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ON THE UNIQUENESS OF A REGULAR THIN NEAR OCTAGON ON 288 VERTICES (OR THE SEMIBIPLANE BELONGING TO THE MATHIEU GROUP M_{12})

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0. INTRODUCTION

A *semisymmetric design* $SSD(v,k,\lambda)$ is a connected incidence structure with v points and v blocks where blocks have size k and there are k blocks on a point while any two different blocks have 0 or λ points in common, and any two distinct points are on 0 or λ blocks (cf. Wild [17]). In case $\lambda=2$ such a structure is called a *semibiplane* (cf. Hughes [11]).
A *partial λ-geometry* (with $\lambda>1$) is a $SSD(v,k,\lambda)$ such that for any nonincident pair (p,B) where p is a point and B a block, there are precisely e blocks on p meeting B. (Then there are also precisely e points on B which are on a block with p. The number e is called the *nexus* of the design. See also Cameron & Drake [5], Drake [7]).
Partial λ-geometries with $\lambda=2$ and $e=3$ have been characterized by Cameron [3] and Brouwer [2], the result being that unique examples exist for $k\in\{3,4,8,24\}$.
Recently I heard a talk by H. Leemans [13] where he characterized partial λ-geometries with $\lambda=2$ and $e=5$ under strong transitivity assumptions, the main results being that the partial λ-geometry with $\lambda=2$, $e=5$ and $k=12$ is unique up to duality, assuming a sufficiently transitive group. The purpose of this note is to remove the conditions on the group of automorphisms.
More precisely, given a partial λ-geometry with $(\lambda,e)=(2,5)$, the standard necessary conditions (cf. [5]) show $k\in\{5,6,10,12,20\}$. Let us look at these possibilities.

1. When $k=5$ we have the symmetric 2-design (biplane) $2-(11,5,2)$. As is well known this design exists and is unique.
2. When $k=6$ we have a resolvable group divisible design $RGD(6,2,3;18)$, i.e., a resolvable transversal design $RT(6,2;3)$, also known as a symmetric (3,6,2)-net. This structure was given e.g. in Hanani [9]; it has been rediscovered many times. It is unique (as is 'well known' - uniqueness will follow as a side result below).
3. We shall see that no example with $k=10$ exists.
4. When $k=12$ there are two nonisomorphic designs (duals of each other). They were discovered by Leonard, who also proved their uniqueness in case the stabilizer of a block in the automorphism group contains $PGL(2,11)$ acting in the natural way. The main purpose of this note is to show that no other solutions exist.
5. Nothing is known in case $k=20$. Most likely it is possible to eliminate this case using the methods of this note, but this looks like a lot of tedious work. Assuming a nice group quickly kills this case.

1. STRONGLY REGULAR GRAPHS AND REGULAR THIN NEAR OCTAGONS

Given a partial λ-geometry, the graph with the points (resp. blocks) as vertices, and pairs of points joined by a block (resp. pairs of blocks with nonempty intersection) as edges is known as the point-graph (resp. block graph) of the geometry. It is easy to verify that these graphs are strongly regular (and have the same parameters). (See [5]. For the definition of a strongly regular graph see e.g. Seidel [15] or Cameron [4].)
In our case we have ($\lambda=2$, $e=5$ and):

k	v	K	λ	μ	r	s	f	g	Comment
5	11	10	9	-	-	-1	-	10	Complete graph K_{11}.
6	18	15	12	15	0	-3	12	5	Complete multipartite graph $K_{6\times 3}$.
10	82	45	24	25	4	-5	40	41	Examples are known, e.g. the block graph of $S(2,5,41)$.
12	144	66	30	30	6	-6	66	77	Examples are known (see below).
20	704	190	54	50	14	-10	285	418	Unknown.

For $k=12$ examples are known derived from a transversal design $T[6,1;12]$ (see Hanani [10]), from a recursive construction using K_{12} and a Hadamard matrix of order 12 (see Goethals & Seidel [18]) and from a regular symmetric Hadamard matrix with constant diagonal (ibid.). Note that any such graph is equivalent to a regular symmetric Hadamard matrix with constant diagonal of order 144 (see [8] and Wallis [16]) and gives rise to a symmetric 2-(144,66,30) design. It is easy to check that our example is not derived from a transversal design: our graph (let us say the point graph) contains precisely 144 12-cliques, corresponding to the blocks; it is not possible to choose 72 of them such that two adjacent points determine a unique line - this would be a 72-coclique in the block graph, while both cliques and cocliques have size at most 12. Neither is it possible to find 84 12-cocliques such that two nonadjacent points determine a unique line as is shown by an explicit check. I do not know whether any of the two strongly regular graphs arising from the two (dual) partial 2-geometries with $k=12$ can be obtained from simpler structures using one of the constructions by Goethals and Seidel.

Given a partial λ-geometry, the (bipartite) incidence graph is a distance regular graph of diameter 4. We have the parameters $V=2v$, $c_1=1$, $c_2=\lambda$, $c_3=e$, $c_4=k$. (For the definition of a distance regular graph, see Biggs [1].) A regular thin near octagon is nothing but a bipartite distance regular graph of diameter 4; the standard parameters are $(s,t_2,t_3,t_4) = (1,c_2-1,c_3-1,c_4-1)$.
Clearly, there is a $1-1$ correspondence between regular thin near octagons and pairs of mutually dual partial λ-geometries.

2. HUSAIN CHAINS AND p-CHAINS

Let Γ be a distance regular graph with $c_2=2$ and diameter at least three. Fix a point Ω and call its k neighbours 'Symbols'. If a and b are Symbols then these points have two common neighbours; one is Ω; call the other ab. Obviously there are $\binom{k}{2}$ points at distance 2 from Ω, the 'Pairs'. Two Symbols determine a unique Pair, and a Pair determines exactly two Symbols. Points at distance 3 from Ω determine a collection of Pairs such that any Symbol is covered 0 or 2 times; that is, we may represent a point at distance 3 from Ω by a union of polygons on the set of Symbols. These unions of polygons are called Husain chains, after Q.M. Husain, who first used them in his investigations of biplanes with $k=5,6,7$.

In this note we are interested in the case $c_3=e=5$. Clearly the Husain chains are now pentagons, and we shall call the points at distance 3 from Ω 'Pentagons'. (Note that not every pentagon on the set of Symbols is determined by a Pentagon.)

Given an edge in a pentagon, there are two edges disjoint from it. This observation gives rise to another kind of chain, let us say p-chains, as we shall see below.

LEMMA 1. *Two intesecting Pairs determine a unique Pentagon.*

PROOF. These pairs have already one common neighbour (a Symbol), so must have exactly one other common neighbour (a Husain chain). □

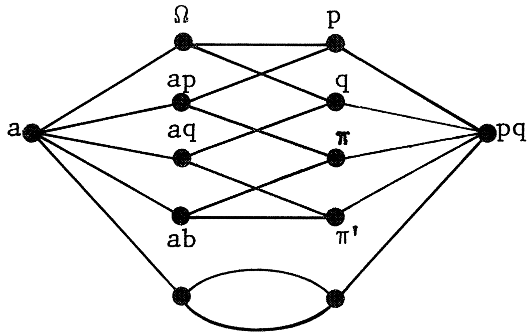
LEMMA 2. *Given a Pair ab and a Symbol c where $c \notin \{a,b\}$, there is a unique Pentagon with edge ab and opposite vertex c.*

PROOF. $d(ab,c)=3$ so ab has $c_3=5$ neighbours at distance 2 from c. Two are the Symbols a,b and two others are the Pentagons on the pairs ab, ac and ab, bc. The fifth neighbour is the required Pentagon. □

Since we shall meet many pentagons and in view of the typographic difficulties of merging text and pictures, it is useful to have a notation. We shall write $(abcde)$ for the Pentagon with edges ab, bc, cd, de and ea. Also e.g. $(ab \cdot d \cdot)$ for the same Pentagon - the notation is unambiguous by Lemma's 1 and 2.

LEMMA 3. *If $(ab \cdot qp)$ is a Pentagon, then so is $(abqp \cdot)$.*

PROOF. The two disjoint Pairs ab and pq have distance two and hence determine two Pentagons. The first is $\pi=(ab \cdot qp)$, and the second cannot be $\pi'=(ab \cdot pq)$, for otherwise look at the points on geodesics from a to pq. We have the picture

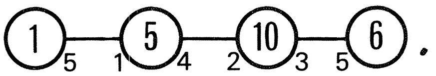


contradiction. □

Each ordered pair of Symbols (a,b) defines a directed graph with indegree and outdegree one on the remaining $k-2$ Symbols: if $(abrqp)$ is a Pentagon, we draw the edges $p \to q \to r$. In this way we obtain a union of directed polygons on $k-2$ Symbols - let us call it the p-chain on the ordered pair (a,b). Clearly, reversing the order of the pair means reversing all arrows of the p-chain.

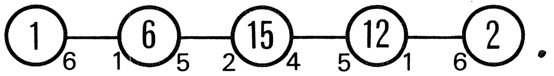
EXAMPLES

1. When $k = 5$ we have the diagram



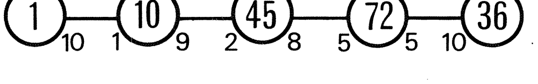
Since there are only five Symbols, Lemma 3 implies that the stabilizer of Ω in $Aut(\Gamma)$ contains $Alt(5)$. But $Alt(5)$ has two orbits of size (½.5! / 10=) 6 on the pentagons, so Γ is uniquely determined - it is the incidence graph of the biplane $2-(11,5,2)$. $Aut(\Gamma) = PGL(2,11)$.

2. When $k = 6$ we have the diagram

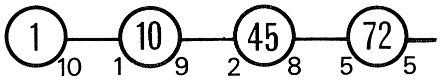


Here p-chains are directed quadrangles. Clearly, by Lemma 3, if one Pentagon is given then all others are determined. Let the set of symbols be $PG(1,5) = \{\infty,0,1,2,3,4\}$ and let (∞0134) be a Pentagon. By Lemma 3, also (∞0342) is a Pentagon, so that the set of Pentagons is invariant under $x \mapsto 3x$. Similarly, since (01234) is a Pentagon, the set of Pentagons is invariant under $x \mapsto x-1$. Finally, also $x \mapsto -\frac{1}{x}$ acts, so that the set of Pentagons is left invariant by $PGL(2,5)$. In view of the Lemma's 1 and 2, and the fact that $PGL(2,5)$ is sharply 3-transitive on the 6 Symbols it follows that no larger group can act. If we define a graph with the Pentagons as vertices and pairs of Pentagons that have one Pair in common as edges, then one easily sees that this is the union of two 6-cliques $\overline{K_{6,6}}$ so that there is only one way to add the two vertices in $\Gamma_4(\Omega)$. This Γ is unique, $Aut(\Gamma)$ is transitive, and $\Gamma_\Omega \doteq PGL(2,5)$.
The two added vertices have mutual distance 4, so Γ is (the unique) antipodal 3-cover of $K_{6,6}$. The automorphism $x \mapsto 2x$ interchanges the two vertices in $\Gamma_4(\Omega) = \{\Omega_1,\Omega_2\}$ so that the stabiliser of $\{\Omega,\Omega_1,\Omega_2\}$ has order 360; it is $\mathbf{Z}_3 \times Sym(5)$.
$Aut(\Gamma) = 3.\ Sym(6).2$ where the 3 stabilizes all twelve antipodal 3-cocliques and the 2 interchanges both bipartite halves of Γ. Neither of the two is factor of a direct product.

3. When $k = 10$ we have the diagram



We shall see that the neighbourhood of Γ with diagram



is uniquely determined - again the Pentagons form an orbit under $PGL(2,9)$.
However, objects in $\Gamma_4(\Gamma)$ cannot exist so that there are no distance regular graphs with parameters $k=10$, $a_1=a_2=a_3=0$, $c_2=2$, $c_3=5$ (and arbitrary diameter).

In this case p-chains are unions of directed polygons on 8 Symbols, i.e. either the union of a 3-gon and a 5-gon or the union of two 4-gons or an 8-gon.

In the first case we find on some ordered pair (x,y) the p-chain $(a_0a_1a_2a_3a_4)$ $(b_0b_1b_2)$. This means that we have the Pentagons $(xya_{i+1}a_ia_{i-1})$, $i\in\mathbf{Z}_5$ and $(xyb_{j+1}b_jb_{j-1})$, $j\in\mathbf{Z}_3$. By Lemma 3 we also have Pentagons $(xa_ia_{i-1}\gamma_ia_{i-2})$, and $(a_{i-2}xa_{i-1}\gamma_i\ \cdot\ )$, $i\in\mathbf{Z}_5$, for certain symbols γ_i. Now the p-chain on (x,a_i) contains the directed paths

$$(ya_{i+2}a_{i+1}\gamma_{i+2}) \text{ and } (a_{i-2}\gamma_ia_{i-1}\cdot\gamma_{i+1})$$

so that $\gamma_i\neq\gamma_{i+2}$ and $\gamma_{i+1}\neq\gamma_{i+2}$ for all $i\in\mathbf{Z}_5$. Also $\gamma\neq a_j$ $(i,j\in\mathbf{Z}_5)$. Thus $\gamma:\mathbf{Z}_5\rightarrow\{b_0,b_1,b_2,y\}$ is injective, contradiction.

In the second case we have the p-chain $(a_0a_1a_2a_3)$ $(b_0b_1b_2b_3)$ on (x,y). We find the same directed paths as before, but now $a_{i-2}=a_{i+2}$ (all indices are in $\mathbf{Z}_4$) so that the two directed paths merge to give $(ya_{i+2}a_{i+1}a_{i-1}y)$ [i.e., $\gamma_i=a_{i+1}$] — a 4-gon. This proves that if we have two 4-gons on one pair then we have two 4-gons on all pairs. It also proves that the set of Symbols $\{x,y,a_0,a_1,a_2,a_3\}$ is closed under forming Pentagons — i.e., carries a subsystem. Thus, we find a Steiner system $S(3,6,10)$, but such systems do not exist (e.g. because λ_1 is not integral). Contradiction.

This shows that all p-chains are directed 8-gons.
Look at the p-chain $(a_0a_1\cdots a_7)$ on (x,y). As before we find directed paths $(ya_{i+2}a_{i+1}\gamma_{i+2})$ and $(a_{i-2}\gamma_ia_{i-1}\cdot\gamma_{i+1})$ in the p-chain on (x,a_i). Consequently $\gamma_i\neq x,y,a_{i-2},a_{i-1},a_i,a_{i+1},a_{i+2}$. Also $\gamma_{i+1}\neq a_{i-2}$ so that $\gamma_i\neq a_{i-3}$. Thus $\gamma_i\in\{a_{i+3},a_{i+4}\}$, and since $\gamma_i\neq\gamma_{i+1}$ we either have γ_{i+3} for all $i\in\mathbf{Z}_8$ or γ_{i+4} for all $i\in\mathbf{Z}_8$. In the first case we find the directed paths

$$(ya_{i+2}a_{i+1}a_{i-3}),\ (a_{i-2}a_{i+3}a_{i-1}\cdot a_{i+4})$$

and there is no way to fill the symbol represented by the dot. Thus $\gamma_i=a_{i+4}$ and we have the p-chain $(ya_{i+2}a_{i+1}a_{i-2}a_{i+4}a_{i-1}a_{i+3}a_{i-3})$ on (x,a_i). Label the ten Symbols as follows: $x=\infty, y=0, a_i=\alpha^i$ where α is a primitive element of F_9. We just showed that one p-chain determines all others, i.e., all Pentagons, and so the set of Pentagons is invariant under $x\mapsto\alpha x$ and $x\mapsto x^{-1}$. If we moreover choose α as a root of $\alpha^2=2\alpha+1$ then the set of Pentagons is also invariant under $x\mapsto 1-x$. Thus:
The set of Pentagons is uniquely determined and consists of the images of $(\infty 0\alpha^2\alpha 1)$ *under* PLG(2,9), *where* $\alpha^2=2\alpha+1$.

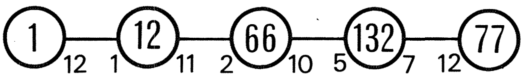
LEMMA 4. *Let* $z\in\Gamma_4(\Omega)$ *be adjacent to the Pentagons* $(abc\cdot\cdot)$ *and* $(bca\cdot\cdot)$. *Then* z *is also adjacent to* $(cab\cdot\cdot)$.

PROOF. Let the two Pentagons be $\pi_1=(abcde)$ and $\pi_2=(bcafg)$. Then we also have Pentagons $\pi_3=(gbaf\ \cdot)$, $\pi_4=(eacd\ \cdot)$, $\pi_5=(bceaf)$ (the latter since the p-chain of (b,c) contains the directed path (g,f,a,e,d)). Between a and z we have five Pairs and five Pentagons, with an incidence giving these the structure of the points and edges of a pentagon. Now the Pentagons π_1 and π_2 join the Pairs ab, ae and ac, af (respectively); it follows that there is exactly one Pentagon adjacent to z joining one of the four pairs of Pairs ab, ac or ab, af or ae, ac or ae, af. But the latter three pairs of Pairs are joined by π_3, π_4, π_5 (respectively), and these cannot be neighbours of z since they have two Pairs in common with π_1 or π_2. Thus z is adjacent to $(cab\cdot\cdot)$. □

Applying this Lemma to the case $k=10$ one easily derives a contradiction. [The details are boring: assume z is joined to $\pi_1=(1\infty 0\alpha^2\alpha)$ and to $\pi_2=(\infty 01\alpha^7\alpha^6)$, then also to $\pi_3=(\infty 10\alpha^5\alpha^3)$. There is a unique pentagon $\pi_4=(\alpha 1\gamma\cdot\cdot)$ joined to z, and trying the five possibilities for γ one sees that only $\gamma=\alpha^4$ is possible. (Note that also $\pi_5=(\gamma 1\alpha^7\cdot\cdot)$ is a neighbour of z.) The map $x\mapsto\frac{1}{1-x}$ leaves the set $\{\pi_1,\pi_2,\pi_3\}$ invariant hence the images of π_4 and π_5 under this map (and its square) are also joined to z. But now one finds six Pairs on α^2 at distance two from z, contradiction.]

3. THE CASE $k=12$

We have the diagram



The situation here will turn out to be as follows: there is a unique distance regular graph Γ; when Ω is chosen in one bipartite half then all p-chains are directed 10-gons, while if Ω is chosen in the other half then all p-chains are unions of two directed 5-gons.

LEMMA 5. *If some p-chain contains a directed pentagon then every p-chain is the union of two directed pentagons.*

PROOF. Suppose the p-chain on (x,y) is $(a_0a_1a_2a_3a_4)$, $(b_0b_1b_2b_3b_4)$. Just as before (in the example $k=10$) we find in the p-chain on (x,a_i) directed paths

$$(ya_{i+2}a_{i+1}\gamma_{i+2}) \text{ and } (a_{i-2}\gamma_i a_{i-1}\cdot\gamma_{i+1})$$

where γ_i is defined by the Pentagon $(xa_ia_{i-1}\gamma_ia_{i-2})$, $i\in\mathbf{Z}_5$. Again the γ_i are mutually distinct, and $a_i\neq\gamma_j$ $(i,j\in\mathbf{Z}_5)$ so that the γ_j form a permutation of the b_j. (Note that $\gamma_{i+2}\neq y$, otherwise we would find (a_iya_{i+1}) in the p-chain on (x,a_{i+2}) and $a_{i+1}=a_{i+4}$, a contradiction.)
By Lemma 3 we have Pentagons $(xa_i\gamma_{i+1}\delta_{i-2}a_{i-1})$ for certain Symbols δ_i $(i\in\mathbf{Z}_5)$. This gives us the directed paths in the p-chain on (x,a_i):

$$(ya_{i+2}a_{i+1}\gamma_{i+2}\delta_{i-1}) \text{ and } (a_{i-2},\gamma_i,a_{i-1},\delta_{i-2},\gamma_{i+1}).$$

Now by inspection $\delta_i\neq y$, a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2} so that the δ_i are among the b_j. Also $\delta_i\neq\gamma_{i+1}$, γ_{i+2}, γ_{i+3}, γ_{i+4} so that $\delta_i=\gamma_i$, and we have the directed paths (for all i):

$$(ya_{i+2}a_{i+1}\gamma_{i+2}\gamma_{i-1}) \text{ and } (a_{i-2}\gamma_ia_{i-1}\gamma_{i-2}\gamma_{i+1}).$$

If the p-chain on (x,a_i) is not the union of two directed pentagons then it contains the edge (γ_{i+1},y) and we have the Pentagon $(xa_ia_{i+2}y\gamma_{i+1})$. By Lemma 3 we find a Pentagon $(xa_{i+2}y\cdot\gamma_{i+1})$ so that for this i we have $\gamma_{i+1}\in\{\gamma_{i+4},\gamma_i\}$, contradiction. □

As we before (for $k=10$) we would like to label the Symbols with the elements of $PG(1,11)=\{\infty\}\cup\mathbf{F}_{11}$ in such a way that the Pentagons form one orbit under $PGL(2,11)$. To this end, assume we are in the situation of Lemma 5. There are Pentagons $(xa_iy\gamma_{i-1}\gamma_{i+2})$ and hence also Pentagons $(xy\gamma_{i-1}\cdot\gamma_{i+2})$ so that if $\gamma_{i+4}=b_j$ then $\gamma_{i+2}=b_{j-2}$. Since we may still choose b_0, we may assume that $\gamma_i=b_i$ $(i\in\mathbf{Z}_5)$. Thus: given the p-chain on (x,y) and the Pentagon $(xa_0a_4b_0a_3)$, the p-chain on (x,a_i) is uniquely determined. But so is the corresponding Pentagon: it is $(xyb_{i-1}b_{i-2}b_{i-3})$. Repeating this argument we see that the set of Pentagons is determined uniquely.
Now label x with ∞, y with 0, a_i with 3^i and b_i with $2\cdot 3^i$ $(i\in\mathbf{Z}_5)$ and we find that the set of

Pentagons consists of the images of (∞0931) under $PGL(2,11)$.

LEMMA 6. *If the p-chains are unions of two directed pentagons then it is possible to label the Symbols in such a way that the Pentagons are the images of* (∞0931) *under* $PGL(2,11)$. □

Next, we show that the points $z \in \Gamma_4(\Omega)$ (considered as sets of twelve Pentagons) are determined uniquely. These 77 points will be seen to form two orbits of sizes 55 and 22 under $PGL(2,11)$.
Consider the point $z \in \Gamma_4(\Omega)$ which is a common neighbour of the Pentagons $\pi_1 = (\infty 0931)$ and $\pi_2 = (\infty 0459)$. (Indeed, these two Pentagons have one Pair in common and hence must have exactly one other common neighbour.) By Lemma 4, z is also adjacent to $\pi_3 = (09\infty 86)$. Considering the five Pentagons adjacent to z and containing the Symbol ∞ we see that there is a Symbol a such that these each contain two (successive) Pairs from $(\infty 1, \infty 0, \infty 9, \infty 8, \infty a)$. Obviously $a \in \{2,3,4,5,6,7,10\}$. Now the map $x \mapsto 9-x$ leaves π_2 invariant and interchanges π_1 and π_3. If we knew that $\Gamma_4(\Omega)$ was invariant under $PGL(2,11)$ it would follow that a is a fix point of this map, i.e., $a=10$. As it is, this involution only halves our work.

If $a=2$ then z is adjacent to the Pentagon (2∞891), but now we see six Pairs on the Symbol 9 at distance two from z, Impossible. Hence also $a=7$ is impossible.

If $a=3$ then z is adjacent to the Pentagons (3∞187) and (3∞874), but these have two Pairs in common, Impossible. Hence also $a=6$ is impossible.

If $a=4$ then z is adjacent to the Pentagons (4,∞,1,6,10) and (4∞857). Looking at the Pairs on 4 (we have seen 40,45 and 4,10,4∞ and 4∞,47) we see that z is adjacent to (0,4,10, · , ·) or to (0,4,7, · , ·). But z cannot be adjacent to (0,4,10,5,9) since this would cover the Pair 09 three times. Thus we find (04712) and also (5,4,10,7,3). Looking at the Pairs on 0 we find (20694), a contradiction since we now have seven Pairs on 4. Hence also $a=5$ is impossible.

This shows that $a=10$.
This result can be formulated: *if z is adjacent to* (∞09 · ·), (09∞ · ·) *and* (9∞0 · ·) *then also to* (10,∞,1,5,6) *and* (10,∞,8,4,3).
The map $x \mapsto 9(1-5x)^{-1}$ interchanges π_1, π_3 and π_2 cyclically, so leaves the hypothesis invariant. We find that z is also adjacent to (7,0,6,1,5), (2,9,5,6,1), (7,0,4,3,8) and (2,9,3,8,4). Looking at the Pairs on the Symbol 1 we find that z is also adjacent to (213··), and this determines all 12 neighbours of z uniquely. Thus:

LEMMA 7. *Suppose that the p-chains are unions of two directed pentagons. Then the set Z of* 55 *points $z \in \Gamma_4(\Omega)$ such that z is adjacent to two Pentagons of the form* (abc · ·) *and* (bca · ·) *is uniquely determined (as set of sets of Pentagons) and forms one orbit under* $PGL(2,11)$. □

REMARK. In this way we obtain a parallelism on the triples from a 12-set: Each point z from the orbit discussed above determines a partition of the set of Symbols into four triples $\{a,b,c\}$ such that z is adjacent to the Pentagons (abc · ·), (bca · ·) and (cab · ·). In the group $PGL(2,11)$ there is a unique element permuting a,b,c in a given 3-cycle; this element has order 3 and four orbits of size 3. This defines the parallelism. (The same construction works in all $PGL(2,q)$ with $q+1 \equiv 0 \pmod 3$, cf. Cameron [3, p. 109].)

LEMMA 8. *Hypothesis as in Lemma* 7. *The remaining set of* 22 *points in* $\Gamma_4(\Omega)$ *is uniquely determined and forms one orbit onder* $PGL(2,11)$.

PROOF. Let $U := \Gamma_4(\Omega) \backslash Z$. We shall determine the neigbours in U of the Pentagon $\pi_1 = (\infty 0931)$. Since π_1 has five neighbours in Z, it has two neighbours in U. Each element z_1 of $\Gamma_4(\Omega)$ adjacent to π_1 is a set of 12 Pentagons. One is π_1, five have an edge in common with π_1 and six are disjoint from π_1.

Let $\pi_j(2 \leqslant j \leqslant 7)$ be these six Pentagons. Since $d(\pi_1,\pi_j)=2$ these two Pentagons have two common neighbours, namely z_1 and z_j $(2 \leqslant j \leqslant 7)$. The z_i are mutually distinct and exhaust $\Gamma_1(\pi_1)\cap\Gamma_4(\Omega)$. Now that we know five of the z_i explicitly (say $\Gamma_1(\pi_1)\cap Z=\{z_3, \cdots ,z_7\}$) we find the 20 Pentagons that are joined to π_1 by exactly one of z_1 and z_2. There is a unique Pentagon $\neq\pi_1$ joined to each of these 20 by z_1 or z_2 namely $\pi_2=(47658)$. Now we know $z_1 \cup z_2$ (viewed as a set of Pentagons), and we know the relation "being joined by either z_1 or z_2", and we have to find two 12-cliques in this graph. Since two 12-cliques cannot have more than 5 points in common, there is at most one way to do this. Using the fact that the known example satisfies our hypothesis (and conclusion), or actually carrying out these computations, we are done. □

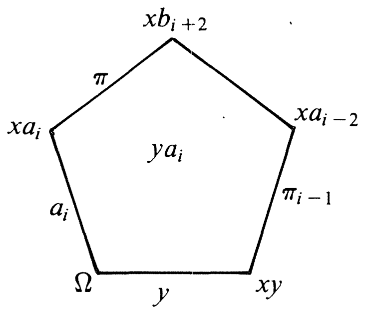
REMARK . Consider an element u of this second orbit $U=\Gamma_4(\Omega)\backslash Z$. It is a set of twelve Pentagons, and we can give it a graph structure by calling two Pentagons adjacent when they have a Pair in common. In this way we obtain a regular graph of valency 5 on 12 vertices; constructing it, or by considering its group ($PGL(2,11)$ induces $Alt(5)$ on this graph) we see that it is the vertex graph of the icosahedron (or equivalently, the face graph of the dodecahedron).
Similarly, we may consider the graph corresponding to an element $z \in Z$. Since z corresponds to a subgroup of order 3 in $PGL(2,11)$, this group induces the normalizer of the subgroup, which is dihedral of order 24. The vertices of this graph may be labeled with $\mathbf{Z}_{12}$ in such a way that two vertices are adjacent iff their difference is in $\{\pm 1,\pm 4,6\}$.

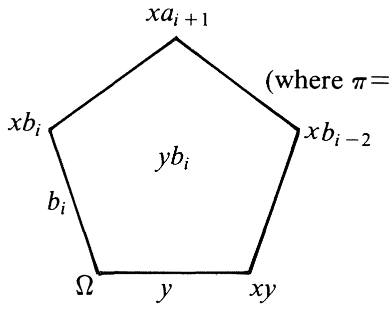
THEOREM. *There is a unique graph* Γ *which is distance regular bipartite of diameter* 4 *with parameters* $k=12$, $c_2=2$, $c_3=5$ *possessing a point* Ω *such that w.r.t.* Ω *the* p*-chains are unions of two directed pentagons. Its group of automorphisms is* $M_{12}\cdot 2$ *and is transitive on each of the two bipartite halves of* Γ. *If we choose a new base point* Ω' *in the other bipartite half of* Γ *then w.r.t.* Ω' *the* p*-chains are directed decagons. The stabilizer in* $Aut(\Gamma)$ *of any point* x_0 *is* $PGL(2,11)$, *transitive on* $\Gamma_1(x_0)$.

PROOF. We have shown the uniqueness of Γ. Now let us see what happens when we choose a new base point Ω'.
Suppose $(a_0a_1a_2a_3a_4)$ is a directed pentagon in the p-chain of (x,y). Let π_i be the Pentagon $(xya_{i+1}a_i a_{i-1})$ $(i \in \mathbf{Z}_5)$. Now choose $\Omega'=xy$. Viewing Γ w.r.t. the new base point Ω' we find the New Symbols x, y and π_i $(i \in \mathbf{Z}_5)$ and the New Pair Ω. This New Pair is contained in the New Pentagons a_i $(i \in \mathbf{Z}_5)$, and $(\pi_0\pi_1\pi_2\pi_3\pi_4)$ is a directed pentagon in the p-chain of the ordered pair of New Symbols (Y,X).
This argument shows:

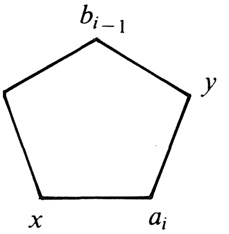
> *If* Ω *and* Ω' *lie in the same bipartite half of* Γ *and some* p*-chain w.r.t.* Ω *contains a directed* j*-gon, then so does some p-chain w.r.t.* Ω'.

This shows that $Aut(\Gamma)$ is transitive on the half containing Ω, but the stabilizer of Ω is $PGL(2,11)$, transitive on the neigbours of Ω, so $Aut(\Gamma)$ is also transitive on the other half.
Now $Aut(\Gamma)$ has the right order $144\cdot|PGL(2,11)|=12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 2$ to be $M_{12}\cdot 2$; also, as we shall see below, it has an imprimitive (transitive) representation on 24 objects with two blocks of size 12, so it can be nothing but $M_{12}\cdot 2$. Since it is the only subgroup with the right index $Aut(\Gamma)_x$ must be $PGL(2,11)$. Finally, suppose $(a_0a_1a_2a_3a_4)$ $(b_0b_1b_2b_3b_4)$ is the p-chain of (x,y) w.r.t. Ω. Choose a new base point $\Omega'=x$. Now Ω, xy, xa_i, xb_i $(i \in \mathbf{Z}_5)$ are the New Symbols. Consider the p-chain w.r.t. Ω' of the ordered pair of New Symbols (Ω,xy). The New Pentagons containing the New Pair $\{\Omega,xy\}=y$ are the ten Pairs ya_i, yb_i $(i \in \mathbf{Z}_5)$, and these look like





(where $\pi=b_{i+2}$



).

Consequently, the p-chain w.r.t. Ω' of (Ω,xy) is the directed decagon

$$(xa_0,xb_2,xa_3,xb_0,xa_1,xb_3,xa_4,xb_1,xa_2,xb_4,xa_0). \quad \square$$

4. THE CASE $K=12$ (cont.) - OTHER TYPES OF P-CHAINS

Having settled the case where a p-chain contains two directed pentagons, let us say the case $5+5$, we still have to examine the cases $3+3+4$, $3+7$, $4+6$ and 10. The former three will turn out to be impossible, the last one leads to the same solution as before.

LEMMA 9. *The case* $3+7$ *does not occur.*

PROOF. Suppose the p-chain of (x,y) is $(a_0a_1a_2a_3a_4a_5a_6)\,(c_0c_1c_2)$. As usual we find on (x,a_i) the directed paths

$$(ya_{i+2}a_{i+1}\gamma_{i+2}) \quad \text{and} \quad (a_{i-2}\gamma_i a_{i-1} \cdot \gamma_{i+1})$$

where γ_i is defined by the Pentagon $(xa_i a_{i-1}\gamma_i a_{i-2})$, $i \in \mathbf{Z}_7$. In the p-chain on (x,a_i) we cannot have an edge (c_0,c_1) since $a_i \neq y$ nor a directed path $(c_0,\cdot,c_1)$ since $a_i \neq c_2$. Thus, the 3 points c_j have mutual distance at least three in the p-chain on (x,a_i), and this p-chain cannot contain a directed path of length 4 without one of these points. Consequently, $\gamma_i \in \{c_0,c_1,c_2\}$ for all $i \in \mathbf{Z}_7$. Also $\gamma_{i+2} \neq \gamma_i,\gamma_{i+1}$, so there is no suitable map $\gamma:\mathbf{Z}_7 \rightarrow \{c_0,c_1,c_2\}$. $\square$

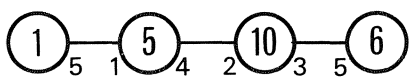
LEMMA 10. *The case* $4+6$ *does not occur.*

PROOF. Suppose the p-chain on (x,y) is $(a_0a_1a_2a_3a_4a_5)\,(c_0c_1c_2c_3)$. Defining γ_i as in the previous Lemma we see that $\gamma_i \neq x,y,a_{i-2},a_{i-1},a_i,a_{i+1},a_{i+2}$. Also $\gamma_{i+2} \neq a_{i-1}$, so each γ_i is one of the c_j. Looking at the directed paths in the p-chain on (x,a_i) (see previous proof) we see that somewhere in this chain two c_j must follow each other. As before (c_j,c_{j+1}) is impossible so we must have an edge (c_j,c_{j+2}). Thus we find a Pentagon $(xa_iyc_{j+2}c_j)$ for each i, but a map $i \mapsto j$ from $\mathbf{Z}_6$ to $\mathbf{Z}_4$ cannot be injective. Contradiction. $\square$

Unfortunately it is not possible to kill the case $3+3+4$ by such local means - the fact that solutions exists for $k=5,6$ means that the occurrence of directed 3-cycles or 4-cycles cannot lead to a contradiction, it only produces a subsystem. A global counting argument kills this case as soon as we know that it always occurs.

LEMMA 11. *It is impossible that all p-chains have type* $3+3+4$.

PROOF. If the p-chain on (x,y) contains the directed 3-gon (uvw) then one immediately verifies (using Lemma 3) that the p-chain on any ordered pair from $\{u,v,w,x,y\}$ contains a directed 3-gon on the remaining three points of this set. In this way we find six Pentagons, and a subgraph of Γ with diagram



in other words, a sub-biplane $2-(11,5,2)$ of the corresponding partial 2-geometry. The total number of such subbiplanes is $\frac{144 \cdot 66 \cdot 2}{11 \cdot 10}$ with is not integral - contradiction. □

LEMMA 12. *If some p-chain is a directed 10-gon then all are.*

PROOF. Suppose the p-chain on (x,y) is $(a_0 \cdots a_9)$. Define γ_i as in the proof of Lemma 9, and we find directed paths as before. Neither $(ya_{i+2}a_{i+1}\gamma_{i+2})$ nor $(a_{i-2}\gamma_i a_{i-1} \cdot \gamma_{i+1})$ can have length 3 (since $\gamma_{i+2} \neq y, \gamma_{i+1}$) and they cannot both be contained in the same 4-gon, so the p-chain on (x,a_i) must be a directed 10-gon. □

LEMMA 13. *If the p-chain w.r.t. Ω are directed 10-gons then the p-chains w.r.t. Ω', a neighbour of Ω, are unions of two directed 5-gons.*

PROOF. Suppose the p-chain on (x,y) is $(a_0 \cdots a_9)$. Just as in the proof of the Theorem in the previous section, look at the p-chain on the ordered pair of New Symbols (Ω,xy) w.r.t. the new base point $\Omega'=x$. We find fragments

$$(xa_0, x\beta, xa_8, \cdot\ , xa_6, \cdot\ , xa_4, \cdot\ , xa_2, \cdot\ , xa_0)$$

(for some β), and

$$(xa_1, \cdot\ , xa_9, \cdot\ , xa_7, \cdot\ , xa_5, \cdot\ , xa_3, \cdot\ , xa_1)$$

so that we either have two 5-gons or one 10-gon. We want to prove $\beta=a_4$ or at least $\beta \neq a_j$ for odd j. The Symbol β is defined by the Pentagon $(\beta xa_0 y\ \cdot)$. Define α by the Pentagon (αxa_0a_2y). Clearly $\alpha \neq x,y,a_0,a_1,a_2,a_8,a_9$: on (x,a_0) we have the fragments $(\beta\alpha ya_2a_1\gamma_2)$ and $(a_8\gamma_0a_9 \cdot \gamma_1)$, part of a directed decagon (and if $\alpha=a_9$ then the p-chain on xa_9 would contain the 4-gon $(ya_1a_0a_2)$ for we have the fragment (a_0a_2y) on (x,α).)
On (x,a_4) we have fragments (ya_6a_5) and $(a_2, \cdot\ , a_3)$ so $\alpha \neq a_4$.
On (x,a_3) we have fragments (ya_5a_4) and $(a_1, \cdot\ , a_2)$ so if $\alpha=a_3$ then these merge and give the fragment $(a_1a_0a_2ya_5a_4)$, but this yields the 4-gon (xya_2a_3) in the p-chain on (a_0,a_1), contradiction. Thus $\alpha \neq a_3$. If $\alpha=a_6$ then $\beta=a_4$ as we wanted. (Note that we have the Pentagon $(\alpha xa_2y\cdot)$ so that we have the fragment $(xa_2, x\alpha, xa_0, x\beta)$ in the p-chain of (Ω,xy) w.r.t. $\Omega'=x$.) Thus we may assume $\alpha \in \{a_5,a_7\}$.
If $\alpha=a_7$ then we have on (x,a_7) the fragment $(a_0a_2ya_9a_8)$ so that (defining α_i by the Pentagon $(\alpha_i xa_i a_{i+2}y)$) we have $\alpha_7=a_{7+5}$. Similarly, if $\alpha=a_5$ then we find $\alpha_5=a_{5+7}$. Thus, by suitably shifting the numbering of the a_i, we may assume $\alpha=a_7$, and now $\alpha_i=a_{7+i}$ for even i, $\alpha_i=a_{5+i}$ for odd i. Now that $\alpha=a_7$ it follows that $\beta=a_5$.
Define γ by the fragment $(\gamma\beta\alpha y)$ on (x,a_0), i.e., by the Pentagon $(xa_0a_7a_5\gamma)$. Clearly $\gamma \neq x,y,a_0,a_1,a_2,a_5,a_7,a_8$, so that $\gamma \in \{a_3,a_4,a_6,a_9\}$. On (x,γ) we have the fragment $(a_0a_7a_5)$ so that $\gamma \neq a_3,a_6,a_4$. Hence $\gamma=a_9$ and we have the fragment $(a_8\gamma_0a_9a_5a_7ya_2a_1\gamma_2)$ on (x,a_0); also $\gamma_1=a_7$. On (x,a_1) we find the 10-gon $(a_4a_6ya_3a_2a_9a_7a_0\cdot\gamma_2)$ so that $\gamma_2 \notin \{a_3,a_4,a_6\}$, contradiction. □

THEOREM *There is a unique bipartite distance regular graph of diameter 4 with parameters $k=12$, $c_2=2$, $c_3=5$.* □

REMARK. Analysing the set of Pentagons for a choice of Ω such that the p-chains are 10-gons one finds that the Pentagons are the images of $(\infty 0571)$ under $PGL(2,11)$ for a suitable labeling of the Symbols.

5. THE CASE $k=12$ (cont.) - STRUCTURE OF THE ASSOCIATED STRONGLY REGULAR GRAPHS

Let for the moment $k \in \{12,20\}$. As already remarked in section 1, if we take the vertices in one bipartite half V' of Γ and call them adjacent whenever they have distance two in Γ then we obtain a strongly regular graph Γ'. Clearly, the vertices in the other bipartite half V'' of Γ are k-cliques in this graph. Now by the Hoffman bound any clique in Γ' has size at most $1=K/(-s)=k$ so that these cliques are maximal. Any point outside a k-clique C is adjacent to precisely 5 points of C.

CLAIM. There are no other k-cliques than the vertices of V''.

PROOF. Choose $\Omega \in V''$ so that the vertices of Γ' are Symbols and Pentagons. If some cliques C contains both Symbols and Pentagons then at most 5 Symbols; but if there are at least 3 Symbols in C, then at most one Pentagon and $|C| \leqslant 6$. If C contains 2 symbols x,y then also all the Pentagons on xy (there are $k-2$ of those), so that C is determined by the pair xy. but we can always choose Ω such that C contains at least two Symbols. □

Thus for $k>6$, the graph Γ' completely determines Γ, and $Aut(\Gamma') \leqslant Aut(\Gamma)$. (This is not true for $k=6$.) Now let $k=12$, and look at the maximal cocliques. By the Hoffman bound these have size at most $144/12=12$, and any point outside a 12-clique S is adjacent to precisely 6 points of S. Suppose S is a 12-clique in Γ'. Let $C_s=\Gamma_1(s)$ for $s \in S$; then the C_s form a partition of the strongly regular graph Γ'' on V'' into maximal cliques. Conversely, any partition of Γ'' into 12 pairwise disjoint 12-cliques arises in this way.
For any vertex Ω of Γ' there are precisely 24 12-cocliques containing Ω. Under $PGL(2,11)$ these fall into two orbits, one of size 2 and one of size 22. Let us call the two 12-cocliques from the small orbit the *special cocliques* for Ω. Now let Γ^A be the graph described in section 3 with p-chains of type $5+5$; let Γ^B be the graph with p-chains of type 10.
In Γ^B the situation is simple: if S is a special coclique for a and $b \in S$, then S is a special coclique for b. It follows that there are precisely 24 special cocliques, and these split (in a unique way) into two partitions of Γ^B. Thus we find the imprimitive representation of $Aut(\Gamma^B)$ on $12+12$ objects, as announced earlier.
(These special cocliques intersect in either 0 or 1 point, i.e., they form a 12×12 grid.) In Γ^A on the other hand, if S is a special coclique for a and $b \in S$, then there is a unique special coclique for b containing a, but it is not S. Consequently, each of the 288 12-cocliques is special for exactly one of its elements. $Aut\ \Gamma^A$ is transitive on these 288 12-cocliques, and the stabilizer of one is $PSL(2,11)$.

These considerations lead to a very simple construction of Γ^B: Look at the Steiner system $S(5,8,24)$ and let D_1 and D_2 be two complementary dodecads, so that there are 132 blocks with 6 points in D_1 and 2 points in D_2, 132 blocks of type $2+6$ and 495 blocks of type $4+4$.
Let the vertices of Γ^B be the ordered pairs $(d_1,d_2) \in D_1 \times D_2$. Call two such pairs (d_1,d_2), (e_1,e_2) nonadjacent whenever either $d_1=e_1$ or $d_2=e_2$ or there is a block B in the Steiner system with $B \cap D_1=\{d_1,e_1\}$ and $B \cap D_2 \supset \{d_2,e_2\}$.

[Note that there are precisely two blocks B', B'' meeting D_1 in $\{d_1,e_1\}$, and we have $B' \cup B'' \supset D_2$. Thus, given d_1, d_2, e_1 there are 5 ways to choose e_2, and any vertex is nonadjacent to $11+11+55=77$ vertices, adjacent to 66 so that we have the right valency.
Note that the definition is symmetric: if $\{d_1,d_2,e_1,e_2\}$ is not covered by a block of type $2-6$, then it is covered by five blocks of type $4-4$ and not by a block of type $6-2$. Consequently, any involution in $M_{12} \cdot 2$ interchanging D_1 and D_2 is an automorphism of Γ^B.
The 24 special cocliques are the 24 points.
The 12-cliques are certain involutions interchanging D_1 and D_2 but cannot be automorphisms, since any automorphism stabilizing a point and all its neighbours (in Γ) must be the identity. Thus, the 12-

cliques form a conjugation class of involutions under conjugation by $M_{12} \cdot 2$, but are not themselves in $M_{12} \cdot 2$.]

Group-theoretically our two graphs Γ^A and Γ^B are defined by subgroups $PGL(2,11)$ of $M_{12} \cdot 2$. Up to conjugacy there are two such subgroups; the first meets M_{12} in a maximal subgroup $PSL(2,11)$ - this yields the rank 4 presentation Γ^A -, and the other meets M_{12} in a subgroup $PSL(2,11)$ that is contained in a M_{11} - this yields Γ^B -. Note that both classes of $PGL(2,11)$'s are maximal in $M_{12} \cdot 2$. (cf. Conway [6].)

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