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(DEPARTMENT OF PURE MATHEMATICS)

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A.E. BROUWER

A NOTE ON THE UNIQUENESS OF THE JOHNSON SCHEME

amsterdam 1983

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Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 05C25, 05B30

A note on the uniqueness of the Johnson scheme

by

A.E. Brouwer

ABSTRACT

Given a graph with $\binom{n}{m}$ vertices, valency $m(n-m)$ such that each edge is in $n - 2$ triangles, and any two nonadjacent vertices have at most 4 common neighbours Dowling proved that it is isomorphic with the graph of m-subsets of an n-set with Johnson distance 1 provided that $n > 2m(m-1) + 4$. Here we improve this bound sufficiently to obtain uniqueness in the desired case $m = 4$, $n = 24$. (Our lower bound for n is $n \geq \max(6m-1, m^2+2m-1)$.)

KEY WORDS & PHRASES: *Johnson scheme, Tetrahedral graph.*

INTRODUCTION

Using the results of Bose & Laskar [1], Dowling [3] showed the uniqueness of the graph of the m-subsets of an n-set with adjacency as having Johnson distance one, under rather mild conditions. Unfortunately I needed a case not covered by his theorem. Therefore I'll show a minor improvement in the basic characterization theorem of Bose & Laskar and use Dowlings proof to obtain the desired result. Since there is not much new here the proofs will be just outlined.

Let G be a graph with valency $k := m(n-m)$, each edge in $\lambda := n-2$ triangles, and any two nonadjacent points having at most 4 neighbours. We shall try to prove that G is derived from the Johnson scheme. (Of course this is not true in general; certainly an assumption like connectedness of G or $v = \binom{n}{m}$ is necessary; also for $m = 2$ and $n = 8$ one has the Chang graphs. One has no reason to expect that no exceptions will occur for larger m.)

1. If $n \geq \frac{1}{2}(m+1)(m+4)$ then G does not contain an $(m+1)$-claw.

PROOF. Let (a,S) be an $(m+1)$-claw. Let there be Y_i vertices in $\Gamma(a)\backslash S$ adjacent to i points in S. Then

$$\sum Y_i = mn - m^2 - (m+1),$$
$$\sum iY_i = (n-2)(m+1),$$
$$\sum \binom{i}{2} Y_i \leq 3\binom{m+1}{2},$$

so $0 \leq \Sigma\frac{1}{2}(i-1)(i-2)Y_i \leq -n + \frac{1}{2}m^2 + \frac{5}{2}m + 1$. □

2. If $n \geq m^2 - m + 2$ then each edge is contained in an m-claw.

PROOF. Counting as under 1 for an S-claw one finds

$$\sum Y_i = mn - m^2 - S,$$
$$\sum iY_i = (n-2)S,$$

so $Y_0 \geq mn - m^2 - (n-1)S > 0$ and the S-claw can be extended as long as $S < m$. □

3. If an edge is contained in an m-claw, and there are no (m+1)-claws, it is in a clique of size at least $n - 3m + 3$.

PROOF. The edge is in $n - 2$ triangles; at most $3(m-1)$ of the third vertices of these triangles have a neighbour in the m-claw; the others are mutually adjacent, otherwise the claw could be enlarged. This yields a clique of size at least $n - 2 - 3(m-1) + 2$. □

4. Let a large clique be a maximal clique of size at least $n - 3m + 3$. Then if $n \geq 6m - 1$ two large cliques have at most one point in common.

PROOF. Otherwise their union has at most $\lambda + 2 = n$ points and their intersection at most 4 points, but $2(n-3m+3) \leq n + 4$ implies $n \leq 6m - 2$. □

5. If $n \geq m^2 + 2m - 1$ then each point is in exactly m large cliques.

PROOF. m cliques cover $m + Y_1 \geq mn - 2m^2 + m$ points (for a maximal m-claw we have $Y_1 = 2Y_0 + Y_1 \geq 2(k-m) - \lambda m = m(n-2m)$), so an $(m+1)^{st}$ can have at most $1 + k - (mn-2m^2+m) = m^2 - m + 1 < n - 3m + 3$ points. □

This shows that if $m \geq 3$ and $n \geq \max(6m-1, m^2+2m-1)$ then G is the pointgraph of a geometry with m lines through each point, where lines intersect in at most one point. The next step is to see that all lines have the same size $n - m + 1$. (This is the average size since $k = m(n-m)$.)

6. Let ℓ_0 and ℓ_1 be two lines through u. Then the number of lines not through u intersecting both ℓ_0 and ℓ_1 is at most $\max(|\ell_0|+1, |\ell_1|+1)$.

PROOF. Let $v \in \ell_0$. Since ℓ_1 is a maximal clique it contains a point $v' \not\sim v$. Since v and v' have at most 4 common neighbours, and u is one of them there pass at most three of the secants through v. Suppose there are in fact three. Then any point v' cannot have a neighbour in $\ell_0\backslash\{u\}$, so there are at most 9 secants. But $|\ell_i| + 1 \geq n - 3m + 4 \geq 9$ (provided that $n \geq 3m+5$). Now assume that no point of $\ell_0 \cup \ell_1$ is incident with three lines intersecting both $\ell_0\backslash\{u\}$ and $\ell_1\backslash\{u\}$. If v is on two secants then any point v' is on at most one and the statement follows. □

7. All lines have size $n - m + 1$.

PROOF. Let ℓ_0 be the shortest line; choose $u \in \ell_0$ and let ℓ_i $(1 \le i \le m-1)$ be the other lines through u. Counting edges not containing u from ℓ_0 to $\cup_{i>0} \ell_i$ we find on one hand $(n-|\ell_0|)(|\ell_0|-1)$ and on the other hand (by 6) at most $\Sigma_{i>0} |\ell_i| + 1 = k + 2m - 1 - |\ell_0|$. This yields a quadratic inequality for $|\ell_0|$ with discriminant $D^2 = (n-2m-2)^2 + 4(n-4m+1) > 0$ (provided that $n \ge m$) and solution $||\ell_0| - \frac{1}{2}(n+2)| \ge \frac{1}{2}D$. For $n \ge 6m - 4$ we find $|\ell_0| \ge \frac{1}{2}(n+2)$ and hence $|\ell_0| \ge \frac{1}{2}(m+2+D) > n - m$. Since $n - m + 1$ is the average line size we must have $|\ell_0| = n - m + 1$. □

8. Let ℓ_0 and ℓ_1 be two lines through u. Then the number of lines not through u intersecting both ℓ_0 and ℓ_1 is at least $n - 3m + 4$ and at most $n - m + 2$.

PROOF. 'At most' follows from the previous two paragraphs. 'At least' follows from 'at most' and $n - 3m + 4 = (m-1)(n-m) - (m-2)(n-m+2)$. (As follows: since $\lambda = (n-m-1) + (m-1)$ we see that if $\ell_0, \ldots, \ell_{m-1}$ are the lines on u, then any point v on one of these lines is adjacent to $m - 1$ points w on another of these lines. Thus the total number of such edges with $v \in \ell_0$ is $(m-1)(n-m)$, and subtracting upper bounds for the number of edges between ℓ_0 and ℓ_j for $j > 1$, we obtain a lower bound for the number of edges between ℓ_0 and ℓ_1.) □

9. Let u be a point not on the line ℓ. Then u is adjacent to at most two points of ℓ.

PROOF. Since any two nonadjacent points have at most 4 common neighbours, and ℓ contains a point nonadjacent to u, there are at most 4 points on ℓ adjacent to u. If there are precisely 4, say v_0, v_1, v_2, v_3 then there are at most 9 lines not through v_0 intersecting both ℓ and uv_0, contradicting 7. If there are 3, say v_0, v_1, v_2 then count edges meeting $\ell \backslash \{v_0, v_1, v_2\}$ and $uv_i \backslash \{v_i\}$ for some $i \in \{0,1,2\}$. There are at most $|\ell| - 3$ such edges, so for some fixed i not more than $\frac{1}{3}(n-m-2)$. But now by 8 we find $\frac{1}{3}(n-m-2)+4 \ge n - 3m + 4$, i.e. $n \le 4m - 1$, contradiction. □

10. Let ℓ_0 and ℓ_1 be two lines on u. Then any point on $\ell_0 \backslash \{u\}$ is adjacent to

precisely one point of $\ell_1\setminus\{u\}$.

PROOF. It is 1 on the average, but never more than 1 by 9. □

Define a new graph G^* with as vertices the lines (large cliques) of G, two lines being adjacent when they intersect.

11. G^* satisfies our assumptions on G with $n^* = n$ and $m^* = m-1$.

PROOF.
(i) G^* is connected iff G is connected.
(ii) G^* has $\binom{n}{m-1}$ vertices iff G has $\binom{n}{m}$ vertices.
(iii) G^* has valency $k^* = (n-m+1)(m-1) = (n^*-m^*)m^*$.
(iv) If ℓ_0 and ℓ_1 are two lines of u, then there are $m-2$ other lines on u and $n-m$ lines not on u meeting ℓ_0 and ℓ_1 so that $\lambda^* = (m-2)+(n-m) = n-2 = n^*-2$.
(v) If ℓ_0 and ℓ_1 are two disjoint lines, then there are at most 4 lines meeting both.

PROOF. If $u \in \ell_0$ is adjacent to point v of ℓ_1, then by 10. u is adjacent to precisely one other point v' of ℓ_1. Let w be a point of $\ell_1\setminus\{v,v'\}$. Then w is adjacent to two points of each of the lines uv and uv' so that we already know all four common neighbours of the nonadjacent vertices u and w. Consequently w is not adjacent to any point of ℓ_0. □ □

THEOREM. Let G be a graph with valency $k = m(n-m)$, each edge in $\lambda = n-2$ triangles and any two nonadjacent points having at most 4 common neighbours. If $n \geq \max(6m-1, m^2+2m-1)$ then each connected component of G is isomorphic to the graph with as vertices the m-subsets of an n-set, two m-sets being adjacent iff they have m-1 points in common.

PROOF. For m = 0 or m = 1 the statement is trivial. Dowling proves the above theorem with the restriction on n replaced by $n > 2m(m-1)+4$ so that his theorem implies ours in the case m = 2 and m = 3. So assume m > 3. By induction and 11, we can label the vertices of G^* with the elements of $\binom{n}{m-1}$ such that intersecting lines have labels with $m-2$ symbols in common.

If u is on lines ℓ_0 and ℓ_1 labelled $L(\ell_0)$, $L(\ell_1)$ then set $L(u) = L(\ell_0) \cup L(\ell_1)$. Clearly $L(u)$ is an m-set. It is independent of the choice of ℓ_0 and ℓ_1, for suppose ℓ_2 is another line on u, and $L(\ell_2) \not\subset L(u)$. Then there is an (m-2)-set M common to $L(\ell_0)$, $L(\ell_1)$ and $L(\ell_2)$. Since $n - m + 2 > m$ there is an line ℓ with $M \subset L(\ell)$ and $u \notin \ell$. Now the three lines ℓ_0, ℓ_1, ℓ_2 on u each meet ℓ, contradicting 9.

Thus L(u) is well defined, and clearly, when u and v are collinear then L(u) and L(v) have L(uv) in common; by counting one sees that conversely if L(u) and L(v) have $m - 1$ symbols in common then u and v are adjacent. □

Now we come to the only conclusion that really interests us:

COROLLARY. *Let* G *be a graph with* $v = \binom{24}{4}$ *vertices, regular of valency* $k = 80$ *where each edge is in* $\lambda = 22$ *triangles and any two nonadjacent vertices have at most* 4 *common neighbours. Then* G *can be labelled such that the vertices are the* 4*-subsets of a* 24*-set and edges are pairs of* 4*-sets with* 3 *elements in common.* □

(This is a key lemma in the proof of the uniqueness of the near hexagon on 759 vertices, see [2].)

REMARK. This result improves Dowling's, but is not very good when one is looking for characterization of the Johnson scheme in terms of its parameters. Aeryung Moon [4] improves our condition on n by replacing it by '$n > 4m$'. Unfortunately she needs all parameters of the association scheme (the proof uses eigenvalue techniques) and her result does not help us in the characterization of the near hexagon derived from the Steiner system S(5,8,24).

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