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A NOTE ON DELBROUCK'S APPROXIMATE SOLUTION TO THE HETEROGENEOUS BLOCKING PROBLEM

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A note on Delbrouck's approximate solution to the heterogeneous blocking problem *)

by

E.A. van Doorn

ABSTRACT

Delbrouck's [2] recent estimates for the call blocking probabilities experienced by heterogeneous traffic streams on a common trunk group are brought to light in a new and simple manner. A link with earlier estimates by Delbrouck [1] is established by means of a Manfield and Downs-type approximation.

KEY WORDS & PHRASES: *teletraffic theory, blocking probability, heterogeneous blocking problem*

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*) This report will be submitted for publication elsewhere.

I INTRODUCTION

The model we consider is that of a trunk group of size N to which k independent streams of calls are offered. Upon arrival at the trunk group each call seizes a free trunk, if available, and keeps it occupied during an exponentially distributed holding time with mean μ^{-1}. Traffic stream ℓ, $\ell = 1,2,\ldots,k$, has mean m_ℓ and peakedness factor z_ℓ. We assume throughout that $z_\ell \geqq 1$. Those calls from stream ℓ for which no free trunk is available upon arrival are lost and constitute the overflow traffic of stream ℓ. The mean of this overflow stream will be denoted by $\tilde{m}_\ell$. The problem is then to give an estimate for the call blocking probability

$$B_\ell = \tilde{m}_\ell / m_\ell \tag{1}$$

experienced by stream ℓ in terms of the parameters N, μ, m_i and z_i, $i = 1,2,\ldots,k$.

Delbrouck [2] has obtained estimates for the quantities B_ℓ on the basis of the assumption that the steady-state probabilities $p(x_1,x_2,\ldots,x_k)$ of having x_i trunks occupied by calls from stream i, $i = 1,2,\ldots,k$, can be written as

$$p(x_1,x_2,\ldots,x_k) = c \prod_{i=1}^{k} p_i(x_i) \ , \tag{2}$$

where c is a normalizing constant and the $p_i(x)$, $x = 0,1,\ldots,$ are probabilities from a negative binomial distribution. The main purpose of this note is to present a simple rationale for Delbrouck's estimates, which avoids the recently criticized ([4]) assumption (2). We also relate the estimates to earlier ones by Delbrouck [1] by means of a Manfield and Downs-type approximation.

II CALCULATION OF BLOCKING PROBABILITIES

Our basic assumption is that in each stream ℓ the calls arrive in batches of size b_ℓ, where

$$\Pr\{b_\ell = i\} = (1 - r_\ell) r_\ell^{i-1} \qquad (i = 1,2,\ldots) \tag{3}$$

$(0 \leqq r_\ell < 1)$, while the arrival of batches is governed by a Poisson process with intensity ν_ℓ. If r_ℓ and ν_ℓ are chosen such that

$$r_\ell = 1 - z_\ell^{-1} \;, \quad \nu_\ell = \mu m_\ell z_\ell^{-1} \;, \tag{4}$$

then the mean and peakedness factor of stream ℓ are equal to m_ℓ and z_ℓ, respectively, (see [3]) which fits in with the setting of the heterogeneous blocking problem as set forth in the introduction.

Let p_i denote the steady-state probability of having i occupied trunks at an arbitrary moment and write

$$P(x) = \sum_{i=0}^{N} p_i x^i \;.$$

Exploiting the fact that Poisson arrivals see time averages ([6]), we can consider the overflow traffic of stream ℓ as a batched Poisson process where the arrival rate of the batches is ν_ℓ (as in the offered stream ℓ), but where the batch size $\tilde{b}_\ell$ is distributed as

$$\Pr\{\tilde{b}_\ell = i\} = \begin{cases} \sum_{j=0}^{N-1} p_j (1 - r_\ell^{N-j}) & \text{if } i = 0 \\ (1 - r_\ell) \sum_{j=0}^{N} p_j r_\ell^{N+i-j-1} & \text{if } i > 0, \end{cases} \tag{5}$$

as can easily be verified. It follows that the average batch size $E(\tilde{b}_\ell) = \Sigma\, i \Pr\{\tilde{b}_\ell = i\}$ in the ℓth overflow stream is given by

$$E(\tilde{b}_\ell) = (1 - r_\ell)^{-1} r_\ell^N P(r_\ell^{-1}) \ . \tag{6}$$

Hence, by (4) and Little's law,

$$\tilde{m}_\ell = \mu^{-1} \nu_\ell E(\tilde{b}_\ell) = m_\ell r_\ell^N P(r_\ell^{-1}) \ , \tag{7}$$

so that

$$B_\ell = \tilde{m}_\ell / m_\ell = r_\ell^N P(r_\ell^{-1}) \ . \tag{8}$$

We can write down the balance equations for the probabilities p_i as

$$(i\mu + \sum_{\ell=1}^{k} \nu_\ell) p_i = (i+1)\mu p_{i+1} + \sum_{j=0}^{i-1} p_j \sum_{\ell=1}^{k} \nu_\ell (1-r_\ell) r_\ell^{i-j-1} \tag{9}$$

$(i = 0,1,\ldots,N-1)$, from which it is easy to see that

$$(i+1) p_{i+1} = \sum_{\ell=1}^{k} a_\ell r_\ell^i \sum_{j=0}^{i} p_j r_\ell^{-j} \tag{10}$$

$(i = 0,1,\ldots,N-1)$, where

$$a_\ell = \nu_\ell / \mu \ . \tag{11}$$

Apart from its depth, the recursive scheme (10) is independent of

N. Thus given the normalized solution $\{p_i\}_0^N$ to (10) for N = n, we can obtain the normalized solution for N = n + 1 by using (10) once for i = n and renormalizing. In view of (8) this gives us the following algorithm for determination of the B_ℓ's:

initialization : $p_0^{(0)} = 1$, $B_\ell^{(0)} = 1$ $(\ell = 1,2,\ldots,k)$

for n = 1,2,...,N : $p_n^{(n)} = \sum_{\ell=1}^{k} a_\ell B_\ell^{(n-1)} / (n + \sum_{\ell=1}^{k} a_\ell B_\ell^{(n-1)})$

$$B_\ell^{(n)} = p_n^{(n)} + (1-p_n^{(n)}) r_\ell B_\ell^{(n-1)} \qquad (\ell = 1,2,\ldots,k)$$

stop : $B_\ell = B_\ell^{(N)}$ $(\ell = 1,2,\ldots,k)$.

With (4), (11) and by making the identifications $\alpha_\ell = a_\ell$, $\beta_\ell = r_\ell$, $E_n = p_n^{(n)}$ and $\sigma(\ell,n) = m_\ell B_\ell^{(n)}$, it is readily seen that this is precisely the scheme suggested by Delbrouck [2, Proposition 1 and (4.2)] for finding estimates for the call blocking probabilities (1). Thus we have actually shown that for given means m_ℓ and peakedness factors $z_\ell \geq 1$, $\ell = 1,2,\ldots,k$, traffic streams can be constructed which, when offered to a common trunk group, experience call blocking probabilities that are exactly determined by Delbrouck's recursive scheme.

III A MANFIELD AND DOWNS-TYPE APPROXIMATION

Manfield and Downs [5] study the model of the introduction and assume in addition that the offered streams are renewal processes with known interarrival time distributions $F_\ell(t)$, $\ell = 1,2,\ldots,k$. They subsequently seek to calculate the call blocking probabilities B_ℓ up to a common multiplicative constant, but have to make two simplifying assumptions in order to get explicit formulas. Their result can be formulated thus: The call blocking probabilities B_ℓ are approximately proportional to the factors

$$1 + \frac{1}{m}\left(\frac{N\psi_\ell(N\mu)}{1 - \psi_\ell(N\mu)} - m_\ell \right) \tag{12}$$

where

$$m = \sum_{\ell=1}^{k} m_\ell$$

and

$$\psi_\ell(s) = \int_{0-}^{\infty} e^{-st} dF_\ell(t) \qquad (\text{Re } s \geqq 0),$$

the Laplace-Stieltjes transform of the interarrival time distribution of the ℓth stream.

Since the batched Poisson processes of the previous section can be thought of as renewal processes with interarrival time distributions

$$F_{\ell}(t) = r_{\ell} + (1 - r_{\ell})(1 - \exp(-\nu_{\ell} t)) \qquad (t > 0), \tag{13}$$

$\ell = 1,2,\ldots,k$, we can use (12) to obtain approximations (up to a multiplicative constant) for the call blocking probabilities whose exact values are determined by the recursive scheme of the previous section. A simple calculation involving (4) and (13) shows that (12) actually reduces to

$$1 + \frac{N}{m}(z_{\ell} - 1) \ . \tag{14}$$

Surprisingly, these are the estimates for the proportionality factors proposed by Delbrouck in [1] for the case where only the means and the peakedness factors of the offered streams are specified.

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