c \in X$, and $|\lambda|$ sufficiently large there exists a Green's function $G(x,\xi,\lambda)$ for the problem

$$(**) \qquad -\eta_{xx} + c(x)\eta = \lambda\eta \quad \text{on} \quad [0,L],$$
$$\eta_x(0) = \eta_x(L) = 0.$$

Denote by T^-, G^- the corresponding Green's functions on $[-L,0]$, and denote

$$T(x,\xi,\lambda) = \begin{cases} T_1(x,\xi,\lambda) & x < \xi \\ T_2(x,\xi,\lambda) & x > \xi, \end{cases}$$

and let T_1^-, T_2^-, G_1, G_2, G_1^-, G_2^- be defined similarly.
Taking $\lambda \ll 0$ real, $\nu = \sqrt{-\lambda}$, it follows

$$T_1(x,\xi,\lambda) = \frac{\cosh \nu x \cosh \nu(L-\xi)}{\nu \sinh \nu L}$$

$$T_2(x,\xi,\lambda) = \frac{\cosh \nu(L-x) \cosh \nu\xi}{\nu \sinh \nu L}$$

$$T_1^-(x,\xi,\lambda) = \frac{\cosh \nu(L+x) \cosh \nu\xi}{\nu \sinh \nu L}$$

$$T_2^-(x,\xi,\lambda) = \frac{\cosh \nu(L+\xi) \cosh \nu x}{\nu \sinh \nu L}.$$

The *existence* of G *implies* $G_{2x}(0,0,\lambda) \neq 0$, $G^-_{1x}(0,0,\lambda) \neq 0$, since otherwise both (e.g.) G_2 and 0 satisfy (**).
The solution of

$$-\eta_{xx} + c(x)\eta = \lambda\eta + k(x) \quad \text{on} \quad [0,L], \quad k \in C[0,L]$$
$$\eta_x(0) = a$$
$$\eta_x(L) = 0$$

is

$$a\,\frac{G_2(x,0,\lambda)}{G_{2_x}(0,0,\lambda)} + \int_0^L G(x,\xi,\lambda)k(\xi)d\xi,$$

similarly, the solution of

$$-\eta_{xx} + c(x)\eta = \lambda\eta + k(x) \quad \text{on} \quad [-L,0],\ k \in C[-L,0]$$
$$\eta_x(0) = b$$
$$\eta_x(L) = 0$$

is

$$b\,\frac{G_1^-(x,0,\lambda)}{G_{1_x}^-(0,0,\lambda)} + \int_{-L}^0 G^-(x,\xi,\lambda)k(\xi)d\xi.$$

So, for the problem $(-A+c-\lambda)\eta = k$, $k \in X$, $\eta \in \mathcal{D}(A)$ and $\eta|_{[-L,0]} = y$, $\eta|_{[0,L]} = z$, we find

$$y(x) = \frac{1}{\mu}(z(0)-y(0))\,\frac{G_1^-(x,0,\lambda)}{G_{1_x}^-(0,0,\lambda)} + \int_{-L}^0 G^-(x,\xi,\lambda)k(\xi)d\xi\,,$$

$$z(x) = \frac{1}{\mu}(z(0)-y(0))\,\frac{G_2(x,0,\lambda)}{G_{2_x}(x,0,\lambda)} + \int_0^L G(x,\xi,\lambda)k(\xi)d\xi,$$

and

$$(z(0)-y(0)) = \{1+\frac{1}{\mu}(\frac{G_1^-(0,0,\lambda)}{G_{1_x}^-(0,0,\lambda)} - \frac{G_2(0,0,\lambda)}{G_{2_x}(0,0,\lambda)})\}^{-1}\ \cdot$$
$$(\int_0^L G(x,\xi,\lambda)k(\xi)d\xi - \int_{-L}^0 G^-(x,\xi,\lambda)k(\xi)d\xi).$$

By the *proof* of lemma (3.2)

$$\{1+\frac{1}{\mu}(\frac{G_1^-(0,0,\lambda)}{G_{1_x}(0,0,\lambda)} - \frac{G_2(0,0,\lambda)}{G_{2_x}(0,0,\lambda)})\}^{-1} \to 1 \quad \text{for} \quad \lambda \to -\infty.$$

So when we use

$$|G(x,\xi,\lambda) - T(x,\xi,\lambda)| \le 4m^2 \; |\lambda|^{-1} h_\nu(x,\xi) \int_0^L |c(x)|dx$$

$$|G^-(x,\xi,\lambda) - T^-(x,\xi,\lambda)| \le 4m^2 \; |\lambda|^{-1} h_\nu(x,\xi) \int_{-L}^0 |c(x)|dx$$

which follows from (3.9) on page 307 in [3], one finds

$$\|\eta\| \le C_1(c) \; \|k\|$$

and since

$$-\eta_{xx} = -c(x)\eta + \lambda\eta + k(x)$$

$$\eta_x(-L) = \eta_x(L) = 0$$

we find

$$\|\eta_x\| \le C_2(c) \; \|k\|$$

Hence we find,

$$(-A+c-\lambda)^{-1} B_1(0) = V \times W \subset X,$$

where

$$B_1(0) = \{K \in X \mid \|k\| \le 1\},$$

and

$$V \subset C[-L,0] \text{ bounded and equicontinuous,}$$

$$W \subset C[0,L] \text{ bounded and equicontinuous.}$$

By Arzéla-Ascoli and Tychonoff, we find $(-A+c-\lambda)^{-1}$ a compact operator. □

A10 Proof of theorem 4.5. Take $\lambda \in S_{a,\phi}$ and let $\nu = \sqrt{-\lambda}$, taking $-\frac{\pi}{2} \le \arg\sqrt{z} < \frac{\pi}{2}$ for the square-root on $\mathbb{C}$. Note $\operatorname{Re} \nu \ge \delta > 0 \; \forall \lambda \in S_{a,\phi}$ (see fig. a4). Since $\|(\lambda-A)^{-1}\| = \|(A-\lambda)^{-1}\|$, let

$$(A-\lambda)^{-1}k = \eta \in \mathcal{D}(A), \; \eta|_{[-L,0]} = y, \; \eta|_{[0,L]} = z.$$

Since

$$-\eta_{xx} = \lambda\eta + k,$$
$$\eta \in \mathcal{D}(A),$$

by the proof of proposition (4.2) and using its notations,

$$y(x) = \left(1 + \frac{2}{\mu}\frac{\cosh \nu L}{\nu \sinh \nu L}\right)^{-1}\left\{\int_0^L T(x,\xi,\lambda)k(\xi)d\xi - \int_{-L}^0 T^-(x,\xi,\lambda)k(\xi)d\xi\right\}\cdot$$

$$\frac{\cosh \nu(L+x)}{\nu \sinh \nu L} + \int_{-L}^0 T^-(x,\xi,\lambda)k(\xi)d\xi.$$

$$z(x) = \left(1 + \frac{2}{\mu}\frac{\cosh \nu L}{\nu \sinh \nu L}\right)^{-1}\left\{\int_0^L T(x,\xi,\lambda)k(\xi)d\xi - \int_{-L}^0 T^-(x,\xi,\lambda)k(\xi)d\xi\right\}\cdot$$

$$\frac{\cosh \nu(L-x)}{\nu \sinh \nu L} + \int_0^L T(x,\xi,\lambda)k(\xi)d\xi.$$

Since

$$\left|\frac{\cosh \nu L}{\nu \sinh \nu L}\right| \le \frac{1}{|\nu||\tanh \mathrm{Re}\nu L|},$$

we can choose a, ϕ such that

$$\left|\frac{\cosh \nu L}{\nu \sinh \nu L}\right| \le \left|\frac{\mu}{4}\right|,$$

and note

$$\left|\frac{\cosh \nu(L-x)}{\nu \sinh \nu L}\right| \le \frac{1}{|\nu||\tanh \mathrm{Re}\nu L|}.$$

Of course, by the symmetry of the problem, we can restrict the analysis to the interval [0,L].

$$\int_0^L T(x,\xi,\lambda)k(\xi)d\xi = \int_0^x \frac{\cosh \nu(L-x)\ \cosh \nu\xi}{\nu \sinh \nu L} k(\xi)d\xi$$

$$+ \int_x^L \frac{\cosh \nu x \cosh \nu(L-\xi)}{\nu \sinh \nu L} k(\xi)d\xi$$

$$\le \|k\| \ \{\int_0^x \frac{\cosh \mathrm{Re}\nu(L-x)\ \cosh \mathrm{Re}\nu.\xi}{|\nu|\ \sinh \mathrm{Re}\nu L} d\xi$$

$$+ \int_x^L \frac{\cosh \mathrm{Re}\nu\ x \cosh \mathrm{Re}\nu(L-\xi)}{|\nu|\ \sinh \mathrm{Re}\nu L} d\xi\}$$

$$= \|k\| \ \{\int_0^x \frac{\cosh \mathrm{Re}\nu(L-x)\ \cosh \mathrm{Re}\nu\xi}{|\nu|\ \sinh \mathrm{Re}\nu L} d\xi + \int_0^{L-x} \frac{\cosh \mathrm{Re}\nu x \cosh \mathrm{Re}\nu\xi}{|\nu|\ \sinh \mathrm{Re}\nu L} d\xi\}$$

$$= \|k\| \ \{\frac{\cosh \mathrm{Re}\nu(L-x)\ \sinh \mathrm{Re}\nu x + \cosh \mathrm{Re}\nu x \ \sinh \mathrm{Re}\nu(L-x)}{|\nu|\ \mathrm{Re}\nu \ \sinh \mathrm{Re}\nu L}\}$$

$$= \|k\| \ \{\frac{2 \sinh \mathrm{Re}\nu L}{|\nu|\ \mathrm{Re}\nu \sinh \mathrm{Re}\nu L}\} = \frac{2}{|\nu|\ \mathrm{Re}\nu} \|k\| .$$

Note that for $\lambda \in S_{a,\phi}$, $\nu = \sqrt{-\lambda}$, $\exists C_1, C_2 \in \mathbb{R}^+$ such that

$\mathrm{Re}\ \nu \ge C_1|\nu|, \quad |\lambda| \ge C_2|\lambda-a|.$ See fig. a5.

Hence

$$\|(A-\lambda)^{-1}\| \le 2(\frac{4}{|\nu|\ \mathrm{Re}\nu})\ |\frac{\mu}{4}| + \frac{2}{|\nu|\ \mathrm{Re}\nu}$$

$$\le \frac{2}{C_1|\nu|^2} (4|\frac{\mu}{4}| + 1)$$

$$\le \frac{M}{|\lambda-a|}$$

where

$$M = \frac{2}{C_1 C_2} (|\mu|+1)$$

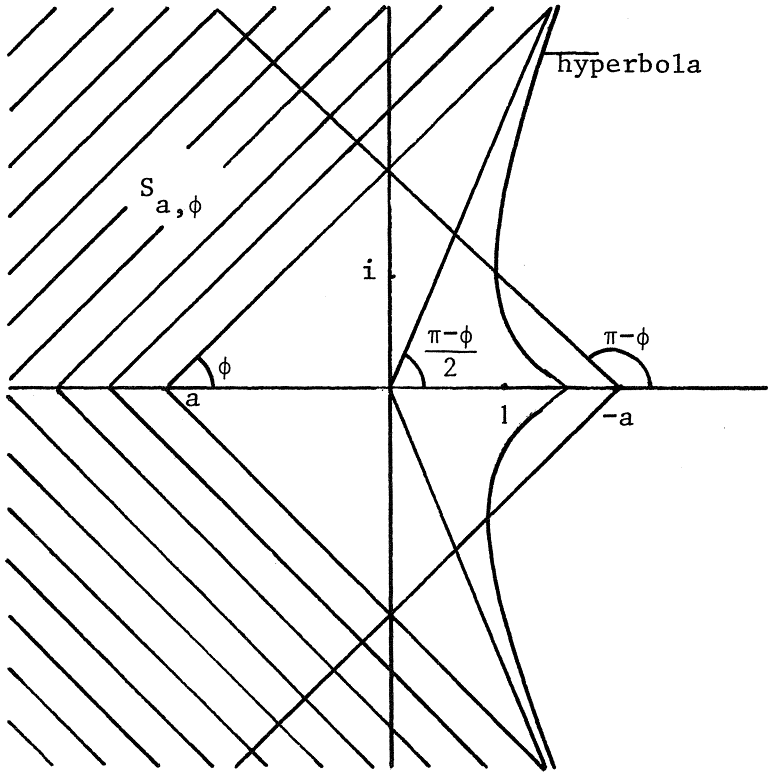


fig. a5

A11 Proof of theorem (4.8).

$$\frac{d}{dt} V(u(.,t)) = \int_{-L}^{0} \{u_x u_{xt} - f(u)u_t\}dx$$

$$+ \int_{0}^{L} \{u_x u_{xt} - f(u)u_t\}dx$$

$$+ \mu u_x(0,t)u_{xt}(0,t).$$

By integration by parts it follows

$$\frac{d}{dt} V(u(.,t)) = u_x(0-,t)u_t(0-,t) - u_x(-L,t)u_t(-L,t)$$

$$- \int_{-L}^{0} \{u_{xx}u_t + f(u)u_t\}dx$$

$$+ u_x(L,t)u_t(L,t) - u_x(0+,t)u_t(0+,t)$$

$$- \int_{0}^{L} \{u_{xx}u_t + f(u)u_t\}dx$$

$$+ \mu u_x(0,t)u_{xt}(0,t)$$

Hence, by (E.P.), we find

$$\frac{d}{dt} V(u(.,t)) = -u_x(0,t)\{u_t(0+,t) - u_t(0-,t)\}$$

$$- \int_{-L}^{0} u_t^2 dx - \int_{0}^{L} u_t^2 dx$$

$$+ \mu u_x(0,t)u_{xt}(0,t)$$

$$= - \int_{-L}^{0} u_t^2 dx - \int_{0}^{L} u_t^2 dx. \qquad \square$$

A12 <u>Proof of lemma 4.9</u>. Since $F(\xi) = -\frac{1}{4}\xi^4 + \frac{1}{2}\xi^3 - \frac{1}{4}\xi^2$, we can define

$$F_M = \max\{F(\xi) \mid \xi \in \mathbb{R}\}.$$

Suppose $K > 0$, $u \in C^2[-L,0] \times C^2[0,L]$, u_x continuous in 0, and suppose $\mu > 0$ and

$$V(u) = \int_{-L}^{0} \{\tfrac{1}{2}u_x^2 - F(u)\}dx + \int_{0}^{L} \{\tfrac{1}{2}u_x^2 - F(u)\}dx + \frac{\mu}{2} u_x^2(0) < K.$$

Then

$$1)\quad \int_{-L}^{0} u_x^2 dx + \int_{0}^{L} u_x^2 dx < 2K + 4LF_M$$

$$2)\quad \int_{-L}^{0} -F(u)du + \int_{0}^{L} -F(u)du < K.$$

By 1) we find

$$\int_{-L}^{0} |u_x| dx < 2K + 4LF_M + L$$

$$\int_{0}^{L} |u_x| dx < 2K + 4LF_M + L$$

and using this, by 2) one finds $C_1 \in \mathbb{R}^+$,

$$\|u\| \le C_1. \qquad \square$$

REFERENCES

[1] CASTEN, R.G. & C.J. HOLLAND, *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. Diff. Equ. 27 (1978), 266-273.

[2] CHAFEE, N., *Asymptotic behaviour for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions*, J. Diff. Equ. 18 (1975), 111-134.

[3] CODDINGTON, E.A. & N. LEVINSON, *Theory of Ordinary Differential Equations*, Academic Press, New York, 1961.

[4] TEN EIKELDER, H.M.M., *A Nonlinear Diffusion Problem Arising in Population Genetics*, Report NA-25, Delft University of Technology, 1979.

[5] FIFE, P.C., *Mathematical aspects of reacting and diffusing systems*, Lect. Notes Biomath. 28, Springer, Berlin, 1979.

[6] FIFE, P.C. & L.A. PELETIER, *Clines induced by variable selection and migration*, Proc. R. Soc. Lond. B214 (1981), 99-123.

[7] HALE, J.K., *Stability and Bifurcation in a Parabolic Equation,* Proc. Symp. Dynamical Systems, Warwick, 1980. To appear.

[8] HALE, J.K. & J. VEGAS, *A Nonlinear Parabolic Equation with Varying Domain*, Arch. Rat. Mech. Anal., to appear.

[9] HENRY, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer, Berlin, 1981.

[10] IOOSS, G. & D.D. JOSEPH, *Elementary Stability and Bifurcation Theory,* Springer, Berlin, 1980.

[11] MATANO, H., *Asymptotic Behaviour and Stability of Solutions of Semilinear Diffusion Equations,* Publ. RIMS, Kyoto Univ. 15 (1979), 401-454.

[12] NAGYLAKI, T., *Clines with variable migration,* Genetics 83 (1976), 867-886.

[13] NAGYLAKI, T., *Selection in One-and Two-locus Systems,* Lect. Notes in Biomath. 15, Springer, Berlin, 1977.

[14] PROTTER, M.H. & H.F. WEINBERGER, *Maximum Principles in Differential Equations,* Prentice Hall, Englewood Cliffs, New Jersey, 1967.

[15] TAYLOR, A.E. & D.C. LAY, *Introduction to Functional Analysis,* second edition, Wiley, New York, 1980.