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HOPF BIFURCATION AND SYMMETRY: TRAVELLING AND STANDING WAVES ON THE CIRCLE

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Hopf bifurcation and symmetry: travelling and standing waves on the circle*)

by

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ABSTRACT

In this paper we consider Hopf bifurcation in the presence of $O(2)$ symmetry. The reaction diffusion equation $u_t = Du_{xx} + f(\mu,u)$ provided with periodic boundary conditions may serve as a model problem. We prove the bifurcation of a torus of standing waves and two circles of travelling waves and we compute the stability (with asymptotic phase) of these periodic solutions, giving explicit formulas. Finally we demonstrate how a small perturbation which breaks part of the symmetry leads to secondary bifurcation.

KEY WORDS & PHRASES: *Hopf bifurcation, symmetry, secondary bifurcation, travelling waves, standing waves, stability*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we study the Hopf bifurcation to periodic orbits at multiple eigenvalues, where the multiplicity is caused by the presence of symmetry, and we investigate the effect of a perturbation which breaks part of the symmetry.

We are motivated by the system of reaction diffusion equations

$$(1.1)\qquad u_t = D(\mu)u_{xx} + f(\mu,u) \quad x \in \mathbb{R},\ \mu \in \mathbb{R},\ u(t,x) \in \mathbb{R}^n,$$

provided with the periodicity conditions:

$$(1.2)\qquad \begin{cases} u(t,0) = u(t,2\pi) \\ u_x(t,0) = u_x(t,2\pi). \end{cases}$$

Due to the periodic boundary conditions, the equations are *rotationally* symmetric: if u(t,x) is a solution, then also u(t,x+ξ) is a solution. Furthermore, *reflection symmetry* is present; together with u(t,x) also u(t,-x) is a solution. These symmetries occurs in many different situations. For instance this is the case in the Couette-Taylor problem, describing the flow of a viscous incompressible fluid filling the space between two concentric cylinders. Independently this has been studied by Iooss [17]. Also the laser equations studied by Peplowski [24] exhibit these symmetries. Our methods are only grounded on the presence of the symmetry groups and therefore similarly apply to all the equations mentioned above.

We will frequently encounter *standing* and *travelling wave* solutions. The existence of these type solutions for (1.1) has been proved by Herschkowitz-Kaufman and Erneux [14,15]. See also Auchmuty [1], Fife [9].

Hopf bifurcation at multiple eigenvalues has been studied in general by Kielhöfer [20]. Here the symmetry yields an (at least) four dimensional invariant subspace for the linearization at a Hopf point. Among the elements of this subspace one can distinguish between travelling waves (e.g. $\zeta^1\sin(kx-\omega t) - \zeta^2\cos(kx-\omega t)$; $\zeta^1,\zeta^2 \in \mathbb{R}^n$) and standing waves (e.g. $\sin kx(\zeta^1\cos \omega t - \zeta^2\sin \omega t)$; $\zeta^1,\zeta^2 \in \mathbb{R}^n$). As the multiplicity is due to symmetry there is more hope to get a complete picture of the bifurcations than in the general case. We utilize the group theoretic methods of Sattinger [25,26], Vanderbouwhede [28] to analyse the bifurcation equations, which we derive in the same spirit as Crandall & Rabinowitz [8] do in the case of Hopf bifurcation of simple eigenvalues. See also the contribution of Othmer in [11].

Keener [19] showed that there is an intimate relation between secondary bifurcation and multiple eigenvalues. In [7], Cowan and Ermentrout show how secondary bifurcation occurs when several pairs of complex conjugate eigenvalues of multiplicity two (due to the same symmetries!) cross the imaginary axis almost simultaneously. Here we will split one pair of multiple eigenvalues by adding to the r.h.s. of (1.1) a term $\eta g(\mu,u,u_x)$, $0 < \eta \ll 1$, which breaks the reflection symmetry. To this equation we will refer as the perturbed one; (1.1) is the unperturbed equation ($\eta=0$).

We will show that under certain assumptions on f (which are generically satisfied) the only time-periodic solutions of (1.1)-(1.2) in a neighbourhood of the zero solution ($f(\mu,0)=0$) are an invariant torus of standing waves and two invariant circles of travelling waves. Two possibilities exist:

(i) the travelling and standing waves bifurcate at the same side (either subcritical or supercritical),

(ii) the travelling waves bifurcate to one side and the standing waves to the other.

If the remaining eigenvalues of the linearized equation lie in the left half plane then (generically) in case (i) one of the wave types is stable and the other is unstable. In case (ii) both types are unstable. These facts complete results of Bajaj [2] and confirm results obtained independently by Peplowski [24]. The stability depends on the direction in which the eigenvalues cross the imaginary axis and the relative size of two numbers (c_1 and c_2), confirming a conjecture of Sattinger [25].

For the perturbed equation we show the possibility for a complex pair of characteristic multipliers of the travelling wave solutions to enter or to escape the unit circle. This leads either to the secondary bifurcation of an attractive invariant torus or to the stabilization of travelling wave solutions. What will happen depends on the relative size of four numbers.

The paper is organized as follows. In section 2 we state the o.d.e. which is obtained after reduction to the center manifold and choose suitable coordinates. We analyse the bifurcation equations and describe the periodic solutions of the unperturbed equation ($\eta=0$). In section 3 we describe the dynamics on the center manifold by averaging the equations. Section 4 deals with the perturbed equation ($\eta\neq 0$). In the first appendix, section 5, we compute all relevant Floquet exponents and in the second appendix, section 6, we make some comments about the reduction of (1.1)-(1.2) to the center manifold and we give formulas for c_1 and c_2 (mentioned above) in terms of the r.h.s. of (1.1).

2. THE ORDINARY DIFFERENTIAL EQUATION ON THE CENTER MANIFOLD

The equation we consider is

$$\frac{dy}{dt} = G(\eta,\mu,y) \quad \eta,\mu \in \mathbb{R}; \quad y \in \mathbb{R}^n, \tag{2.1}$$

and we make the following hypotheses:

H_1 $G \in C^r$, $r \geq 6$; $G(\eta,\mu,0) = 0$.

H_2 the linear part of the unperturbed vector field ($\eta=0$) is given by

$$A(\mu) = \begin{pmatrix} \alpha(\mu) & -\omega(\mu) & 0 & 0 \\ \omega(\mu) & \alpha(\mu) & 0 & 0 \\ 0 & 0 & \alpha(\mu) & -\omega(\mu) \\ 0 & 0 & \omega(\mu) & \alpha(\mu) \end{pmatrix}, \tag{2.2}$$

where $\alpha(0) = 0$, $\omega(0) = \omega_0 \neq 0$. Furthermore we assume that the transversality condition holds: Re $\alpha'(0) \neq 0$;

H_3 let for $\theta \in [0,2\pi)$ $M(\theta)$ be the rotation matrix

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

We assume that G is covariant with respect to $M(\theta)$, i.e.,

$$M(\theta)G(\eta,\mu,y) = G(\eta,\mu,M(\theta)y),$$

H_4 let $\tilde{L}$ be the matrix

$$\tilde{L} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

The unperturbed vector field is also covariant with respect to the matrix $\tilde{L}$, i.e.:

$$\tilde{L}\, G(0,\mu,y) = G(0,\mu,\tilde{L}y).$$

H_3 is due to the rotation symmetry and H_4 renders the reflection symmetry of the unperturbed system.

At this point we change to complex variables $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4$ and define $H\colon \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ by $H^1 = G^1 + iG^2$ and $H^2 = G^3 + iG^4$. Equivalently to solving (2.1) we solve for $z = (z_1,z_2)^T$ the equation

(2.3) $$\dot{z} = H(\eta,\mu,z).$$

We make this transformation for computational reasons. For instance we can now describe the rotation symmetry briefly as follows:
Let for $\theta \in [0,2\pi)$ $\Gamma(\theta)$ be the complex matrix

(2.4) $$\Gamma(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

then H_3 is equivalent with

(2.5) $$\Gamma(\theta)H(\eta,\mu,z) = H(\eta,\mu,\Gamma(\theta)z).$$

At $\eta = 0$ H is covariant with respect to

(2.6) $$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(2.7) $$LH(0,\mu,z) = H(0,\mu,Lz).$$

As a consequence of (2.5) one easily derives

<u>LEMMA 2.1</u>. *Let in the Taylor series expansion of* H^i *with respect to* z $A^i_{k\ell mn}(\eta,\mu)$ *be a nonvanishing coefficient of* $z_1^k \bar{z}_1^\ell z_2^m \bar{z}_2^n$ *then for* i = 1: $k - \ell - m + n = 1$ *and for* i = 2: $k - \ell - m + n = -1$.

The next lemma follows from (2.7).

<u>LEMMA 2.2</u>. *Let* $A^i_{k\ell mn}$ *be as in the previous lemma. Then*

$$A^1_{k\ell mn}(0,\mu) = A^2_{mnk\ell}(0,\mu).$$

After these prelimemaries concerning the symmetry conditions we proceed doing the quite standard preparations to apply the method of Liapunov Schmidt. In the following $2\pi\omega^{-1}$ will be the unknown period of the periodic solution, we write $s = \omega t$ and look for 2π-periodic solutions of

(2.8) $$\omega\frac{dz}{ds} = H(\eta,\mu,z).$$

Let X be the space of continuously differentiable 2π-periodic functions from $\mathbb{R}$ into $\mathbb{C}^2$ provided with the supremum norm

$$\|u\|_X = \sup_{s\in[0,2\pi)} \left|\frac{du(s)}{ds}\right| + \sup_{s\in[0,2\pi)} |u(s)|.$$

Let Z be the space of continuous 2π-periodic functions from $\mathbb{R}$ into $\mathbb{C}^2$ provided with the supremum norm

$$\|u\|_Y = \sup_{s\in[0,2\pi)} |u(s)|.$$

Let Bz be the linear part of H at $\mu = 0$ and $\eta = 0$, i.e.

$$B = \begin{pmatrix} i\omega_0 & 0 \\ 0 & i\omega_0 \end{pmatrix}.$$

Define $\mathbb{L}_0$: X → Z by

$$\mathbb{L}_0 u(s) = -\omega_0 \frac{du(s)}{ds} + Bu(s).$$

Then, equivalently to looking for $2\pi\omega^{-1}$-periodic solutions of equation (2.1) we look for 2π periodic solutions of the equation

$$(2.9) \qquad \mathbb{L}_0 u + F(\eta,\mu,\omega,u) = 0,$$

where

$$(2.10) \qquad F(\eta,\mu,\omega,u) = (\omega_0-\omega) \frac{du}{ds} + H(\eta,\mu,u) - Bu.$$

Indeed if $u(s)$ satisfies (2.9) then $(\mathrm{Re}\, u_1(\omega t), \mathrm{Im}\, u_1(\omega t), \mathrm{Re}\, u_2(\omega t), \mathrm{Im}\, u_2(\omega t))^T$ is a real $2\pi/\omega$ periodic solution of (2.1).

We define in X the elements $\phi_1 = \phi_1(s) = (e^{is},0)^T$ and $\phi_2 = L\phi_1$. From the definition of $\mathbb{L}_0$ and the Fredholm alternative for periodic solution we obtain:

LEMMA 2.3.

$$N(\mathbb{L}_0) = \{u \in X \mid u(s) = z_1\phi_1(s) + z_2\phi_2(s);\ z_1,z_2 \in \mathbb{C}\}$$

$$R(\mathbb{L}_0) = \{u \in Z \mid \int_0^{2\pi} e^{-is}\, u(s)ds = (0,0)^T\}.$$

In Z we define the inner product [,] by

$$[f,g] = \frac{1}{2\pi} \int_0^{2\pi} f(s).\ \overline{g(s)}ds.$$

Consequently $R(\mathbb{L}_0) = \{u \in Z \mid [u,\phi_1] = [u,\phi_2] = 0\}$. As we pointed out before we will use coördinates that reflect the symmetry. S will be the representation of $SO(2) = \mathbb{R}\,(\mathrm{mod}\ 2\pi)$ over Z defined by

$$(2.11) \qquad (S(\xi)u)(s) = u(s+\xi), \quad 0 \le \xi < 2\pi.$$

On $N(\mathbb{L}_0)$ S induces a two dimensional complex representation of SO(2). Let $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ denote the element $z_1\phi_1(s) + z_2\phi_2(s)$ in $N(\mathbb{L}_0)$. It follows that

$$S(\xi)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \left[\begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{i\xi} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right].$$

On the other hand also Γ defines a two dimensional complex representation of SO(2) over $N(\mathbb{L}_0)$:

$$\Gamma(\theta)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \left[\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right].$$

These observations immediately lead to the following

LEMMA 2.3. *There is in* $N(\mathbb{L}_0)$ *a one-to-one correspondence between* $\{[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}] \mid |z_1|^2 + |z_2|^2 \neq 0;\ z_1, z_2 \in \mathbb{C}\}$ *and* $\{\varepsilon\, S(\xi)\Gamma(\theta)\ (\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2) \mid \varepsilon \in \mathbb{R}_+\setminus\{0\},\ \alpha \in [0,\frac{\pi}{2}],\ \xi,\theta \in [0,2\pi)\}$.

Let P be the projection of X onto $N(\mathbb{L}_0)$ defined by

$$Pu = [u,\phi_1]\phi_1 + [u,\phi_2]\phi_2.$$

By the same formula we extend the action of P to elements of Z. This extension we will again denote by P. Then $u \in R(\mathbb{L}_0)$ iff $Pu = 0$. Without loosing generality (confert [5, section 5.5]) we look for solutions u of equation (2.9) of the form

$$u = \varepsilon(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi), \quad P\chi = 0.$$

P commutes with $\mathbb{L}_0$. Therefore the equation (2.9) is equivalent to the pair of equations

$$\text{(2.12a)} \qquad \mathbb{L}_0\ \varepsilon\chi + (I-P)F(\eta,\mu,\omega,\varepsilon(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi)) = 0,$$

$$\text{(2.12b)} \qquad PF(\eta,\mu,\omega,\varepsilon(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi)) = 0.$$

As $\mathbb{L}_0$ is an isomorphism of $N(P)$ onto $R(\mathbb{L}_0)$ we first solve equation (2.12a). We devide by ε and look for solutions of the equation $\mathcal{F} = 0$, where

$$F(\eta,\mu,\omega,\varepsilon,\alpha,\chi) = \begin{cases} \chi + \frac{1}{\varepsilon}\mathbb{L}_0^{-1}(I-P)F(\eta,\mu,\omega,\varepsilon(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi)) & \varepsilon \neq 0, \\ \chi + \mathbb{L}_0^{-1}(I-P)F_u(\eta,\mu,\omega,0)(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi)) & \varepsilon = 0. \end{cases}$$

(i) $F \in C^{r-1}$; (ii) $F(0,0,\omega_0,0,\alpha,0) = 0$; (iii) $F_\chi(0,0,\omega_0,0,\alpha,0) = \mathrm{Id}$. Therefore an application of the Implicit Function Theorem (I.F.T.) and some standard arguments yield

THEOREM 2.4. *There exist positive numbers* $\tilde{\eta},\tilde{\mu},\tilde{\omega},\tilde{\varepsilon}$ *and a* C^{r-1} *function* χ^* *defined on* $\Lambda = \{(\eta,\mu,\omega,\varepsilon,\alpha) \in \mathbb{R}^5 \mid |\eta| < \tilde{\eta},\ |\mu| < \tilde{\mu},\ |\omega-\omega_0| < \tilde{\omega},\ |\varepsilon| < \tilde{\varepsilon},\ \alpha \in [0,\frac{\pi}{2}]\}$ *which takes values in* $N(P) \subset Z$ *such that* (i) $\chi^*(0,0,\omega_0,0,\alpha) = 0$; (ii) $\partial\chi^*/\partial\varepsilon\ (0,0,\omega_0,0,\alpha) = 0$; (iii) $F(\eta,\mu,\varepsilon,\alpha,\chi^*(\eta,\mu,\varepsilon,\alpha)) = 0$. *Moreover this solution is unique in a small neighbourhood of the origin in* $\Lambda \times N(P)$.

We summarize the foregoing in:

THEOREM 2.5. *There exists a positive number* r_0 *such that if* u *satisfies (2.9) and* $0 < \|u\| \leq r_0$ *then the there exist unique* θ,ξ *(depending on* u*) and* $(\eta,\mu,\omega,\varepsilon,\alpha) \in \Lambda$ *such that* $u = \varepsilon\Gamma(\theta)S(\xi)(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi^*(\eta,\mu,\omega,\varepsilon,\alpha))$ *and*

(2.13) $$PF(\eta,\mu,\omega,\varepsilon(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2 + \chi^*(\eta,\mu,\omega,\varepsilon,\alpha))) = 0.$$

Furthermore if $(\eta,\mu,\omega,\varepsilon,\alpha) \in \Lambda$ *satisfies (2.13) then for all* $\theta,\xi \in [0,2\pi)$ $\Gamma(\theta)S(\xi)u$ *is a solution of equation (2.9).*

$PF = 0$ iff $K = (K^1,K^2)^T = 0$ where $K^i = [F,\phi_i]$ for $i \in \{1,2\}$. In the following we will look for zero's of the so-called bifurcation equations:

(2.14) $$K(\eta,\mu,\omega,\varepsilon,\alpha) = 0.$$

As it has been remarked for instance by Vanderbouwhede [28], in general the bifurcation equations obtained by the Liapunov-Schmidt reduction inherite the symmetry properties of the original equation, provided that the projection operators used in the reduction procedure commute with the symmetry operators, which is the case here. We work this out for the mapping K.

LEMMA 2.6.

(i) $LK(\eta=0,\mu,\omega,\varepsilon,\alpha) = K(\eta=0,\mu,\omega,\varepsilon,\frac{\pi}{2}-\alpha)$

(ii) $K^1(\eta,\mu,\omega,\varepsilon,\alpha=0) = 0$

(iii) $K^2(\eta,\mu,\omega,\varepsilon,\alpha=\frac{\pi}{2}) = 0$.

PROOF.

(i) It suffices to prove that $K^1(\eta=0,\mu,\omega,\varepsilon,\alpha) = K^2(\eta=0,\mu,\omega,\varepsilon,\frac{\pi}{2}-\alpha)$.
$K^1(\eta=0,\mu,\omega,\varepsilon,\alpha) = [F(\eta=0,\mu,\omega,\varepsilon(\sin\alpha\,\phi_1+\cos\alpha\,\phi_2+\chi^*(\eta=0,\mu,\omega,\varepsilon,\alpha)),\phi_1] =$
$[LF(\eta=0,\mu,\omega,\varepsilon(\sin\alpha\,\phi_1+\cos\alpha\,\phi_2+\chi^*(\eta=0,\mu,\omega,\varepsilon,\alpha)),L\phi_1] =$
$[F(\eta=0,\mu,\omega,\varepsilon(\sin\alpha\,\phi_2+\cos\alpha\,\phi_1+L\chi^*(\eta=0,\mu,\omega,\varepsilon,\alpha)),\phi_2]$. L commutes with Π_0 and P. If we apply L to equation (2.12a) then from the uniqueness of χ^* we derive that $L\chi^*(\eta=0,\mu,\omega,\varepsilon,\alpha) = \chi^*(\eta=0,\mu,\omega,\varepsilon,\frac{\pi}{2}-\alpha)$. This proves (i).

(ii) Let $\alpha = 0$. Define a projection operator Q of Z into itself by the formula

$$Qu = \frac{1}{2\pi}\int_0^{2\pi} S(\theta)\Gamma(\theta)u\,d\theta \tag{2.15}$$

Q commutes with P and F, $Q\phi_2 = \phi_2$, $Q\phi_1 = 0$ and $[Qu,v] = [u,Qv]$. If χ^* satisfies (2.12a) then also $Q\chi^*$ satisfies this equation because $\alpha = 0$. Therefore $\chi^* = Q\chi^*$. Hence
$K^1(\eta,\mu,\omega,\varepsilon,\alpha=0) = [F(\eta,\mu,\omega,\varepsilon\phi_2+\chi^*(\eta,\mu,\omega,\varepsilon,\alpha=0)),\phi_1] =$
$[QF(\eta,\mu,\omega,\varepsilon\phi_2+\chi^*(\eta,\mu,\omega,\varepsilon,\alpha=0)),\phi_1]$
$[F(\eta,\mu,\omega,\varepsilon\phi_2+\chi^*(\eta,\mu,\omega,\varepsilon,\alpha=0)),Q\phi_1] = 0.$

(iii) follows combining (i) and (ii). □

REMARK. We have used the fact that $S(-\theta)\Gamma(\theta)\phi_1 = \phi_1$ and $S(\theta)\Gamma(\theta)\phi_2 = \phi_2$. Returning to the reaction diffusion equation (1.1) we see that $S(\theta)$ corresponds to a translation in the time variable, whereas $\Gamma(\theta)$ corresponds to a translation in the space variable. Therefore we will call ϕ_1 and ϕ_2 travelling wave solutions.

LEMMA 2.7. *Denote the first terms in the Taylor series of* H *as follows:*

$$H^1(\eta,\mu,z) = (i\omega_0+a\mu+b_1\eta)z_1+c_1z_1|z_1|^2+c_2z_1|z_2|^2+c_3z_1^2z_2$$
$$+\,c_4|z_1|^2\bar{z}_2+c_5|z_2|^2\bar{z}_2+c_6\,\bar{z}_1\,\bar{z}_2^2+\text{h.o.t.}^{(1)},$$

$$H^2(\eta,\mu,z) = (i\omega_0 + a\mu + b_2\eta)z_2 + c_1 z_2|z_2|^2 + c_2 z_2|z_1|^2 + c_3 z_2^2 z_1 + c_4|z_2|^2\,\bar{z}_1 + c_5|z_1|^2\,\bar{z}_1 + c_6\,\bar{z}_2\,\bar{z}_1^2 + \text{h.o.t.}^{(1)},$$

where

$$\text{h.o.t.}^{(1)} \in O((|\mu| + |\eta|)(|z_1| + |z_2|)^3 + (|\mu| + |\mu|)^2(|z_1| + |z_2|) + (|z_1| + |z_2|)^5).$$

Then the expansion of the bifurcation function is given by

$$(2.16)\quad \begin{cases} K^1(\eta,\mu,\omega,\varepsilon,\alpha) = \varepsilon \sin\alpha\{i(\omega-\omega_0) + a\mu + b_1\eta + c_1\varepsilon^2\sin^2\alpha + c_2\varepsilon^2\cos^2\alpha + \text{h.o.t.}^{(2)}\} \\ K^2(\eta,\mu,\omega,\varepsilon,\alpha) = \varepsilon \cos\alpha\{i(\omega-\omega_0) + a\mu + b_2\eta + c_1\varepsilon^2\cos^2\alpha + c_2\varepsilon^2\sin^2\alpha + \text{h.o.t.}^{(2)}\}, \end{cases}$$

where

$$\text{h.o.t.}^{(2)} \in O(\varepsilon^2(|\mu| + |\eta| + |\omega-\omega_0|) + (|\mu| + |\eta|)^2 + \varepsilon^4).$$

<u>PROOF</u>.

$$K^1(\eta,\mu,\omega,\varepsilon,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} e^{-is}F^1(\eta,\mu,\omega,\varepsilon\sin\alpha\,\phi_1 + \varepsilon\cos\alpha\,\phi_2 + \varepsilon\chi^*)ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-is}\{(\omega-\omega_0)\frac{d}{ds}(\varepsilon\sin\alpha e^{is} + \varepsilon\chi^{*1}(s)) + H^1(\eta,\mu,\varepsilon\sin\alpha\,\phi_1 + \varepsilon\cos\alpha\,\phi_2 + \varepsilon\chi^*(s)) - i\omega_0\varepsilon\sin\alpha e^{is}\}ds.$$

Because second order terms in the expansion of H with respect to z are missing, the expansion of χ^* with respect to ε starts at order ε^2. The result follows from the definitions above and Lemma 2.6. □

The remainder of this section is devoted to the analysis of the points where K vanishes for the unperturbed equation ($\eta=0$). The stability of the corresponding periodic solutions shall be determined by evaluating the Floquet exponents. We discuss the perturbed equation ($\eta\neq 0$) in section 4.

THEOREM 2.8. *Let* $\eta = 0$. *Suppose that*

(i) Re $a > 0$,

(ii) Re $c_1 < 0$,

(iii) Re $(c_1 \pm c_2) \neq 0$,

(iv) Re $c_2 \neq 0$.

Then the only possible values of α *at points in a neighbourhood of* $(\mu,\omega,\varepsilon) = (0,\omega_0,0)$ *where* K *vanishes are* $\alpha = 0$, $\alpha = \frac{\pi}{4}$, $\alpha = \frac{\pi}{2}$. *There exists* $\bar{\varepsilon}$ *positive such that for* $\alpha \in \{0,\pi/2\}$, $0 \le \varepsilon \le \bar{\varepsilon}$ *there exist unique* $\mu = \mu(\varepsilon)$, $\omega = \omega(\varepsilon)$ *such that* $K(\eta=0, \mu(\varepsilon), \omega(\varepsilon), \varepsilon, \alpha \in \{0,\pi/2\}) = 0$, *and for* $\alpha = \pi/4$, $0 \le \varepsilon \le \bar{\varepsilon}$ *there exist unique* $\bar{\mu} = \mu(\varepsilon)$, $\bar{\omega} = \bar{\omega}(\varepsilon)$ *such that* $k(\eta=0, \bar{\mu}(\varepsilon), \bar{\omega}(\varepsilon), \varepsilon, \alpha=\pi/4) = 0$. *Moreover*

$$\mu(\varepsilon) = \frac{-\text{Re } c_1}{\text{Re } a}\varepsilon^2 + O(\varepsilon^3),$$

$$\bar{\mu}(\varepsilon) = \frac{-\text{Re}(c_1+c_2)}{\text{Re } a}\varepsilon^2 + O(\varepsilon^3).$$

PROOF. Define $\xi = \varepsilon\sin\alpha$ and $\rho = \varepsilon\cos\alpha$. At $\eta = 0$ we write (2.16) as

$$\begin{cases} \xi \cdot g(\mu,\omega,\xi,\rho) = 0 \\ \rho \cdot g(\mu,\omega,\rho,\xi) = 0. \end{cases}$$

If $\xi = 0$ ($\alpha=0$) we must solve (if $\varepsilon\neq 0$) the equation

$$g(\mu,\omega,\rho,0) = 0.$$

Let $\rho^2 = r$ and $f(\mu,\omega,r) = g(\mu,\omega,\rho,0)$. The real part of f does not depend on ω. By the IFT we obtain $r = r(\mu)$ such that Re $f(\mu,\omega,r(\mu)) = 0$. The imaginary part of f is linear in ω, and we obtain ω as a function of μ such that Im $f(\mu,\omega(\mu),r(\mu)) = 0$. Finally as Re $a \neq 0$ we can also write μ as a function of ε, and the expansion above follows easily. The case $\rho = 0$ ($\alpha=\pi/2$) goes similarly, exchanging the roles of ρ and η. If $\xi \neq 0$ and $\rho \neq 0$ let $\xi^2 = r_1^2$ and $\rho^2 = r_2^2$ and

$$F(\mu,\omega,r_1,r_2) = \begin{pmatrix} g(\mu,\omega,\xi,\rho) \\ g(\mu,\omega,\rho,\xi) \end{pmatrix}.$$

The real part of F does not depend on ω. Re $F(0,\omega,0,0) = 0$ and

$$D_{r_1,r_2} \operatorname{Re} F(0,\omega,0,0) = \begin{pmatrix} \operatorname{Re} c_1 & \operatorname{Re} c_2 \\ \operatorname{Re} c_2 & \operatorname{Re} c_1 \end{pmatrix}.$$

As $\operatorname{Re}(c_1^2 - c_2^2) \neq 0$ we obtain $r_1 = r_1(\mu)$ and $r_2 = r_2(\mu)$ such that $\operatorname{Re} F(\mu,\omega,r_1(\mu),r_2(\mu)) = 0$. We can also solve the equation $0 = H(\mu,r) = \operatorname{Re} F(\mu,\omega,r,r)$ and obtain r as a function of μ. By the uniques part of the IFT we conclude that $r_1(\mu) = r_2(\mu)$. Consequently $\operatorname{Im} F(\mu,\omega,r_1(\mu),r_1(\mu)) = 0$ reduces to a single equation, linear in ω, and we obtain ω as a function of μ. Also here we obtain μ as a function of ε because Re $a \neq 0$.

REMARK 1. The periodic solutions which bifurcate from the ρ,ξ-axis are invariant under $S(\theta)\Gamma(\theta)$, $S(\theta)\Gamma(-\theta)$ respectively. Therefore we will call these solutions travelling wave solutions. Compare the remark below Lemma 2.6. The periodic solutions which bifurcate from the diagonal are invariant under L. These are the standing wave solutions.

REMARK 2. With each travelling wave solution u corresponds a circle of solutions $S(\xi)u$, $0 \leq \xi < 2\pi$. With each standing wave solution corresponds a torus of solutions $S(\xi)\Gamma(\theta)u$, $0 \leq \xi,\theta < 2\pi$.

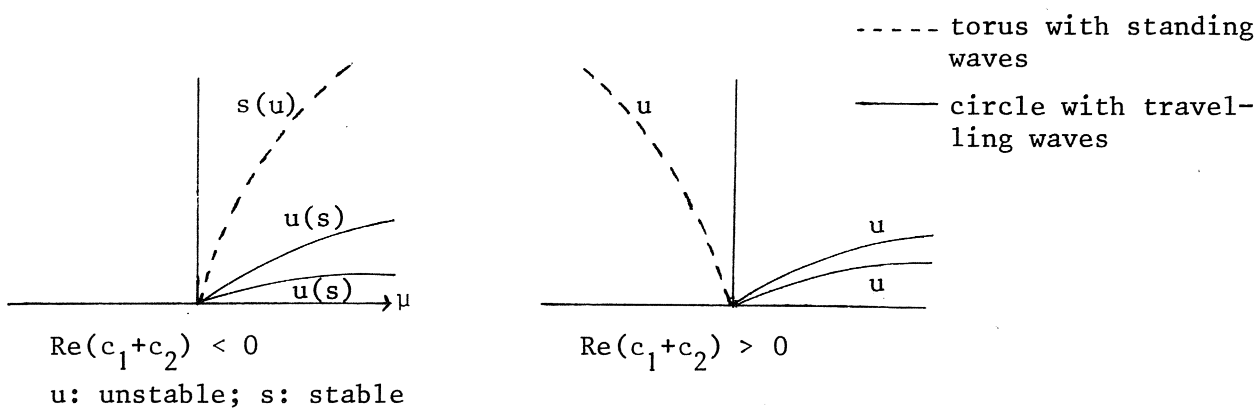


figure 1.

REMARK 3. The first two assumptions replace without loss of generality: $\operatorname{Re} a \neq 0$, $\operatorname{Re} c_1 \neq 0$.

In the next theorem we give asymptotic expressions for the four characteristic multipliers of the periodic solutions. It is obvious that two of them are zero for the standing wave solutions ($\alpha=\pi/4$). Both

$$s \mapsto \lim_{\theta\to 0} \frac{\Gamma(\theta)u(\varepsilon)(s)-u(\varepsilon)(s)}{\theta},$$

and

$$s \mapsto \lim_{\xi\to 0} \frac{S(\xi)u(\varepsilon)(s)-u(\varepsilon)(s)}{\xi}$$

are independent 2π-periodic solutions of the linearized equation. For the travelling wave solutions ($\alpha=0$, $\alpha=\frac{\pi}{2}$) those two solutions are not independent. Generically only one characteristic multiplier will be equal to zero. In section 3 we will use this theorem to give a complete description of the dynamics on the center manifold. The proof of this theorem is straightforward but lengthy. We closely follow Crandall & Rabinowitz [8]. The proof is postponed to appendix 1 (section 5).

THEOREM 2.9. *For $\alpha = \pi/4$, the two non trivial characteristic exponents are given by:*

$$\begin{cases} \kappa_1 = \mathrm{Re}(c_1+c_2)\varepsilon^2 + O(\varepsilon^3) \\ \kappa_2 = \mathrm{Re}(c_1-c_2)\varepsilon^2 + O(\varepsilon^3) \end{cases}.$$

For $\alpha \in \{0,\frac{\pi}{2}\}$, the three non trivial characteristic exponents are given by:

$$\begin{cases} \kappa_1 = 2\mathrm{Re}c_1\varepsilon^2 + O(\varepsilon^3) \\ \kappa_2 = (\mathrm{Re}(c_2-c_1) + i\ \mathrm{Im}(c_2-c_1))\varepsilon^2 + O(\varepsilon^3) \\ \kappa_3 = (\mathrm{Re}(c_2-c_1) - i\ \mathrm{Im}(c_2-c_1))\varepsilon^2 + O(\varepsilon^3). \end{cases}$$

This result shows that generically the stability is as indicated in figure 1. It also indicates how to construct examples of a diffusive Lotka-Volterra system with (at least) three species exhibiting stable periodic (in both time and space) solutions. This would complete results of Kishimoto [21].

3. DYNAMICS ON THE CENTER MANIFOLD: THE UNPERTURBED EQUATION

The o.d.e. on the center manifold inherits the symmetry properties of the original equation. The method of averaging is used to bring the vector field by a smooth coordinate transformation, smoothly depending on the parameters, in a form which is covariant with respect to the linear action of the unperturbed equation at $\mu = 0$ up to finite order. It is not a priori obvious that by the averaging procedure the symmetry properties which are already present do not disappear. Fortunately this is not the case as we shall prove later on.

The symmetry which is present from the beginning reduces the complexity of the normal forms. If one could justify the averaging up to all order then one would obtain the bifurcation function as the r.h.s. of the differential equation.

First we deal with the unperturbed equation ($\eta=0$) and start the averaging procedure with the equation (compare Lemma 2.7)

$$(3.1)\qquad \begin{cases} \dot z_1 = (i\omega_0+a\mu)z_1 + c_1 z_1|z_1|^2 + c_2 z_1|z_2|^2 + c_3 z_1^2 z_2 \\ \qquad + c_4|z_1|^2 \bar z_2 + c_5|z_2|^2 \bar z_2 + c_6 \bar z_1 \bar z_2^2 + \text{h.o.t.} \\ \dot z_2 = (i\omega_0+a\mu)z_2 + c_1 z_2|z_2|^2 + c_2 z_2|z_1|^2 + c_3 z_2^2 z_1 \\ \qquad + c_4|z_2|^2 \bar z_1 + c_5|z_1|^2 \bar z_1 + c_6 \bar z_2 \bar z_1^2 + \text{h.o.t.}, \end{cases}$$

where

$$\text{h.o.t.} \in O(|\mu|(|z_1|+|z_2|)^3 + |\mu|^2(|z_1|+|z_2|) + (|z_1|+|z_2|)^5).$$

For convenience we will write equation (3.1) as

$$(3.2)\qquad \dot z = (1 + \frac{a(\mu)}{i\omega_0})Bz + H_3(z) + \text{h.o.t.},$$

the subindex indicating the degree of the homogeneous polynomial in z_1, $\bar z_1$, z_2, $\bar z_2$.

If we apply the smooth near identity transformation

$$z = \xi + K(\xi), \quad K(\xi) \in O(|\xi|^2)$$

then in the new coordinates ξ the equation

$$\dot{z} = F(z)$$

transforms to

$$\dot{\xi} = F(\xi+K(\xi)) + D\tilde{K}(\xi+K(\xi))F(\xi+K(\xi)) := \tilde{F}(\xi).$$

where by definition $\tilde{K}$ satisfies

$$\xi = z + \tilde{K}(z).$$

LEMMA 3.1. *If* K *is covariant with respect to the matrix* M*, i.e.* $MK(\xi) = K(M\xi)$, *then the same is true for* $\tilde{K}$.

PROOF.

$$\tilde{K}(Mz) = \tilde{K}(M(\xi+K(\xi)) = \tilde{K}(M\xi+K(M\xi)) =$$

$$-K(M\xi) = -MK(\xi) = M\tilde{K}(\xi+K(\xi)) = M\tilde{K}(z). \quad \square$$

LEMMA 3.2. *If the vector field* F *is covariant with respect to the matrix* M *and the same is true for the coordinate transformation, then also the vector field* $\tilde{F}$ *is covariant with respect to* M.

PROOF. It follows from Lemma 3.1, the explicit formula for $\tilde{F}$ above and the relation $MDK(\xi) = DK(M\xi)M$. $\square$

Cubic avaraging is based on the fact that a change of variables

$$z = \xi + K_3(\xi),$$

where K_3 is a homogeneous cubic, can be performed so that the following two conditions are satisfied

(i) $\tilde{H}(\mu,z)$, the vectorfield after transformation, is equivariant with respect to $\Gamma(\theta)$, $0 \leq \theta < 2\pi$, and L,

(ii) $\tilde{H}_3(z)$ is equivariant with respect to the matrix $\exp(B\tau)$, $0 \leq \tau < 2\pi$.

The transformation which settles these conditions is given by

$$(3.2)\qquad \begin{cases} z_1 = \xi_1 + c_3\xi_1^2\xi_2 + c_4|\xi_1|^2\bar{\xi}_2 + c_5|\xi_2|^2\bar{\xi}_2 + c_6\bar{\xi}_1\bar{\xi}_2^2 \\ z_2 = \xi_1 + c_3\xi_2^2\xi_1 + c_4|\xi_2|^2\bar{\xi}_1 + c_5|\xi_1|^2\bar{\xi}_1 + c_6\bar{\xi}_2\bar{\xi}_1^2. \end{cases}$$

After this transformation we are left with the system of equations

$$(3.3)\qquad \begin{aligned} \dot{z}_1 &= (i\omega_0+a\mu)z_1 + c_1z_1|z_1|^2 + c_2z_1|z_2|^2 + \text{h.o.t.} \\ \dot{z}_2 &= (i\omega_0+a\mu)z_2 + c_1z_2|z_2|^2 + c_2z_2|z_1|^2 + \text{h.o.t.}, \end{aligned}$$

where

$$\text{h.o.t} \in O(|\mu|(|z_1|+|z_2|)^3+|\mu|^2(|z_1|+|z_2|)+(|z_1|+|z_2|)^5).$$

We introduce two sets of polar coordinates

$$z_1 = r_1e^{i\theta_1}, \quad z_2 = r_2e^{i\theta_2}$$

and in the $r_1 - r_2$ plane we rescale, with $|\mu|^{\frac{1}{2}}$, i.e. for $\mu \neq 0$ we introduce new variables $\bar{r}_1$ and $\bar{r}_2$ defined by

$$r_1 = \bar{r}_1|\mu|^{\frac{1}{2}} \text{ and } r_2 = \bar{r}_2|\mu|^{\frac{1}{2}}.$$

We again replace $\bar{r}_1$ and $\bar{r}_2$ by r_1 and r_2 respectively. Tranforming equation (3.3) to the new variables, rescaling the time with $|\mu|$ yields for $\mu \neq 0$

$$(3.4)\qquad \begin{cases} \dot{r}_1 = r_1(\text{Re}\, a\,\text{sgn}\{\mu\} + \text{Re}\, c_1r_1^2 + \text{Re}\, c_2r_2^2) + O(|\mu|) \\ \dot{r}_2 = r_2(\text{Re}\, a\,\text{sgn}\{\mu\} + \text{Re}\, c_1r_2^2 + \text{Re}\, c_2r_1^2) + O(|\mu|), \end{cases}$$

uniformly on compact sets as $\mu \to 0$.

Taking the limit for $\mu \to 0$ we obtain equation (3.5). Notice that the right hand side is a scaled version of the real part of the bifurcation function truncated at order three (in ε).

$$(3.5)\qquad \begin{cases} \dot{r}_1 = r_1(\text{Re}\, a\,\text{sgn}\{\mu\} + \text{Re}\, c_1r_1^2 + \text{Re}\, c_2r_2^2) \\ \dot{r}_2 = r_2(\text{Re}\, a\,\text{sgn}\{\mu\} + \text{Re}\, c_1r_2^2 + \text{Re}\, c_2r_1^2). \end{cases}$$

We assume that the conditions of Theorem 2.8 are satisfied and distinguish between the following three cases

1. $Re(c_1+c_2) < 0$, $Re(c_1-c_2) < 0$
2. $Re(c_1+c_2) < 0$, $Re(c_1-c_2) > 0$
3. $Re(c_1+c_2) > 0$, $Re(c_1-c_2) < 0$.

In each case we also distinguish between μ positive and μ negative. Fig. 2 depicts the phase portrait of this limit system in each case.

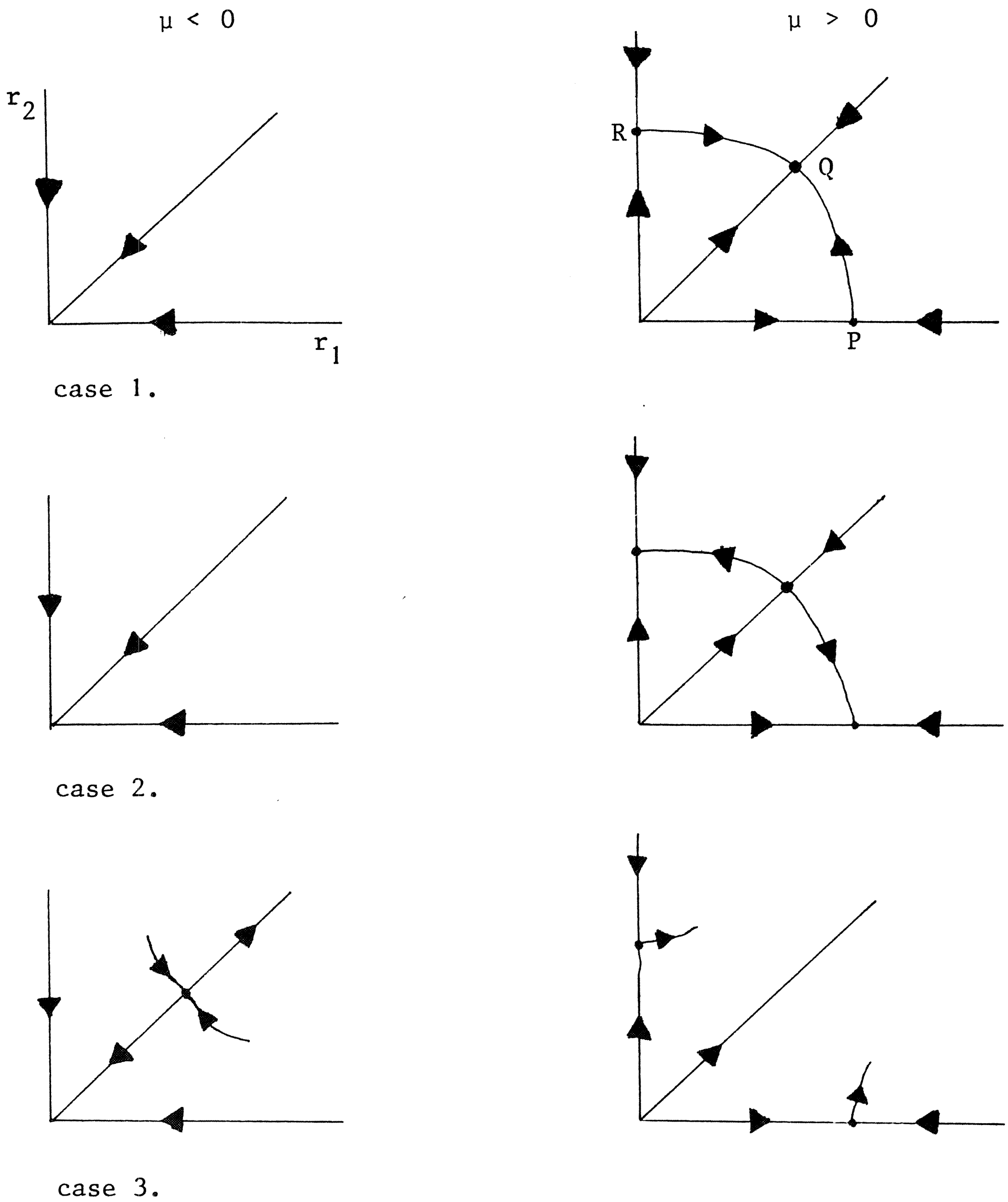


fig. 2

In these pictures, the distance from the fixed points to the origin is of order 1, thus of the order $|\mu|^{\frac{1}{2}}$ in the unscaled coordinates. The geometric behaviour of the flow of the vector field (3.4) is only slightly different. The r_1 and r_2 axis are no longer invariant, the diagonal however still is. The points P and R represent in four space an invariant circle, the point Q represents an invariant two torus. From Theorem 2.8 and 2.9 we infer that there exists for μ small enough a fixed but small neighbourhood of P, say U, such that U contains an invariant circle. This invariant circle has a two dimensional unstable and a one dimensional stable manifold. Each point of the boundary of U which is not on the stable manifold will enter (under the flow) in finite time a neighbourhood V of Q. Depending on the initial condition the orbit will approach one of the periodic orbits on the two torus of periodic orbits which is contained in V. I.e. the torus is asymptotically stable with asymptotic phase [12]. Similar arguments apply to case 2 and case 3.

4. THE PERTURBED EQUATION: THE SECONDARY BIFURCATION OF AN INVARIANT TORUS FROM THE TRAVELLING WAVE SOLUTIONS

We start to analyze the perturbed equation in great detail assuming H5 - H7. At the end of this section we will list all possible situations depending on the relative size of c_1, c_2, b_1, b_2.

H5 $\quad$ Re $a > 0$

H6 $\quad$ Re $c_1 <$ Re $c_2 < 0$

H7 $\quad$ Re $b_1 <$ Re b_2.

In the bifurcation equations we replace $\varepsilon \sin\alpha$, $\varepsilon\cos\alpha$ by ξ,ρ respectively. In Lemma 2.7 we have derived that (see (2.16))

$$(4.1)\qquad \begin{cases} \xi(i(\omega-\omega_0)+a\mu+b_1\eta+c_1\xi^2+c_2\rho^2+\text{h.o.t.}) = 0 \\ \rho(i(\omega-\omega_0)+a\mu+b_2\eta+c_2\xi^2+c_1\rho^2+\text{h.o.t.}) = 0 \\ \text{h.o.t.} \in O(|\mu|+|\eta|+|\omega-\omega_0|)(|\xi|+|\rho|)^2 + (|\mu|+|\eta|)^2 + (|\xi|+|\rho|)^4). \end{cases}$$

If we ignore the higher order terms, than the zeros of the real parts of equation (4.1) are the intersections of the two curves

$$\{(\xi,\rho) \in \mathbb{R}^2 \mid \xi.L_1(\mu,\eta) = 0\}$$
$$\{(\xi,\rho) \in \mathbb{R}^2 \mid \rho.L_2(\mu,\eta) = 0\},$$

where

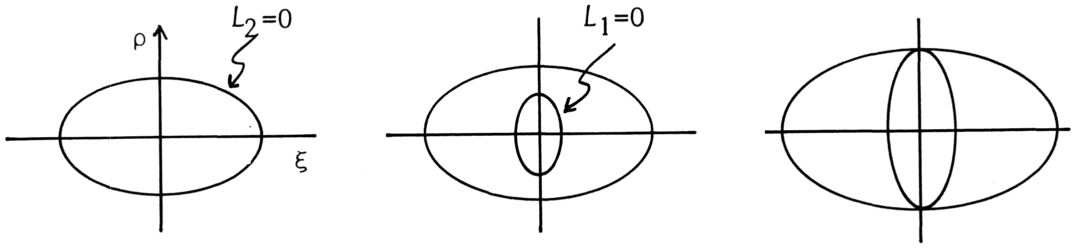
$$L_1(\mu,\eta) = \mathrm{Re}\, a\, \mu + \mathrm{Re}\, b_1\, \eta + \mathrm{Re}\, c_1\, \xi^2 + \mathrm{Re}\, c_2\, \rho^2,$$
$$L_2(\mu,\eta) = \mathrm{Re}\, a\, \mu + \mathrm{Re}\, b_2\, \eta + \mathrm{Re}\, c_2\, \xi^2 + \mathrm{Re}\, c_1\, \rho^2.$$

The curves $L_1(\mu,0) = 0$ and $L_2(\mu,0) = 0$ represent for μ positive two ellipses intersecting transversally along the diagonals $\xi = \pm\rho$. For η positive the curve $L_2 = 0$ represents an ellips if μ is larger then $-\eta\, \mathrm{Re}\, b_2/\mathrm{Re}\, a$. The ellipse represented by $L_1 = 0$ starts growing at $\mu = -\eta\, \mathrm{Re}\, b_1/\mathrm{Re}\, a$ and is tangent to the other ellips at $(\xi,\rho) = (0,\pm\eta\rho_0)$ at $\mu = \eta\hat{\mu}_0$, where

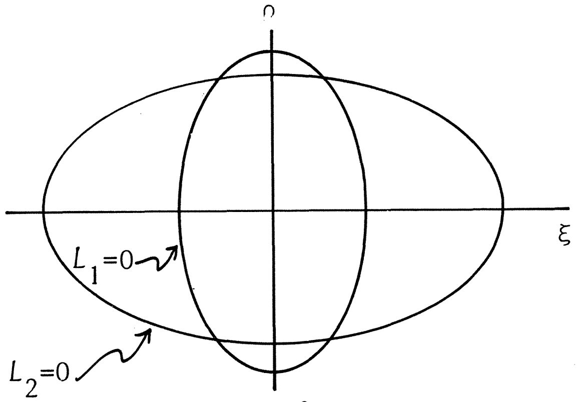
$$(4.2) \qquad \rho_0^2 = \frac{\mathrm{Re}(b_1-b_2)}{\mathrm{Re}(c_1-c_2)},$$

$$(4.3) \qquad \hat{\mu}_0 = \frac{\mathrm{Re}\, c_1\, \mathrm{Re}\, b_1 - \mathrm{Re}\, b_2\, \mathrm{Re}\, c_2}{\mathrm{Re}\, a\, \mathrm{Re}(c_2-c_1)}.$$

This is illustrated in figure 3.



$-\eta\frac{\mathrm{Re} b_2}{\mathrm{Re} a} < \mu < -\eta\frac{\mathrm{Re} b_1}{\mathrm{Re} a}$ $\qquad$ $-\eta\frac{\mathrm{Re} b_1}{\mathrm{Re} a} < \mu < \eta\hat{\mu}_0$ $\qquad$ $\mu = \eta\hat{\mu}_0$



$\mu > \eta\hat{\mu}_0$

These geometric considerations suggest the rescaling

(4.4) $\mu \mapsto \hat{\mu}\eta.$

Of course the results of section 2 concerning the travelling wave solutions, corresponding to the intersection of the ellipses with the positive ξ and ρ axis, are applicable. We omit the details of the calculations except for the results on the characteristic multipliers. That is part of appendix 1. We consider $\hat{\mu}$ as a variable and expand η as a function of ε (which measures the amplitude of the periodic solution) and $\hat{\mu}$.

<u>THEOREM 4.1</u>. *Let* I *be an interval* $[i_-, i_+]$ *such that* $-\frac{\mathrm{Re}\, b_2}{\mathrm{Re}\, a} < i_- < \hat{\mu}_0$ *and* $i_+ > \hat{\mu}_0$. *There exist a positive constant* $\bar{\varepsilon}$ *and* C^{r-1}*-functions* $(\varepsilon,\hat{\mu}) \mapsto \eta(\varepsilon,\hat{\mu})$; $(\varepsilon,\hat{\mu}) \mapsto \omega(\varepsilon,\hat{\mu})$; $(\varepsilon,\hat{\mu}) \mapsto u(\varepsilon,\hat{\mu})$ *defined for* $0 \le \varepsilon \le \bar{\varepsilon}$ *and* $\hat{\mu} \in I$ *which satisfy for all* $\hat{\mu} \in I$: $\eta(0,\hat{\mu}) = 0$; $\omega(0,\hat{\mu}) = \omega_0$; $u(0,\hat{\mu}) = 0$ *such that*

$$\begin{cases} \mathbb{J}_0 u + F(\eta,\eta\hat{\mu},\omega,u) = 0 \\ [u(\varepsilon,\hat{\mu}),\phi_2] = \varepsilon;\ [u(\varepsilon,\hat{\mu}),\phi_1] = 0. \end{cases}$$

The characteristic exponents of the periodic solution $u(s)$ *are given by* $\{0, 2\mathrm{Re}\, c_1\varepsilon^2 + O(\varepsilon^3); k(\varepsilon,\hat{\mu}); \overline{k(\varepsilon,\hat{\mu})}\}$, *where*

$$k(\varepsilon,\hat{\mu}) = k_2(\hat{\mu})\varepsilon^2 + O(\varepsilon^3) \qquad (\textit{uniformly for } \hat{\mu} \in I),$$

$$k_2(\hat{\mu}) = \mathrm{Re}(c_2-c_1) + \frac{\mathrm{Re}\, c_1}{\mathrm{Re}(b_2+a\hat{\mu})}\mathrm{Re}(b_2-b_1) + i\{\mathrm{Im}(c_2-c_1) + \frac{\mathrm{Re}\, c_1}{\mathrm{Re}(b_2+a\hat{\mu})}\mathrm{Im}(b_2-b_1)\}.$$

The expansion of η *as a function of* ε *starts at order* ε^2*:*

$$\eta(\varepsilon,\hat{\mu}) = \eta_2(\hat{\mu})\varepsilon^2 + O(\varepsilon^3) \qquad (\textit{uniformly for } \hat{\mu} \in I),$$

$$\eta_2(\hat{\mu}) = \frac{-\mathrm{Re}\, c_1}{\mathrm{Re}(b_2+a\hat{\mu})}.$$

The travelling wave solution described above corresponds to the intersection of $L_2 = 0$ with the positive ρ-axis. From the above theorem we see that $\mathrm{Re}(k_2(\hat{\mu}_0))$ vanishes and $\frac{d}{d\hat{\mu}}\mathrm{Re}(k_2(\hat{\mu}_0)) > 0$. This corresponds in figure 3 to the fact that the two ellipses intersect at $\mu = \eta\hat{\mu}_0$ and if we follow the point of intersection in the first quadrant we find that its ξ-component leaves the ρ-axis with positive speed if μ is increased with positive speed. This is the essential information which we exploit in the next theorem.

<u>THEOREM 4.1</u>. *There exist* $\bar{\eta} > 0$ *and a* C^{r-2}*-mapping* $\eta \to \hat{\mu}^*(\eta)$ *from* $[0,\bar{\eta}]$ *into* $\mathbb{R}$ *such that* $\hat{\mu}^*(0) = \hat{\mu}_0$*; if we fix* η *in the interval* $(0,\bar{\eta}]$ *and we follow the above mentioned travelling wave solutions corresponding to this* η *as* $\hat{\mu}$ *varies, then two characteristic exponents cross the imaginary axis with positive speed if* $\hat{\mu}$ *passes through* $\hat{\mu}^*(\eta)$ *with positive speed.*

<u>PROOF</u>. The expansion of η starts with

$$\eta(\varepsilon,\hat{\mu}) = \eta_2(\hat{\mu}_0)\varepsilon^2 + O(|\hat{\mu}-\hat{\mu}_0||\varepsilon|^2 + |\varepsilon|^3),$$

and as $\eta_2(\hat{\mu}_0)$ is positive we find that

$$\varepsilon = \sqrt{\frac{\eta}{\eta_2(\hat{\mu}_0)}} + O(|\hat{\mu}-\hat{\mu}_0||\eta| + |\eta|^{\frac{3}{2}}).$$

This expression we substitute in the expansion of κ:

$$\operatorname{Re}\kappa(\hat\mu,\eta) = \rho(\hat\mu-\hat\mu_0)\eta + O(|\eta|^{\frac{3}{2}}+|\hat\mu-\hat\mu_0|^2|\eta|),$$

$$\rho = \frac{\operatorname{Re} a.\operatorname{Re}(b_2-b_1)}{\operatorname{Re}(b_2+a\hat\mu_0)} > 0.$$

Define ψ by

$$\psi(\hat\mu,\eta) = \begin{cases} \dfrac{\operatorname{Re}\kappa(\hat\mu,\eta)}{\eta}, & \eta > 0, \\ \dfrac{\partial}{\partial\eta}\operatorname{Re}\kappa(\hat\mu,0) & \eta = 0. \end{cases}$$

By the Implicit Function Theorem we find the mapping $\hat\mu^*$ as above, satisfying $\operatorname{Re}\kappa(\hat\mu^*(\eta),\eta) = 0$. The last part of the theorem follows from the identity $\frac{\partial}{\partial\hat\mu}\operatorname{Re}\kappa(\hat\mu,\eta) = \rho\eta + O(|\eta|^{\frac{3}{2}})$. □

COROLLARY 4.2. *If* $\operatorname{Im}\kappa(\mu^*(\eta),\eta) \neq 0$ *then an attractive invariant torus bifurcates from the travelling wave solutions as* $\hat\mu$ *passes through* $\hat\mu^*(\eta)$.
For a proof we refer to Iooss [16] or Langford [22].

REMARK. One usually encounters a non-strongly resonance condition which the characteristic multipliers have to satisfy at the bifurcation point. This condition is necessary to bring the equations into normal form up to a sufficient order. Due to the symmetry a lot of terms in the Taylor expansion are not present and we do not have to impose this condition.

In the remainder of this section we study the dynamical system on the center manifold. The remarks we made in the previous section concerning the averaging procedure carry over to the perturbed equation ($\eta\neq 0$). After linear and cubic overaging we are left with the system of equations

$$(4.5)\quad \begin{cases} \dot z_1 = (i\omega_0+a\mu+b_1\eta)z_1 + c_1z_1|z_1|^2 + c_2z_1|z_2|^2 + \text{h.o.t.} \\ \dot z_2 = (i\omega_0+a\mu+b_2\eta)z_2 + c_1z_2|z_2|^2 + c_1z_2|z_1|^2 + \text{h.o.t.} \\ \text{h.o.t.} \in O((|\mu|+|\eta|)(|z_1|+|z_2|)^3 + (|\mu|+|\eta|)^2(|z_1|+|z_2|) \\ \qquad + (|z_1|+|z_2|)^5). \end{cases}$$

We introduce two sets of polar coordinates

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2},$$

and in the $r_1 - r_2$ plane we rescale with $\sqrt{\eta}$; i.e. for $\eta > 0$ we introduce new variables $\hat{r}_1$ and $\hat{r}_2$ defined by

$$r_1 = \hat{r}_1\sqrt{\eta} \text{ and } r_2 = \hat{r}_2\sqrt{\eta}.$$

and suppress the hats. Again we rescale μ with a factor η as in (4.4). Transforming equation (4.5) to the new variables and rescaling the time with η yields for $\eta > 0$:

$$\text{(4.6)} \qquad \begin{cases} \dot{r}_1 = r_1(\operatorname{Re} a\,\hat{\mu} + \operatorname{Re} b_1 + \operatorname{Re} c_1\, r_1^2 + \operatorname{Re} c_2\, r_2^2) + O(\eta) \\ \dot{r}_2 = r_2(\operatorname{Re} a\,\mu + \operatorname{Re} b_2 + \operatorname{Re} c_2\, r_1^2 + \operatorname{Re} c_1\, r_2^2) + O(\eta). \end{cases}$$

Taking the limit as $\eta \to 0$ we obtain the limit system

$$\text{(4.7)} \qquad \begin{cases} \dot{r}_1 = r_1(\operatorname{Re} a\,\hat{\mu} + \operatorname{Re} b_1 + \operatorname{Re} c_1\, r_1^2 + \operatorname{Re} c_2\, r_2^2) \\ \dot{r}_2 = r_2(\operatorname{Re} a\,\hat{\mu} + \operatorname{Re} b_2 + \operatorname{Re} c_2\, r_1^2 + \operatorname{Re} c_1\, r_2^2). \end{cases}$$

This system has on the r_2-axis the fixed point P

$$P = \left(0,\left[\frac{-\operatorname{Re}(a\hat{\mu}+b_2)}{\operatorname{Re} c_1}\right]^{\frac{1}{2}}\right) = (0,\bar{r}_2),$$

for $\hat{\mu} > -\dfrac{\operatorname{Re} b_2}{\operatorname{Re} a}$.

The eigenvalues of the linearization around this fixed point are

$$\text{(4.8)} \qquad \begin{cases} \lambda_1(P) = \bar{r}_2^2\left\{\operatorname{Re}(c_2-c_1) + \dfrac{\operatorname{Re} c_1 \operatorname{Re}(b_2-b_1)}{\operatorname{Re}(a\hat{\mu}+b_2)}\right\} \\ \lambda_2(P) = 2\bar{r}_2^2 \operatorname{Re} c_1. \end{cases}$$

Therefore the fixed point is stable for $\dfrac{-\operatorname{Re} b_2}{\operatorname{Re} a} < \hat{\mu} < \hat{\mu}_0$ and unstable for $\hat{\mu} > \hat{\mu}_0$. Note that at $\hat{\mu} = \hat{\mu}_0$, $\bar{r}_2^2 = \rho_0^2$ (see (4.2)). Now we vary $\hat{\mu}$ around $\hat{\mu}_0$. The fixed point that bifurcates from the r_2-axis we will call R. If we parametrize $\hat{\mu}$ by $\xi \in \mathbb{R}_+$ as

$$\hat{\mu} = \hat{\mu}_0 - \frac{\mathrm{Re}(c_1+c_2)}{\mathrm{Re}\ a}\,\xi^2,$$

then

$$R = (\xi,(\rho_0^2+\xi^2)^{\frac{1}{2}}).$$

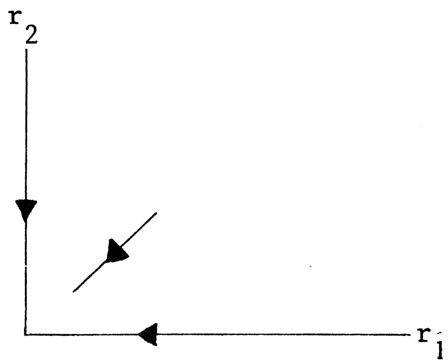
The eigenvalues of the linearization around this fixed point are

$$(4.9)\qquad \begin{cases} \lambda_1(R) = 2\rho_0^2\,\mathrm{Re}\ c_1 + O(\xi^2) \\ \lambda_2(R) = 2\dfrac{((\mathrm{Re}\ c_1)^2-(\mathrm{Re}\ c_2)^2)}{\mathrm{Re}\ c_1}\,\xi^2 + O(\xi^4). \end{cases}$$

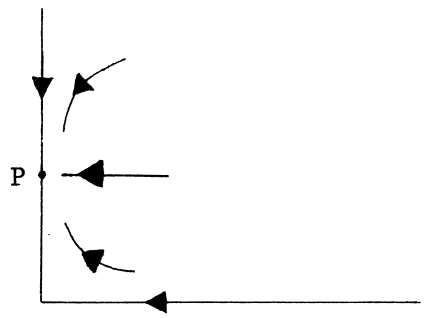
Finally on the r_1-axis we have for $\hat{\mu}$ larger than $\frac{-\mathrm{Re}\ b_1}{\mathrm{Re}\ a}$ the fixed point

$$Q = ([\frac{-\mathrm{Re}(a\hat{\mu}+b_1)}{\mathrm{Re}\ c_1}]^{\frac{1}{2}},0),$$

which is unstable. Figure 4 depicts the phase portrait when μ varies from values less then $\frac{-\mathrm{Re}\ b_2}{\mathrm{Re}\ a}$ to values larger then $\hat{\mu}_0$. Figure 5 shows the corresponding bifurcation diagram.



$\hat{\mu} < \frac{-\mathrm{Re}\ b_2}{\mathrm{Re}\ a}$



$\frac{-\mathrm{Re}\ b_2}{\mathrm{Re}\ a} < \hat{\mu} < \frac{-\mathrm{Re}\ b_1}{\mathrm{Re}\ a}$

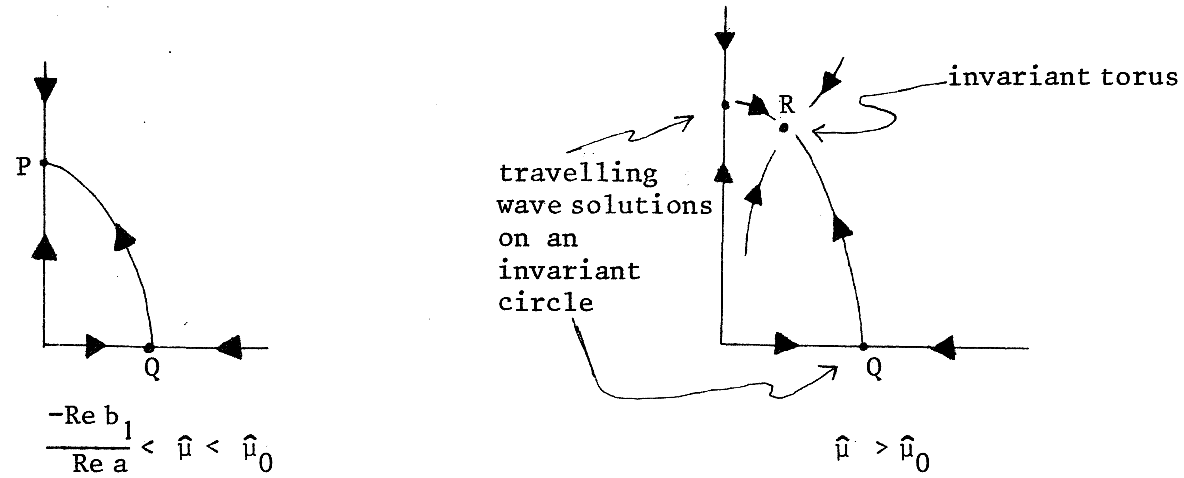


figure 4.

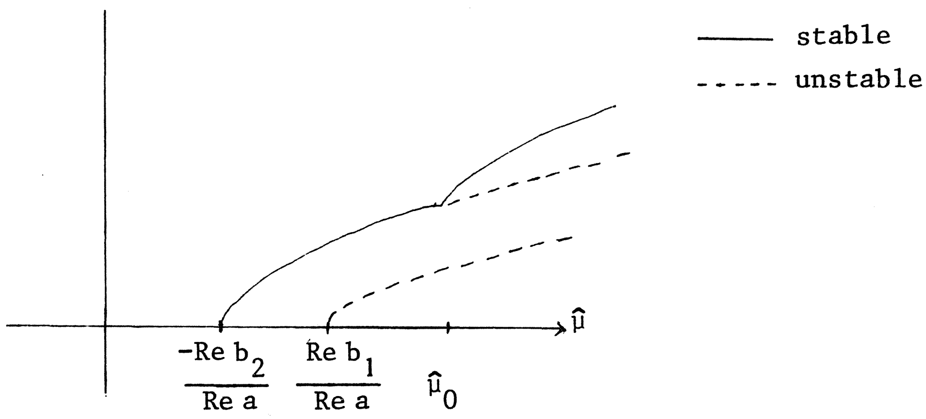


figure 5.

The qualitative behaviour near the zero solution for η small but positive is the same as depictured above. There exist a positive constant $\gamma = \gamma(\eta)$ such that annulus $A(\eta)$

$$(4.10) \qquad A(\eta) \begin{cases} (1-\gamma(\eta))\xi \le r_1 \le (1+\gamma(\eta))\xi \\ (1-\gamma(\eta))(\rho_0^2+\xi^2)^{\frac{1}{2}} \le r_2 \le (1+\gamma(\eta))(\rho_0^2+\xi^2)^{\frac{1}{2}} \end{cases}$$

remains positively invariant under the flow, so that the invariant torus lies entirely within the annulus. All solutions starting in a neighbourhood of the origin, which do not lie on the unstable manifold of a travelling wave solution, are attracted by the invariant torus. Except for a positive factor, the real parts of the characteristic multipliers of the travelling wave solutions on the r_2-axis, are given by $\lambda_1(P)+O(\eta)$, $\lambda_2(P)+O(\eta)$ (see formula (4.8)). The corresponding values for the travelling wave solutions on the r_1-axis are obtained by interchanging the roles of b_1 and b_2.

If instead of $H_6 - H_7$ the conditions

H_6' Re c_2 < Re c_1 < 0,

H_7' Re b_2 < Re b_1,

hold then the bifurcating torus is unstable and the annulus is negatively invariant. In the next figure we depict the various bifurcations and their limit behaviour depending on the real parts of b_1, b_2, c_1, c_2. The scaling of variables is as before. Throughout we assume that Re c_1 < 0 and Re a > 0.

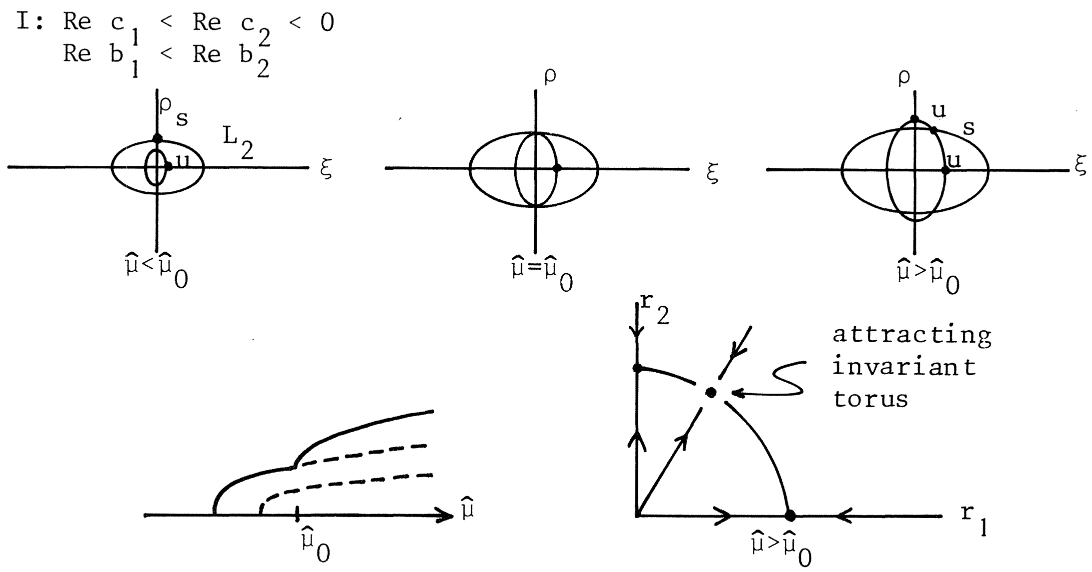


figure 6a

II: Re c_2 < Re c_1 < 0

Re b_2 < Re b_1

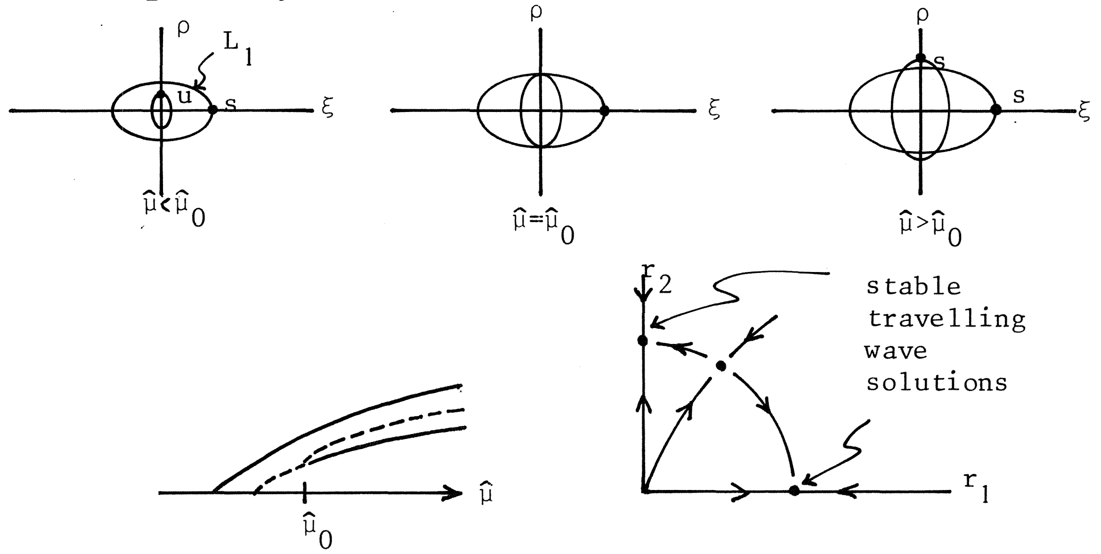


figure 6b

III: Re c_2 > 0

Re b_1 < Re b_2

Re(c_1+c_2) < 0

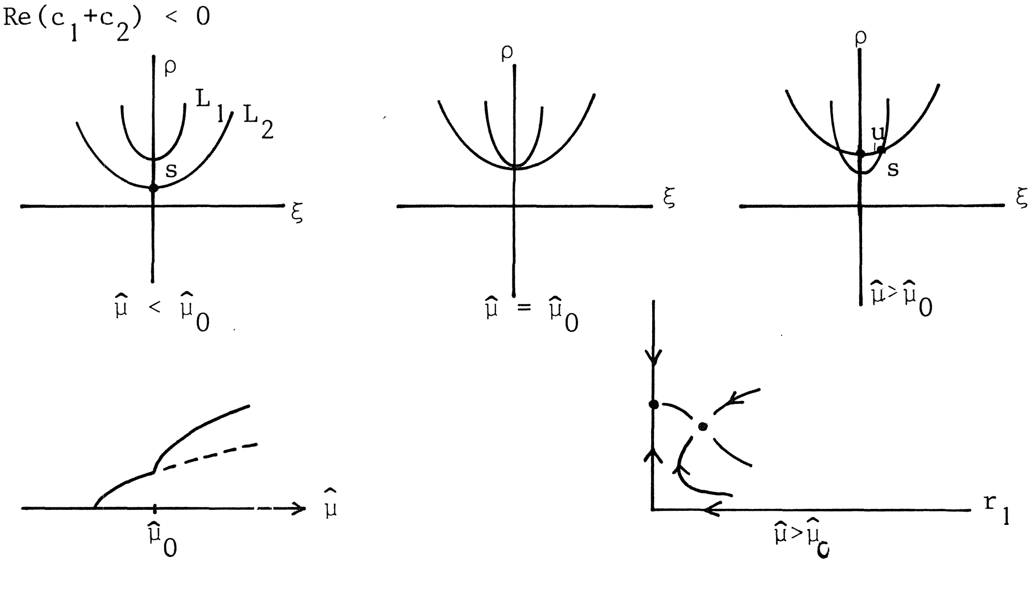


figure 6c

IV: $\mathrm{Re}\, c_2 > 0$
$\mathrm{Re}\, b_1 < \mathrm{Re}\, b_2$
$\mathrm{Re}\,(c_1+c_2) > 0$

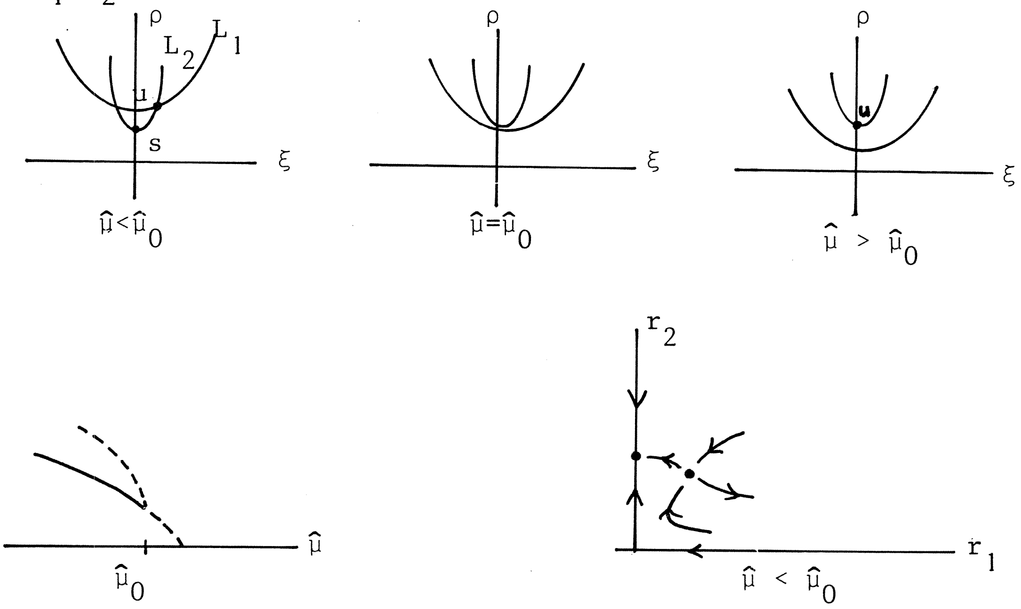


figure 6d

Exchanging the roles of b_1 and b_2 boils down to exhanging ξ, r_1 with ρ, r_2 respectively.

5. APPENDIX 1.

I: Proof of Theorem 2.10.

We introduce for $\alpha \in [0,\pi/2]$ the series expansions with respect to ε for the solution $(u(\varepsilon),\mu(\varepsilon),\omega(\varepsilon))$ of equation (2.9)

$$(5.1)\qquad \begin{aligned} u(\varepsilon) &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + O(\varepsilon^3),\\ \omega(\varepsilon) &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3),\\ \mu(\varepsilon) &= \mu_1\varepsilon + \mu_2\varepsilon^2 + O(\varepsilon^3).\end{aligned}$$

The parameterization by ε has been choosen such that

$$(5.2)\qquad [u(\varepsilon),\phi_1] = \varepsilon \sin\alpha, \quad [u(\varepsilon),\phi_2] = \varepsilon\cos\alpha,$$

in agreement with the definition of u above (2.12).
Therefore

$$(5.3)\qquad \begin{aligned} &[u_1,\phi_1] = \sin\alpha, \quad [u_1,\phi_2] = \cos\alpha,\\ &[u_n,\phi_i] = 0, \quad n \ge 2, \quad i \in \{1,2\}.\end{aligned}$$

Collecting terms of order ε^2 in (2.9), and taking the inner product with ϕ_1 and ϕ_2 yields

$$(5.4)\qquad \mu_1 = \omega_1 = 0, \quad u_1 = \sin\alpha\begin{pmatrix} e^{is}\\ 0\end{pmatrix} + \cos\alpha\begin{pmatrix} 0\\ e^{is}\end{pmatrix}.$$

Collecting the terms of order ε^3 we obtain

$$(5.5)\qquad \mathbb{L}_0 u_3 + \mu_2 H_{\mu z}(0)u_1 + \frac{1}{6}H_{zzz}(0)(u_1^3) - \omega_2\frac{du_1}{ds} = 0,$$

writing 0 to indicate $\eta = 0$, $\mu = 0$ and $u = 0$. The solvability conditions are

$$(5.6)\qquad \begin{cases} \sin\alpha\{\mu_2 a - i\omega_2 + \sin^2\alpha\, c_1 + \cos^2\alpha\, c_2\} = 0,\\ \sin\alpha\{\mu_2 a - i\omega_2 + \cos^2\alpha\, c_1 + \sin^2\alpha\, c_2\} = 0,\end{cases}$$

a, c_1 and c_2 are defined in the expansion of H^1 in Lemma 2.7 as the coefficients of μz_1, $z_1|z_1|^2$ and $z_1|z_2|^2$ respectively.

The characteristic exponents are the eigenvalues of the equation

$$\mathbb{J}_0 v + F_u(0,\mu,\omega,u)v = \kappa v. \tag{5.7}$$

See for instance [8]. The Implicit Function Theorem yields the existence of four characteristic exponents. Here we compute the first terms in the expansion with respect to ε of the exponents. We distinguish between the cases $\alpha = \pi/4$ and $\alpha \in \{0,\frac{\pi}{2}\}$.

case 1: $\alpha = \frac{\pi}{4}$.

Both

$$v^1(\varepsilon) = \frac{\sqrt{2}}{\varepsilon} \lim_{\theta\to 0} \{\frac{\Gamma(\theta)u(\varepsilon)-u(\varepsilon)}{\theta}\} = \begin{pmatrix} ie^{is} \\ -ie^{is} \end{pmatrix} + O(\varepsilon^2),$$

and

$$v^2(\varepsilon) = \frac{\sqrt{2}}{\varepsilon} \lim_{\theta\to 0} \{\frac{S(\theta)u(\varepsilon)-u(\varepsilon)}{\theta}\} = \begin{pmatrix} ie^{is} \\ ie^{is} \end{pmatrix} + O(\varepsilon^2)$$

are independent solutions of (5.7) with $\kappa = 0$. Define

$$v^3(\varepsilon) = \begin{pmatrix} e^{is} \\ e^{is} \end{pmatrix} + \psi(\varepsilon),$$

$$v^4(\varepsilon) = \begin{pmatrix} e^{is} \\ e^{-is} \end{pmatrix} + \rho(\varepsilon),$$

where

$$\psi(0) = \rho(0) = [\psi,\phi_i] = [\rho,\phi_i] = 0, \quad i \in \{1,2\}.$$

To obtain the other two characteristic exponents we solve the set of equations

$$\begin{cases} \mathbb{J}(\varepsilon)v^3(\varepsilon) - \gamma(\varepsilon)v^1(\varepsilon) - \delta(\varepsilon)v^2(\varepsilon) - \kappa(\varepsilon)v^3(\varepsilon) - \lambda(\varepsilon)v^4(\varepsilon) = 0 \\ \mathbb{J}(\varepsilon)v^4(\varepsilon) - p(\varepsilon)v^1(\varepsilon) - q(\varepsilon)v^2(\varepsilon) - r(\varepsilon)v^3(\varepsilon) - s(\varepsilon)v^4(\varepsilon) = 0. \end{cases} \tag{5.8}$$

Where by definition $\mathbb{L}(\varepsilon)v$ is the left hand side of (5.7) with $\mu = \mu(\varepsilon)$, $\omega = \omega(\varepsilon)$ and $u = u(\varepsilon)$. The characteristic exponents are the eigenvalues of the 4×4 real matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma(\varepsilon) & \delta(\varepsilon) & \kappa(\varepsilon) & \lambda(\varepsilon) \\ p(\varepsilon) & q(\varepsilon) & r(\varepsilon) & s(\varepsilon) \end{pmatrix}. \tag{5.9}$$

We will write $\gamma(\varepsilon) = \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + O(\varepsilon^3)$ etc. Collecting the first order terms in ε, taking the inner product with ϕ_1 and ϕ_2 yields

$$\gamma_1 = \delta_1 = \kappa_1 = \lambda_1 = p_1 = q_1 = r_1 = s_1 = 0. \tag{5.10}$$

The second order terms in ε lead to:

$$\begin{aligned} \mathbb{L}_0\psi_2 - \omega_2 \frac{d}{ds} v^3(0) + \mu_2 H_{\mu z}(0)v^3(0) + \tfrac{1}{2}H_{zzz}(0)(u_1^2, v^3(0)) \\ - \gamma_2 v^1(0) - \delta_2 v^2(0) - \kappa_2 v^3(0) - \lambda_2 v^4(0) = 0 \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathbb{L}_0\rho_2 - \omega_2 \frac{d}{ds} v^4(0) + \mu_2 H_{\mu z}(0)v^4(0) + \tfrac{1}{2}H_{zzz}(0)(u_1^2, v^4(0)) \\ - p_2 v^1(0) - q_2 v^2(0) - r_2 v^3(0) - s_2 v^4(0) = 0. \end{aligned} \tag{5.12}$$

The inner product of the left hand sides of (5.11) and (5.12) with both ϕ_1 and ϕ_2 must vanish. By straightforward computation we get:

$$\begin{aligned} &\gamma_2 = \lambda_2 = 0, \quad \kappa_2 + i\delta_2 = c_1 + c_2 \\ &q_2 = r_2 = 0, \quad s_2 + ip_2 = c_1 - c_2. \end{aligned} \tag{5.13}$$

Therefore the second order terms of the characteristic exponents are the eigenvalues of the matrix

$$(5.14)\quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{Im}(c_1+c_2) & \mathrm{Re}(c_1+c_2) & 0 \\ \mathrm{Im}(c_1-c_2) & 0 & 0 & \mathrm{Re}(c_1-c_2) \end{pmatrix}$$

and the result follows.

case 2: $\alpha = \frac{\pi}{2}$ (the case $\alpha=0$ goes similarly).

In this case $v^1(\varepsilon)$ and $v^2(\varepsilon)$ as defined above are not independent. We define

$$v^1(\varepsilon) = \frac{1}{\varepsilon} \lim_{\theta\to 0} \{\frac{\Gamma(\theta)u(\varepsilon)-u(\varepsilon)}{\theta}\} = \begin{pmatrix} ie^{is} \\ 0 \end{pmatrix} + O(\varepsilon^2),$$

$$v^2(\varepsilon) = \begin{pmatrix} 0 \\ ie^{is} \end{pmatrix} + \psi^2(\varepsilon),$$

$$v^3(\varepsilon) = \begin{pmatrix} e^{is} \\ 0 \end{pmatrix} + \psi^3(\varepsilon),$$

$$v^4(\varepsilon) = \begin{pmatrix} 0 \\ e^{is} \end{pmatrix} + \psi^4(\varepsilon),$$

where $\psi^j(0) = [\psi^j,\phi_i] = 0$, $j \in \{2,3,4\}$ and $i \in \{1,2\}$. Along the same lines as above we find that the order ε terms of the characteristic exponents vanish and the order ε^2 terms are the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{Re}(c_2-c_1) & 0 & -\mathrm{Im}(c_2-c_1) \\ 2\mathrm{Im}c_1 & 0 & 2\mathrm{Re}c_1 & 0 \\ 0 & \mathrm{Im}(c_2-c_1) & 0 & \mathrm{Re}(c_2-c_1) \end{pmatrix}$$

and the result follows. □

II: Characteristic exponents in Theorem 4.1.

Repeating the first part of I with $\alpha = 0$, $\eta = \eta(\varepsilon)$, $\mu = \eta(\varepsilon)\hat{\mu}$, $\omega = \omega(\varepsilon)$, find that $\eta_1 = \omega_1 = 0$, $u(\varepsilon) = \varepsilon\binom{0}{e^{is}} + O(\varepsilon^3)$ and η_2, ω_2 satisfy

(5.15) $$\eta_2(\hat{\mu}a+b_2) - i\omega_2 + c_1 = 0.$$

Define

$$v^1(\varepsilon) = -\frac{1}{\varepsilon}\lim_{\theta\to 0}\frac{\Gamma(\theta)u(\varepsilon)-u(\varepsilon)}{\theta} = \begin{pmatrix}0\\ ie^{is}\end{pmatrix} + O(\varepsilon^2)$$

$$v^2(\varepsilon) = \begin{pmatrix}ie^{is}\\ 0\end{pmatrix} + \varepsilon\psi_1^2 + \varepsilon^2\psi_2^2 + O(\varepsilon^3)$$

$$v^3(\varepsilon) = \begin{pmatrix}e^{is}\\ 0\end{pmatrix} + \varepsilon\psi_1^3 + \varepsilon^2\psi_2^3 + O(\varepsilon^3)$$

$$v^4(\varepsilon) = \begin{pmatrix}0\\ e^{is}\end{pmatrix} + \varepsilon\psi_1^4 + \varepsilon^2\psi_2^4 + O(\varepsilon^3)$$

where $[\psi_j^k,\phi_i] = 0$ for $k \in \{2,3,4\}$ and $i,j \in \{1,2\}$.
We solve the equations

(5.16) $$\mathbb{L}(\varepsilon)v^j - \sum_{k=1}^{4}\gamma^{jk}(\varepsilon)v^k(\varepsilon) = 0, \quad \gamma^{jk}(\varepsilon) = \varepsilon\gamma_1^{jk} + \varepsilon^2\gamma_2^{jk} + O(\varepsilon^3),$$

where now

$$\mathbb{L}(\varepsilon) = \mathbb{L}_0 + F_u(\eta(\varepsilon),\eta(\varepsilon)\hat{\mu},\omega(\varepsilon),u(\varepsilon)).$$

At order ε we find

(5.17) $$\gamma_1^{jk} = \psi_1^j = 0 \quad j \in \{2,3,4\} \text{ and } k \in \{1,2,3,4\}.$$

At order ε^2, taking the innerproduct of (5.16) with ϕ_1 and ϕ_2 leads to the solvability conditions

$$(5.18)\quad \begin{cases} i(\eta_2(\hat{\mu}a+b_1)-i\omega_2) + ic_2 - i\gamma_2^{22} - \gamma_2^{23} = 0, \\ -i\gamma_2^{21} - \gamma_2^{24} = 0, \\ \eta_2(\hat{\mu}a+b_1) - i\omega_2 + c_2 - i\gamma_2^{32} - \gamma_2^{33} = 0, \\ -i\gamma_2^{31} - \gamma_2^{34} = 0, \\ \eta_2(\hat{\mu}a+b_2) - i\omega_2 + 3c_1 - i\gamma_2^{42} - \gamma_2^{43} = 0 \\ -i\gamma_2^{41} - \gamma_2^{44} = 0. \end{cases}$$

If we combine this with formula (5.15) then the result follows easily. □

6. APPENDIX 2: REDUCTION TO THE CENTER MANIFOLD

We write (1.1)-(1.2) as an abstract Cauchy problem. Let X be the space of 2π-periodic continuous functions of $\mathbb{R}$ into $\mathbb{R}^n$. We characterize the operator A by

$$(6.1) \qquad \mathcal{D}(A) = \{u \in X \mid u \in C^2(\mathbb{R};\mathbb{R}^n) \ \& \ u' \in X\},$$

$$(6.2) \qquad Au = u_{xx}.$$

Let $D(\mu)$ be the diagonal matrix $(d_1(\mu),\dots,d_n(\mu))$, $d_i > 0$. The substitution operator $F\colon \mathbb{R} \times X$ into X is defined by

$$(6.3) \qquad F(\mu,u)(x) = f(\mu,u(x)).$$

Equivalently to (1.1)-(1.2) we consider

$$(6.4) \qquad \begin{cases} \dfrac{du}{dt} = D(\mu)Au + F(\mu,u) \\ u_0 \in X. \end{cases}$$

It is not hard to see that $D(\mu)A$ is the generator of a contracting analytic semigroup. See [3, p. 59] or the discussion below.

By direct computation we derive that

$$\lambda u - D(\mu)A = h \quad (h \in X,\ \lambda \geq 0)$$

implies that u^i, the i-th component of u, is the 2π-periodic function given by

$$(6.5) \qquad u^i(x) = \sqrt{\frac{d_i}{4\lambda}} \left\{ \frac{e^{-2\pi x\sqrt{\lambda/d_i}}}{1-e^{-2\pi\sqrt{\lambda/d_i}}} \int_0^{2\pi} e^{-(2\pi-\tau)\sqrt{\lambda/d_i}}\, h(\tau)d\tau \right.$$
$$+ \frac{e^{2\pi x\sqrt{\lambda/d_i}}}{1-e^{2\pi\sqrt{\lambda/d_i}}} \int_0^{2\pi} e^{-\tau\sqrt{\lambda/d_i}}\, h(\tau)d\tau$$
$$\left. + \int_0^{2\pi} e^{-|2\pi x-\tau|\sqrt{\lambda/d_i}}\, h(\tau)d\tau \right\}, \quad 0 \leq x \leq 2\pi.$$

This shows that indeed $D(\mu)A$ satisfies the conditions of the Hille-Yosida Theorem [3]. The mapping F is as smooth as f. $F_u(\mu,0)$ is bounded linear operator, therefore

(6.6) $$L(\mu) = D(\mu)A + F_u(\mu,0)$$

is the generator of an analytic semigroup [18]. The equations (6.1) define a dynamical system. By definition $S(t)u_0$, $t \geq 0$, will denote the solution $u(t)$ that starts at u_0. The complex number λ is an eigenvalue of $L(\mu)$ iff there exists $k \in \mathbb{N} \cup \{0\}$ such that $\det(-k^2D(\mu) + F_u(\mu,0)-\lambda I) = 0$. Let by definition for $k \in \mathbb{N} \cup \{0\}$

(6.7) $$E(k) = \{\lambda \in \mathbb{C} \mid \det(-k^2D(\mu)+F_u(\mu,0)-\lambda I) = 0\}.$$

We make the following hypothesis on the linearized equations:
There exist $k_0 > 0$, $\omega_0 > 0$, $\mu_0 \in \mathbb{R}$, $\alpha,\delta > 0$, $\zeta = \zeta_1 + i\zeta_2 \in \mathbb{C}^n$ such that

(α) $E(k_0) - \{i\omega_0,-i\omega_0\} \subseteq \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda < -\alpha\}$

(β) $E(k) \subseteq \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda < -\alpha\}$, $k \neq k_0$, $|\mu-\mu_0| \leq \delta$

(γ) $-k_0^2D(\mu_0) + F_u(\mu_0,\cdot)-i\omega_0 I$ has a one dimensional generalized nullspace spanned by $\{\zeta\}$.

Let span$\{\zeta^*\}$ be the generalized nulspace of $-k_0^2D(\mu_0) + F_u^*(\mu_0,\cdot) + i\omega_0 I$, normalized such that $\langle\zeta,\zeta^*\rangle_{\mathbb{C}^n} = 1 = 1 - \langle\zeta,\bar{\zeta}^*\rangle_{\mathbb{C}^n}$ and define

(6.8) $$v_1 = e^{ik_0x}\zeta; \quad v_2 = e^{-ik_0x}\zeta$$
$$v_2^* = e^{ik_0x}\zeta^*; \quad v_2^* = e^{-ik_0x}\zeta^*.$$

From the hypotheses it follows that the generalized nullspace of $(L(\mu_0)-i\omega_0)$ is two dimensional and spanned by $\{v_1,v_2\}$. In X we use the innerproduct $\langle\ ,\ \rangle = \langle .,.\rangle_X$ defined by

$$\langle u,v\rangle_X = \frac{1}{2\pi}\int_0^{2\pi} u(x).\overline{v(x)}dx.$$

The projection operator P of X onto $X_0 = N(L(\mu_0)-i\omega_0)\oplus N(L(\mu_0)+i\omega_0)$ along

$\mathcal{R}(L(\mu_0)-i\omega_0)\cap\mathcal{R}(L(\mu_0)+i\omega_0)$ is given by

$$(6.9)\qquad Pu = \langle u,v_1^*\rangle v_1 + \langle u,v_2^*\rangle v_2 + \text{c.c.}$$

The center manifold theorem states the following

THEOREM 6.1. *There exist positive constants* K, δ, r, ε, γ *and a mapping* C *from* $\Lambda = \{\mu \mid |\mu-\mu_0| \le \delta\} \times X_0$ *into* X *such that*

(i) Im C *is tangent to* X_0 *at* 0, *i.e.,* $C(\mu,0) = 0$; $D_\phi C(0,0)\psi = \psi$, $\psi \in X_0$,

(ii) Im C *is local-invariant, i.e., if* $\phi \in X_0$ *satisfies* $\|\phi\| \le r$ *and for* $0 \le \tau \le t$ $\|S(\tau)\phi\| \le r$ *then* $C(\mu,PS(t)\phi) = S(t)\phi$,

(iii) Im C *is conditional-attractive, i.e., if for* $0 \le \tau \le t$ $\|S(\tau)u_0\| \le r$ *then* $\|S(t)u_0 - S(t)C(\mu,Pu_0)\| \le Ke^{-\gamma t}$.

We do not give a proof of this theorem, but we mention Carr [4], Henry [13] Chow and Hale [6] and the references given there. Here we only show that then center manifold inherits the symmetry properties of the equations (1.1)-(1.2).

The center manifold is defined as the image of the mapping C:

$$C(\mu,\phi) = u(\mu,\phi)(0)$$

where $\mu \in \Lambda$, $\phi \in X_0$ and u satisfies the integral equation

$$\begin{aligned} u(s) = e^{L(\mu_0)s}\phi + \int_{-\infty}^{s} e^{L(\mu_0)(s-\tau)} &(I-P)\{(L(\mu)-L(\mu_0))u(\tau) \\ &+ \tilde{F}(\mu,u(\tau)) - \tilde{F}_u(\mu,0)u(\tau)\}d\tau \\ + \int_0^s e^{L(\mu_0)(s-\tau)} &P\{(L(\mu)-L(\mu_0))u(\tau) + \tilde{F}(\mu,u(\tau)) - \tilde{F}_u(\mu,0)u(\tau)\}d\tau. \end{aligned}$$

The $\sim$ means that we modify the nonlinearity outside a ball of radius r. Let $\xi: \mathbb{R}_+ \to \mathbb{R}$ be a C^∞-function such that (i) $\xi(y) = 1$ for $0 \le y \le 1$; (ii) $0 \le \xi(y) \le 1$ for $1 \le y \le 2$; $\xi(y) = 0$ for $y \ge 2$. For some positive parameter r, which is in the construction choosen small enough to satisfy suitable bounds we consider

$$\tilde{F}(\mu,u) - \tilde{F}_u(\mu,0)u = (F(\mu,u) - F_u(\mu,0)u)\xi(\tfrac{\|u\|}{r}).$$

The projection operator P, $L(\mu)$, $F(\mu,u)$ and $\xi(\frac{\|u\|}{r})$ are all of them covariant with respect to rotation and reflection. From the integral equation above we see that the same will be true for the mapping C.

Let z_1 and z_2 be the coefficients of v_1 and v_2 in Pu (see (6.9)). On the center manifold these coefficients satisfy in a neighbourhood of the origin the ordinary differential equation

$$(6.10)\qquad \begin{cases} \dfrac{dz_1}{dt} = \langle D(\mu)AC(\mu,z_1v_1+z_2v_2+c.c.) + F(\mu,C(\mu,z_1v_1+z_2v_2+c.c.)),v_1^*\rangle \\ \dfrac{dz_2}{dt} = \langle D(\mu)AC(\mu,z_1v_1+z_2v_2+c.c.) + F(\mu,C(\mu,z_1v_1+z_2v_2+c.c.)),v_2^*\rangle . \end{cases}$$

It is easy to see that this set of equations has the symmetry properties as stated in the formulas (2.5) and (2.7). Define

$$(6.11)\qquad a(\mu) = \langle L(\mu)v_1,v_1^*\rangle \qquad b(\mu) = \langle L(\mu)\bar{v}_2,v_1^*\rangle .$$

The linear part of (6.10) is given by

$$(6.12)\qquad \frac{d}{dt}\begin{pmatrix} z_1 \\ \overline{z_1} \\ z_2 \\ \overline{z_2} \end{pmatrix} = \begin{pmatrix} a(\mu) & 0 & 0 & b(\mu) \\ 0 & \overline{a(\mu)} & \overline{b(\mu)} & 0 \\ 0 & b(\mu) & a(\mu) & 0 \\ \overline{b(\mu)} & 0 & 0 & \overline{a(\mu)} \end{pmatrix}\begin{pmatrix} z_1 \\ \overline{z_1} \\ z_2 \\ \overline{z_2} \end{pmatrix} .$$

Let us denote this matrix by $B(\mu)$. At $\mu = \mu_0$ $B(\mu_0)$ becomes the matrix diag $(i\omega_0,-i\omega_0,i\omega_0,-i\omega_0)$. There exists a smooth matrix valued mapping $\mu \to T(\mu)$ which is of the same form as $B(\mu)$, i.e. also covariant with respect to rotation and reflection, such that

(i) $\qquad T(\mu_0) = I$

(ii) $\qquad T(\mu)^{-1}B(\mu)T(\mu)$ is a diagonal matrix.

Starting point in section 2 are the equations (6.10) which one obtains after applying this smooth linear coordinate transformation.

As we promised in the introduction we will give a formula for c_1 and c_2 in terms of the r.h.s. of (6.4). The computations are straightforward. The Taylor series expansion of (6.10) up to order three involves the expansion of the center manifold up to order two. See [10] for the details of calculation in the case of a Hopf bifurcation at simple eigenvalues. Define

$$(6.13)\quad \begin{cases} L_0 = -k_0^2 D(\mu_0) + F_u(\mu_0,0) \\ a = \tfrac{1}{2}(2i\omega_0 - L_0)^{-1} F_{uu}(\mu_0,0)(v_1,v_1) \\ b = -L_0^{-1} F_{uu}(\mu_0,0)(\bar{v}_1,v_1) \\ c = (2i\omega_0 - L_0)^{-1} F_{uu}(\mu_0,0)(v_1,v_2) \\ d = -L_0^{-1} F_{uu}(\mu_0,0)(v_1,\bar{v}_2) \\ e = -L_0^{-1} F_{uu}(\mu_0,0)(v_2,\bar{v}_2), \end{cases}$$

then

$$(6.14)\quad \begin{aligned} c_1 = {} & \tfrac{1}{2}\langle F_{uuu}(\mu_0,0)(v_1,v_1,\bar{v}_1),v_1^*\rangle + \\ & \langle F_{uu}(\mu_0,0)(\bar{v}_1,a),v_1^*\rangle + \\ & \langle F_{uu}(\mu_0,0)(v_1,b),v_1^*\rangle. \end{aligned}$$

$$(6.15)\quad \begin{aligned} c_2 = {} & \langle F_{uuu}(\mu_0,0)(v_1,v_2,\bar{v}_2),v_1^*\rangle + \\ & \langle F_{uu}(\mu_0,0)(\bar{v}_2,c),v_1^*\rangle + \\ & \langle F_{uu}(\mu_0,0)(v_2,d),v_1^*\rangle + \\ & \langle F_{uu}(\mu_0,0)(v_1,e),v_1^*\rangle. \end{aligned}$$

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