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SOME OBSERVATIONS CONCERNING THE ZERO-CURVES
OF THE REAL AND IMAGINARY PARTS OF RIEMANN'S ZETA FUNCTION

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Some observations concerning the zero-curves of the real and imaginary parts of Riemann's zeta function

by

J. van de Lune

ABSTRACT

It is shown here that the supremum σ_0 of the set $\{\sigma \in \mathbb{R} \mid \operatorname{Re} \zeta(\sigma+it) < 0$ for some $t \in \mathbb{R}\}$ is given by the (unique) solution of the equation

$$\sum_p \operatorname{asin}(p^{-\sigma}) = \frac{\pi}{2}, \quad (\sigma > 1)$$

where p runs through the primes.
For $\sigma = \sigma_0$ we have $\operatorname{Re} \zeta(\sigma+it) > 0$ for all $t \in \mathbb{R}$.

Using all primes $< 10^9$, we found (numerically) that $\sigma_0 > 1.192$.
Moreover, a method is presented for the numerical determination of t-values such that $\operatorname{Re} \zeta(1+it) < 0$. As a result we have for example:
$\operatorname{Re} \zeta(1+i*682,112.92) \cong -.003$.
The paper concludes with an informal discussion of how to find values of t such that the "signed modulus" Z(t) behaves quite "unusual". As an example we mention the result

$$Z(t) < -453.9 \quad \text{for} \quad t = 725,177,880,629,981.914,597.$$

Finally, some values of t are listed in the vicinity of which Gram's law and/or Rosser's rule are violated.

KEY WORDS & PHRASES: *Zero-curves, Riemann's zeta function, Riemann hypothesis, Lehmer's phenomenon, exceptions to Gram's law and/or Rosser's rule.*

0. INTRODUCTION

Looking at the values of Re $\zeta(1+it)$, $t > 0$, as tabulated by HASELGROVE & MILLER [5], one will notice that Re $\zeta(1+it)$ is positive for all listed values of t in the interval $[0,100]$. Since Re $\zeta(1+it)$ does not even come close to zero in this range (the minimal value (in this range) being $\cong .32$ at $t \cong 14.2$), one may wonder whether Re $\zeta(1+it)$ is ever negative for some $t \in \mathbb{R}$. One may compare this question with Gram's observation that even Re $\zeta(\frac{1}{2}+it)$ is preponderantly positive (cf. EDWARDS [3; Section 6.5]).

I am aware that Theorem 11.9 in TITCHMARSH [11; p. 256] answers the above question in the *affirmative*. However, then the question remains: How far to the right of $\sigma = 1$ does Re $\zeta(s)$ assume negative values and how can one actually find values of t such that Re $\zeta(1+it)$ is negative? In other words: How far to the right do the zero-curves of Re $\zeta(s)$ penetrate the halfplane $\sigma > 1$.

It should be noted here that the observation that Re $\zeta(1+it)$ is "usually" positive is not too much of a surprise. First of all, for $\sigma > 1$, we have

$$\text{Re } \zeta(\sigma+it) = 1 + \sum_{n=2}^{\infty} \frac{\cos(t \log n)}{n^{\sigma}}$$

and it is to be expected that it will take some "time" before the (positive) leading term (=1) has been overwhelmed by the remaining ("rather erratic") terms of the series for Re $\zeta(s)$. Moreover, since $|\text{Re } \zeta(\sigma+it)| \leqq \zeta(\sigma)$ for $\sigma > 1$, we may, for $u > 0$, consider the Laplace transform $\Phi_{\sigma}(u)$ of Re $\zeta(\sigma+it)$:

$$\Phi_{\sigma}(u) = \int_0^{\infty} e^{-ut} \text{ Re } \zeta(\sigma+it)dt =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \frac{u}{u^2+(\log n)^2}, \quad u > 0$$

from which it is clear that, for all $\sigma > 1$, $\Phi_{\sigma}(u) > 0$ for all $u > 0$. In addition to this it is easy to show that

$$\lim_{u \downarrow 0} u.\Phi_{\sigma}(u) = 1, \quad \sigma > 1$$

so that Re $\zeta(\sigma+it)$ has the (positive) Abel-Laplace-limit 1 as $t \to \infty$. One may show that this statement also holds true for $\sigma = 1$ and, since Re $\zeta(1+it)$ is a "rather small function" of t (as $t\to\infty$), (cf. TITCHMARSH [11; p. 42]), we have a clear (though only heuristic) indication that Re $\zeta(1+it)$ is preponderantly positive.

For reasons of just giving it a try I have evaluated Re $\zeta(1+it)$ for quite a number of (more or less randomly chosen) t-values, resulting in the rather disappointing fact that (in this way) I did *not* find one single t for which Re $\zeta(1+it)$ was negative. A "champion-observation" was:

$$\mathrm{Re}\ \zeta(1+i * 8646.23) \cong 0.043.$$

Nevertheless, below I shall produce some numerical values of t for which Re $\zeta(1+it) < 0$. *Implicitly* I will determine "precisely" how far to the right the zero-curves of Re $\zeta(s)$ penetrate the halfplane $\sigma > 1$.

In addition to this I will give some examples of the remarkable erratic behaviour of the "signed modulus" (cf. EDWARDS [3]) $Z(t) := e^{i\theta(t)}\zeta(\frac{1}{2}+it)$, in the neighbourhood of some t-values for which $\zeta(1+it)$ assumes "unusual" values.

1. HOW FAR DO THE ZERO LINES OF Re $\zeta(s)$ PENETRATE THE HALFPLANE $\sigma > 1$?

For $\sigma > 1$ we have by definition

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n^{-\sigma}\exp(-it\log n)$$

so that

$$\mathrm{Re}\ \zeta(s) = \sum_{n=1}^{\infty} n^{-\sigma}\cos(t\log n) \geq 1 - \sum_{n=2}^{\infty} n^{-\sigma}$$

which is easily seen to be *positive* for $\sigma \geq 2$. Hence, the zero curves of Re $\zeta(s)$ do not penetrate the halfplane $\sigma > 1$ arbitrarily far (to the right).

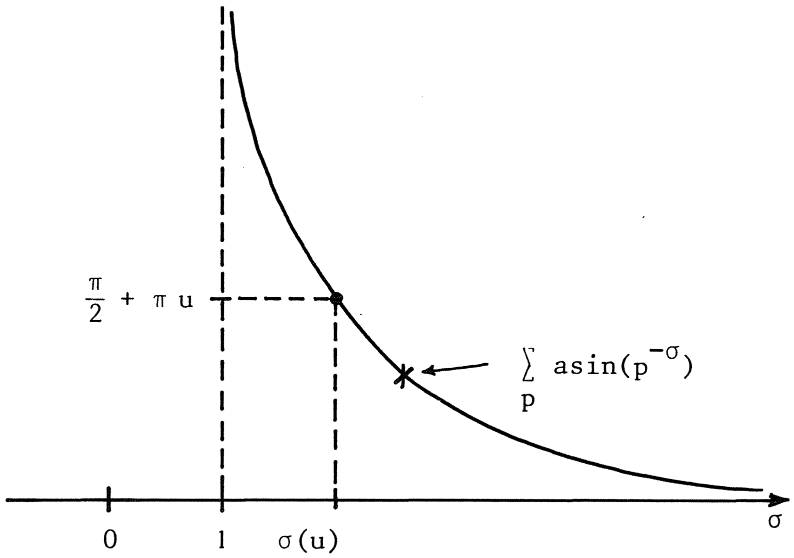
From the above explicit representation of Re $\zeta(s)$ it is not immediately clear whether Re $\zeta(s) = 0$ at all for any s in the halfplane $\sigma > 1$.

In order to see that the zero-curves of Re $\zeta(s)$ enter the halfplane

$\sigma > 1$ indeed (bluntly ignoring TITCHMARSH's Theorem 11.9 referred to above) I first define the real function $\sigma(u)$ for $u > -\frac{1}{2}$ as the unique solution of the equation (in σ)

$$\sum_p \operatorname{asin}(p^{-\sigma}) = \frac{\pi}{2} + \pi u$$

where p runs through the set of prime numbers. For $u > -\frac{1}{2}$ this equation has a unique solution indeed, since for real σ the series $\Sigma_p \operatorname{asin}(p^{-\sigma})$ converges only for $\sigma > 1$ and its sum *decreases* from $+\infty$ to 0 as σ increases from 1 to $+\infty$. Clearly, $\sigma(u)$ is a decreasing function of u, assuming all positive values.



In what follows I will mainly be interested in $\sigma_0 := \sigma(0)$ and $\sigma_1 := \sigma(1)$.

INTERMEZZO

Before proceeding I want to make some remarks about $\sigma(u)$ for *large* values of u. Since $\Sigma_p\, p^{-1}$ is very slowly divergent it is to be expected that if u is large then $\sigma(u)$ will be very close to 1. This is a consequence of the well known fact that

$$\sum_{p \le x} p^{-1} = \log\log x + \mathcal{O}(1), \quad x \to \infty.$$

Hence, for *large* (positive) u we have (temporarily writing σ instead of $\sigma(u)$)

$$\begin{aligned} \frac{\pi}{2} + \pi u &= \sum_p \operatorname{asin}(p^{-\sigma}) = \\ &= \sum_p p^{-\sigma} + \sum_p (\operatorname{asin}(p^{-\sigma}) - p^{-\sigma}) \cong \\ &\cong \sum_p p^{-\sigma} + \sum_p (\operatorname{asin}(p^{-1}) - p^{-1}). \end{aligned} \tag{1.1}$$

On the other hand we obtain (for $\sigma > 1$ and close to 1) from the Euler product for $\zeta(s)$

$$\begin{aligned} \log \zeta(\sigma) &= -\sum_p \log(1-p^{-\sigma}) = \\ &= \sum_p p^{-\sigma} - \sum_p (\log(1-p^{-\sigma}) + p^{-\sigma}) \cong \\ &\cong \sum_p p^{-\sigma} - \sum_p (\log(1-p^{-1}) + p^{-1}) \end{aligned}$$

so that, recalling that

$$\zeta(\sigma) = \frac{1}{\sigma-1} + \gamma + o(1), \quad \sigma \downarrow 1$$

we have

$$\sum_p p^{-\sigma} \cong \log(\frac{1}{\sigma-1} + \gamma) + \sum_p (\log(1-p^{-1}) + p^{-1})$$

where γ is Euler's constant. Combining this with (1.1) we thus have

$$\frac{\pi}{2} + \pi u \cong \log(\frac{1}{\sigma-1} + \gamma) + \sum_p (\log(1-p^{-1}) + \operatorname{asin}(p^{-1})) \cong \log(\frac{1}{\sigma-1} + \gamma) - C$$

where

$$C := -\sum_p (\log(1-p^{-1}) + \operatorname{asin}(p^{-1})) \cong 0.283\,465$$

and hence, as a first approximation (for *large* u)

$$\sigma(u) \cong 1 + \left(e^{\pi(u+\frac{1}{2})+C-\gamma}\right)^{-1} .$$

For $u = 0$ this approximation yields

$$\sigma(0) \cong 1.172$$

whereas, using all primes $< 10^9$, I arrived computationally at the inequality $\sigma(0) > 1.192$. I devote a few words and a picture to the procedure by which this lower bound for $\sigma(0)$ was obtained.

Define

$$f(x) = \frac{\pi}{2} - \sum_{p} \operatorname{asin}(p^{-x}), \quad x > 1$$

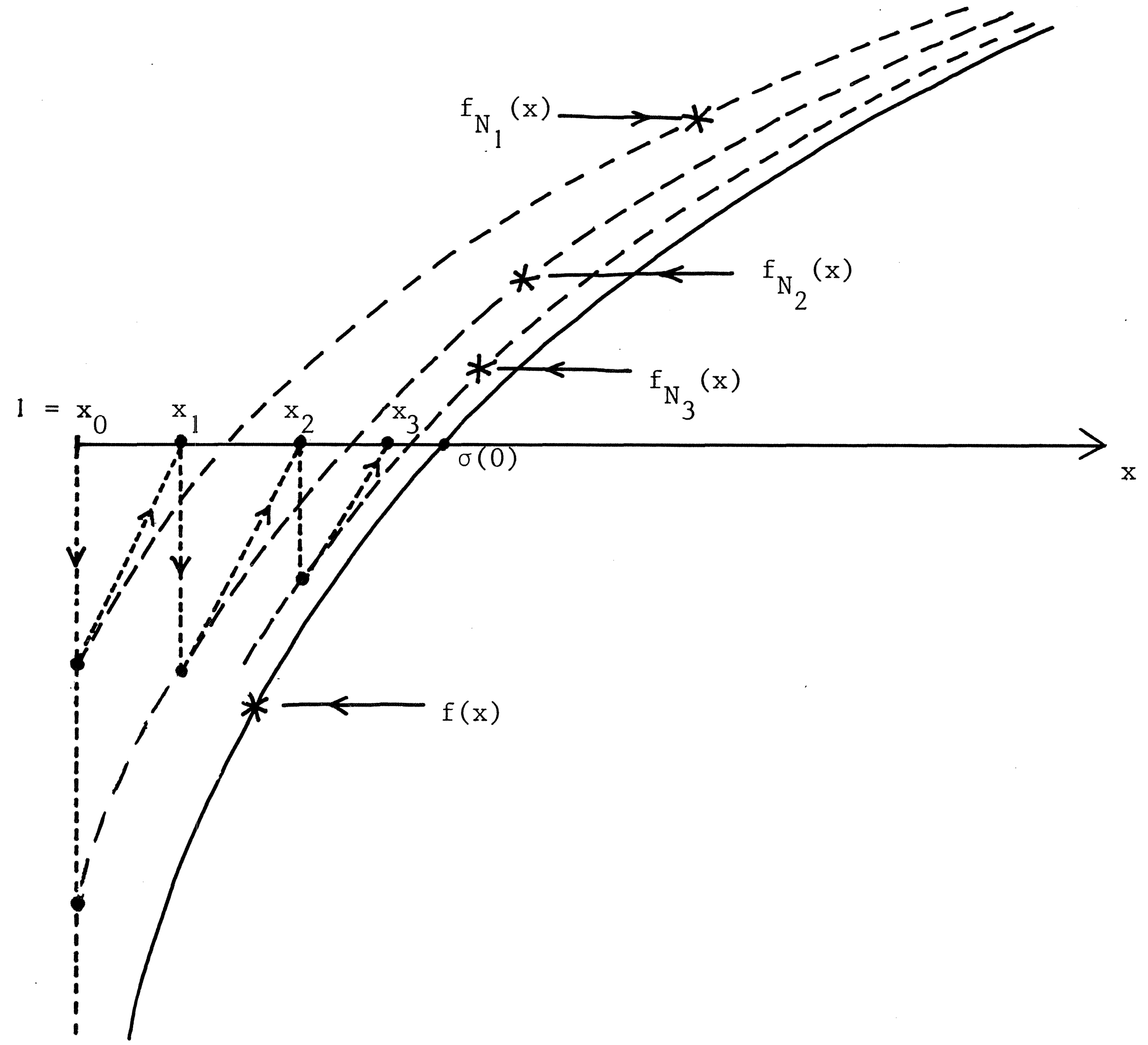
and, for $N \in \mathbb{N}$, define

$$f_N(x) = \frac{\pi}{2} - \sum_{p \leqq N} \operatorname{asin}(p^{-x}), \quad x \geqq 1.$$

Then both $f(x)$ and $f_N(x)$ are *increasing* and *concave* (in x) and $f_N(x) > f(x)$ for $x > 1$. The following picture reveals the numerical (Newton) approach to the problem in question.

(Note that $31 \leqq N_1 < N_2 < N_3 < \ldots;\ 1 = x_0 < x_1 < x_2 < x_3 \ldots,$

where $$x_{k+1} := x_k - \frac{f_{N_{k+1}}(x_k)}{f'_{N_{k+1}}(x_k)} \quad . \qquad)$$



The complete listing of the corresponding FORTRAN-program will be given in an Appendix. It seems to me that this is an appropriate instance to express my gratitude to A.E. Brouwer, J. Jansen and E. Wattel for their most valuable suggestions when writing this program.

For u = 1 the above approximation yields

$$\sigma(1) \cong 1.0068$$

which will most probably be quite close to its true value.

I now continue the discussion of the main subject of this section by *fixing* a σ *between* $\sigma_1 := \sigma(1)$ and $\sigma_0 := \sigma(0)$. From the Euler product for $\zeta(s)$ we obtain (for $\sigma > 1$)

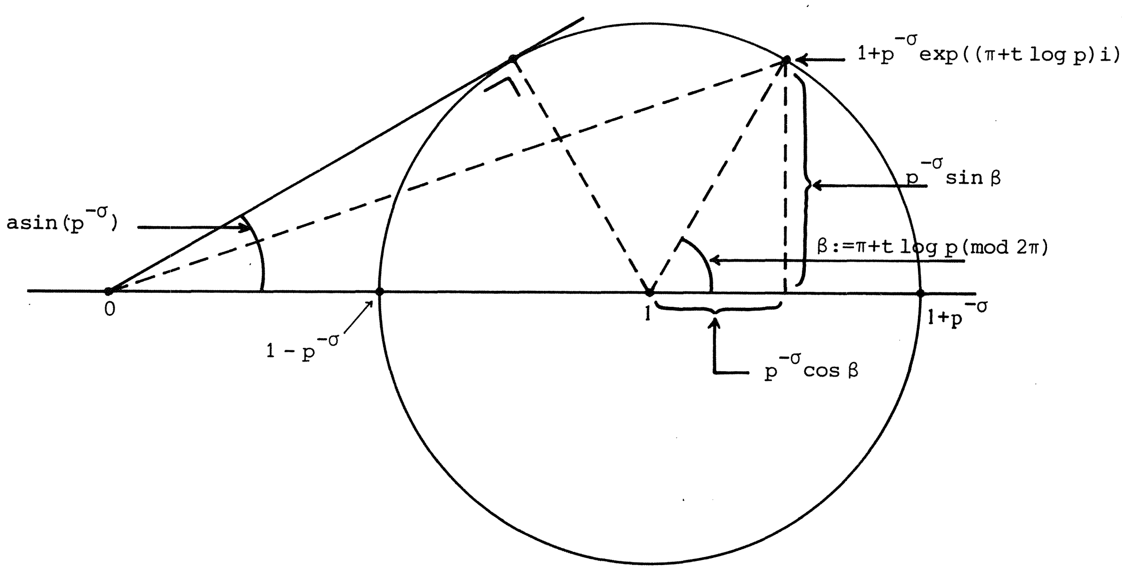
$$\arg \zeta(s) = -\sum_p \arg(1-p^{-s}) = \sum_p \arg(1-p^{-\bar{s}}) =$$

($\bar{s}$ denoting the complex conjugate of s)

$$= \sum_p \arg(1+p^{-\sigma}\exp((\pi+t\log p)i)) =$$

$$= \sum_p \operatorname{atan} \frac{\sin(t\log p)}{-p^{\sigma}+\cos(t\log p)} .$$

In order to see this quickly, just consult the following picture.



Indeed, from this picture we infer directly that

$$\arg(1+p^{-\sigma}\exp((\pi+t\log p)i)) =$$

$$= \operatorname{atan}\frac{p^{-\sigma}\sin\beta}{1+p^{-\sigma}\cos\beta} = \operatorname{atan}\frac{\sin\beta}{p^{\sigma}+\cos\beta} =$$

$$= \operatorname{atan}\frac{\sin(\pi+t\log p)}{p^{\sigma}+\cos(\pi+t\log p)} =$$

$$= \operatorname{atan}\frac{\sin(t\log p)}{-p^{\sigma}+\cos(t\log p)}.$$

From the above picture it is also clear that

$$-\operatorname{asin}(p^{-\sigma}) \le \operatorname{atan}\frac{\sin(t\log p)}{-p^{\sigma}+\cos(t\log p)} \le \operatorname{asin}(p^{-\sigma})$$

so that, for $\sigma_1 < \sigma < \sigma_0$,

$$|\arg\zeta(s)| \le \sum_p \operatorname{asin}(p^{-\sigma}) < \sum_p \operatorname{asin}(p^{-\sigma_1}) = \frac{3}{2}\pi.$$

Consequently, for $\sigma_1 < \sigma < \sigma_0$, we need not worry that

$$|\arg\zeta(s)| \geqq \frac{3}{2}\pi.$$

Now choose an ε satisfying (note that σ is fixed now)

$$0 < \varepsilon < \frac{1}{3}\sum_p (\operatorname{asin}(p^{-\sigma})-\operatorname{asin}(p^{-\sigma_0}))$$

and choose an N such that

$$\sum_{p>N} \operatorname{asin}(p^{-\sigma}) < \varepsilon$$

so that

$$\arg\zeta(s) > \sum_{p\le N} \operatorname{atan}\frac{\sin(t\log p)}{-p^{\sigma}+\cos(t\log p)} - \varepsilon.$$

According to Kronecker's theorem there exist arbitrarily large values of t (> 0) such that for every term in the sum $\Sigma_{p\leq N}$ we have

$$\operatorname{atan} \frac{\sin(t \log p)}{-p^{\sigma}+\cos(t \log p)} > \operatorname{asin}(p^{-\sigma}) - \frac{\varepsilon}{\pi(N)} .$$

Consequently, for such t we have

$$\arg \zeta(s) > \sum_{p\leq N} \operatorname{asin}(p^{-\sigma}) - 2\varepsilon =$$

$$= \sum_{p} \operatorname{asin}(p^{-\sigma}) - \sum_{p>N} \operatorname{asin}(p^{-\sigma}) - 2\varepsilon >$$

$$> \sum_{p} \operatorname{asin}(p^{-\sigma}) - 3\varepsilon.$$

It follows that (recall the definition of σ_0)

$$\arg \zeta(s) > \frac{\pi}{2} + \left(\sum_{p} \operatorname{asin}(p^{-\sigma}) - \sum_{p} \operatorname{asin}(p^{-\sigma_0})\right) - 3\varepsilon >$$

$$> \frac{\pi}{2} + 3\varepsilon - 3\varepsilon = \frac{\pi}{2}.$$

Resuming, we have shown that for any $\sigma \in (\sigma_1, \sigma_0)$ there exist arbitrarily large t (> 0) such that

$$\frac{\pi}{2} < \arg \zeta(\sigma+it) < \frac{3}{2}\pi.$$

Hence, since for any $z \in \mathbb{C}$, $z \neq 0$,

(!) $$\operatorname{Re}(z) < 0 \iff \frac{\pi}{2} < \arg(z) (\operatorname{mod} 2\pi) < \frac{3}{2}\pi$$

we have shown the existence of infinitely many $s = \sigma + it$ with $\sigma > 1$, such that $\operatorname{Re} \zeta(s) < 0$. Since σ was chosen arbitrarily between σ_1 and σ_0 it easily follows that the zero-lines of $\operatorname{Re} \zeta(s)$ penetrate the halfplane $\sigma > 1$ as far as $\sigma = \sigma_0 = \sigma(0)$.

For $\sigma > \sigma_0$ we have by the definition of σ_0

$$\sum_{p} \operatorname{asin}(p^{-\sigma}) < \frac{\pi}{2}$$

and it is an easy consequence that

$$\mathrm{Re}\ \zeta(s) > 0 \text{ for } \sigma > \sigma_0.$$

Finally we prove that $\mathrm{Re}\ \zeta(\sigma_0+it) \neq 0$ for all t.
Suppose that $\mathrm{Re}\ \zeta(\sigma_0+it_0) = 0$. Then

$$\frac{\pi}{2} = |\arg\ \zeta(\sigma_0+it_0)| \leq \sum_p \operatorname{atan} \frac{|\sin(t_0 \log p)|}{p^{\sigma_0}-\cos(t_0 \log p)} \leq$$

$$\leq \sum_p \operatorname{asin}(p^{-\sigma_0}) = \frac{\pi}{2}$$

so that for all primes p

$$\operatorname{atan} \frac{|\sin(t_0 \log p)|}{p^{\sigma_0}-\cos(t_0 \log p)} = \operatorname{asin}(p^{-\sigma_0}).$$

From this it is easily deduced that

$$\cos(t_0 \log p) = p^{-\sigma_0} \text{ for all primes } p.$$

Hence

$$t_0 \log p = k_p \cdot 2\pi + \operatorname{acos}(p^{-\sigma_0}), \text{ for some } k_p \in \mathbb{Z}$$

and

$$t_0 \log q = k_q \cdot 2\pi + \operatorname{acos}(q^{-\sigma_0}), \text{ for some } k_q \in \mathbb{Z}$$

so that (taking p and q as neighbouring primes) for $n_{p,q} := k_p - k_q$,

$$t_0 \log \frac{p}{q} = n_{p,q} \cdot 2\pi + \operatorname{acos}(p^{-\sigma_0}) - \operatorname{acos}(q^{-\sigma_0}).$$

Letting $p \to \infty$ and observing that $\frac{p}{q} \to 1$ it follows that eventually $n_{p,q} = k_p - k_q = 0$ so that k_p is eventually constant (k_0, say) and hence

$$\log p = \frac{1}{t_0}(k_0 \cdot 2\pi + \operatorname{acos}(p^{-\sigma_0}))$$

for all large p, which is a palpable absurdity.

2. HOW TO FIND $s = \sigma + it$, $\sigma > 1$, SUCH THAT Re $\zeta(s) < 0$?

It stands to reason that when searching for an $s = \sigma + it$, $\sigma > 1$, such that Re (s) < 0, we should try to find a t such that for some "fairly large" N the sum (note that I replace $\sigma > 1$ by $\sigma = 1$)

$$S_N(t) := \sum_{p \le N} \operatorname{atan} \frac{\sin(t \log p)}{-p+\cos(t \log p)}$$

is either larger than $\frac{\pi}{2}$ or less than $-\frac{\pi}{2}$. In actual numerical computations we need not worry about "overshoot", i.e. we need not worry that when finding a t such that $S_N(t) > \frac{\pi}{2}$ (or $< -\frac{\pi}{2}$) we would be confronted with the possibility of accidentally having found a t such that $S_N(t) > \frac{3\pi}{2}$ (or $S_N(t) < -\frac{3\pi}{2}$). In any of these cases we would have

$$\sum_{p \le N} \operatorname{asin}(p^{-1}) > \frac{3\pi}{2} \cong 4.712.$$

However, it would require a tremendous number of primes to make such an inequality true. Define, for some $N \in \mathbb{N}$,

$$f_1(t) := \frac{\pi}{2} - S_N(t)$$

and

$$f_2(t) := \frac{\pi}{2} + S_N(t).$$

Then we are searching for a t such that either $f_1(t) < 0$ or $f_2(t) < 0$. It is clear that $f_1(0) = f_2(0) = \frac{\pi}{2}$ (> 0) so that we are essentially interested in finding the positive down-zeros of either f_1 or f_2. In view of a possible application of the *maximum slope principle* (see [9]) we compute

$$\frac{d}{dt} S_N(t) = \sum_{p \le N} \frac{1}{1+\left(\frac{\sin(t \log p)}{-p+\cos(t \log p)}\right)^2} *$$

$$* \frac{(-p+\cos(t \log p))\log p \cos(t \log p) + \log p \sin(t \log p)\sin(t \log p)}{(-p+\cos(t \log p))^2} =$$

$$= \sum_{p \le N} \frac{(\log p)(1-p\cos(t \log p))}{p^2-2p\cos(t \log p)+1}$$

and conclude after some calculus that

$$\left|\frac{d}{dt} S_N(t)\right| \le \sum_{p\le N} \frac{\log p}{p-1} \quad (=: M_N) \text{ for all } t.$$

Using this maximal slope we may treat $f_1(t)$ and $f_2(t)$ simultaneously by means of the "algorithm" defined by

$$t_0 = 0, \; t_{n+1} = t_n + \frac{\min(f_1(t_n), f_2(t_n))}{M_N} \quad \text{if } n \geq 0.$$

We still have to choose our "fairly large" N. In order to make sure that f_1 and/or f_2 have (real) zeros indeed we certainly need to choose N such that

$$\sum_{p\le N} \operatorname{asin}\left(\frac{1}{p}\right) \ge \frac{\pi}{2}$$

and one may verify that this inequality requires $N \geq 31$.

Indeed, we have

$$\frac{\pi}{2} - \sum_{p\le 29} \operatorname{asin}\left(\frac{1}{p}\right) \cong + .0051$$

and

$$\frac{\pi}{2} - \sum_{p\le 31} \operatorname{asin}\left(\frac{1}{p}\right) \cong - .0271.$$

It is to be expected that, with N = 31, we are not going to find a zero of either f_1 or f_2 very quickly.

In order to find a remedy for this we may

(i) take N larger than 31 and/or

(ii) replace $\frac{\pi}{2}$ in the definition of f_1 and f_2 by a somewhat smaller number.

Actually I did both. I replaced f_1 and f_2 by

$$f_1^*(t) := A - S_N(t)$$

and

$$f_2^*(t) := A + S_N(t)$$

and after performing some numerical experiments with various values of A and N, I decided to take ($\frac{\pi}{2}$ >) $A \geq 1.45$ and $N \geq 43$. In order not to waste too much computer time the corresponding computer program printed a message when either $f_1^*(t)$ or $f_2^*(t)$ was < .1. The heuristic reason for lowering the constant $\frac{\pi}{2}$ in the definition of f_1 and f_2 is of course: Do not let the first few primes do *all* the "dirty work".

When a "down-zero" of f_1^* or f_2^* was found I determined the local minimum nearby and evaluated Re $\zeta(1+it)$ in an interval around the corresponding t by means of the Euler-Maclaurin summation formula (see EDWARDS [3; Chapter 6]) and/or by a Riemann-Siegel type formula for $\zeta(1+it)$ as described in HASELGROVE & MILLER [5; p. XIX]. (It should be noted that the formula given by Haselgrove & Miller contains some slight errors!) On January 22, 1979, for the first time I found a t such that Re $\zeta(1+it) < 0$:

$$\text{Re}\ \zeta(1+i * 38\,468\,816.1) \cong -.107.$$

This result was a consequence of a rather low local minimum of f_2^*. Subsequent investigations have shown that Re $\zeta(1+it) < 0$ for the following values of t (I do not claim that the following table is exhaustive):

TABLE I

t	Re $\zeta(1+it) \cong$	t	Re $\zeta(1+it) \cong$
682 112.92	-.003	59 564 375.45	-.010
1 466 782.07	-.001	100 489 439.10	-.034
3 548 283.42	-.019	200 229 743.80	-.047
6 164 063.00	-.026	300 044 243.20	-.017
7 766 995.03	-.074	350 691 975.10	-.014
8 350 473.49	-.002	500 797 651.60	-.072
23 079 622.39	-.008	603 389 001.03	-.059
38 468 816.11	-.108	752 294 743.98	-.085
40 124 822.40	-.036	800 757 394.81	-.043
40 656 048.60	-.037	1 910 738 309.548	-.181
47 686 011.07	-.008	3 634 344 284.40	-.055

A more systematic search, starting at $t = 100$, has made me believe that $t \cong 682112.92$ is the "smallest" t such that $\mathrm{Re}\,\zeta(1+it) < 0$. In order to speed up the search for low minima of f_1 and/or f_2 one may replace these functions by functions of the following type:

$$\frac{\pi}{2} \pm \left(\sum_{p\le M} \operatorname{atan} \frac{\sin(t \log p)}{-p+\cos(t \log p)} - \sum_{M<p\le N} \frac{\sin(t \log p)}{p} \right)$$

and then apply the "accellerated maximal slope principle" (as described in [9]) to the "long sum" $\Sigma_{M<p\le N}$, where $M = 20$ and $N = 400$, say.

3. TESTING THE RIEMANN HYPOTHESIS

3.1. The Lehmer phenomenon

It is well known that Riemann's zeta function is meromorphic on $\mathbb{C}$ (having only one simple pole at $s = 1$) and satisfies the functional equation

$$\zeta(s) = 2(2\pi)^{s-1} \sin \frac{\pi s}{2}\, \Gamma(1-s)\zeta(1-s).$$

This equation leads quite naturally to the definition of a real function $Z(t)$ of the real variable t

$$Z(t) := \pi^{-\frac{1}{2}it} \frac{\Gamma(\frac{1}{4} + \frac{1}{2}it)}{|\Gamma(\frac{1}{4} + \frac{1}{2}it)|}\, \zeta(\tfrac{1}{2}+it)$$

and it is this function which plays a crucial role in most present day numerical work on $\zeta(s)$ in relation to the location and/or separation of the so called non-trivial zeros of $\zeta(s)$. The non-trivial zeros of $\zeta(s)$ are those in the strip $0 < \sigma < 1$. It is not hard to show that all other zeros of $\zeta(s)$ are $s = -2, -4, -6, \ldots$ etc., the so called trivial zeros.

There are infinitely many non-trivial zeros which (due to the functional equation) lie on $\sigma = \frac{1}{2}$ or, if not, come in pairs $\beta+\gamma i$, $1-\beta+\gamma i$ (since $\zeta(\bar{s}) = \overline{\zeta(s)}$ we may restrict ourselves to the upper halfplane), symmetrically about the line $\sigma = \frac{1}{2}$.

RIEMANN [10] conjectured that in fact they all lie on the line $\sigma = \frac{1}{2}$. For more details on this sketchy background information on $\zeta(s)$ we refer to

EDWARDS [3], HASELGROVE [5], TITCHMARSH [11] and INGHAM [4].

As to the numerical verification of the Riemann hypothesis (in a given range) I refer to [1], [2], [7] and [8].

From the definition of Z it is clear that the zeros of $\zeta(s)$ on $\sigma = \frac{1}{2}$ coincide with the real zeros of Z and the localization of the zeros of $\zeta(\frac{1}{2}+it)$ reduces to the problem of determining the (real) zeros of the real function $Z(t)$ (by counting the number of sign changes). However, the behaviour of $Z(t)$ tells us more. It has been shown (see EDWARDS [3; p.176]) that: "If there were a point at which the graph of $Z(t)$ came near to $Z = 0$ but did not actually cross it (that is, if Z had a small positive local minimum or a small negative local maximum) then the Riemann hypothesis would be contradicted".

Needless to say that this phenomenon has never been observed. However, we know of "critical" situations. These critical situations were discovered by LEHMER [7] and to get some idea of what this is all about, we take a look at the following table.

<u>Table II</u>

t	$Z(t) \cong$	t	$Z(t) \cong$
17,142.00	3.537947	17,144.30	-1.065744
17,142.10	4.999834	17,144.40	-1.191793
17,142.20	5.992144	17,144.50	-1.149738
17,142.30	6.448149	17,144.60	-0.919866
17,142.40	6.366785	17,144.70	-0.517136
17,142.50	5.809052	17,144.80	0.009444
17,142.60	4.886151	17,144.90	0.583447
17,142.70	3.741426	17,145.00	1.112700
17,142.80	2.529189	17,145.10	1.504075
17,142.90	1.393062	17,145.20	1.678685
17,143.00	0.447674	17,145.30	1.585713
17,143.10	-0.234572	17,145.40	1.211782
17,143.20	-0.629179	17,145.50	0.584976
17,143.30	-0.758070	17,145.60	- 0.227934
17,143.40	-0.680939	17,145.70	- 1.130061
17,143.50	-0.461433	17,145.80	- 2.007975
17,143.60	-0.250518	17,145.90	- 2.748040
17,143.70	-0.069857	17 146.00	- 3.253245
17,143.80	0.002045	17,146.10	- 3.457327
17,143.90	-0.061060	17,146.20	- 3.334863
17,144.00	-0.250129	17,146.30	- 2.904948
17,144.10	-0.524499	17,146.40	- 2.228399
17,144.20	-0.820683	17,146.50	- 1.398749

We see that Z(t) has a "barely" positive local maximum at t $\cong$ 17143.8. Usually one observes that not too far from such a point, Z(t) shows a strong oscillatory behaviour. This is called a Lehmer phenomenon. There is no precise definition of this notion and I do not feel the need to make up one. (For similar striking phenomena I refer to the pictures in [8].) At this moment I consider it as more relevant to ask whether one can "force Z(t) to produce a Lehmer phenomenon". To some extend I have succeeded in doing so, although I do not claim having been exhaustive.

My main goal was to predict t-intervals on which Z(t) will (very probably) behave quite unusual. More specifically I wanted to predict t-intervals on which Z(t) is such that either

(i) $|Z(t)|$ is "very" large at some points, or

(ii) $|Z(t)|$ is "very" small throughout the interval, or

(iii) $|Z(t)|$ oscillates violently.

In order to get some idea of what I call "unusual behaviour" I first present an example of what I consider to be the "usual behaviour" of Z(t).

Table III

t	Z(t) $\cong$	t	Z(t) $\cong$
223334443.50	-1.026677	223334444.75	-6.535380
223334443.55	-.770170	223334444.80	-7.042659
223334443.60	-.446187	223334444.85	-6.909384
223334443.65	-.132839	223334444.90	-6.083566
223334443.70	.091280	223334444.95	-4.609723
223334443.75	.164632	223334445.00	-2.626647
223334443.80	.056951	223334445.05	-.350287
223334443.85	-.222398	223334445.10	1.956180
223334443.90	-.624431	223334445.15	4.019987
223334443.95	-1.068926	223334445.20	5.599368
223334444.00	-1.460356	223334445.25	6.518619
223334444.05	-1.707836	223334445.30	6.693356
223334444.10	-1.745897	223334445.35	6.141471
223334444.15	-1.551259	223334445.40	4.978012
223334444.20	-1.152170	223334445.45	3.395086
223334444.25	-.627862	223334445.50	1.630160
223334444.30	-.097583	223334445.55	-.071095
223334444.35	.299603	223334445.60	-1.492752
223334444.40	.430687	223334445.65	-2.479189
223334444.45	.195760	223334445.70	-2.954746
223334444.50	-.448470	223334445.75	-2.929415
223334444.55	-1.474187	223334445.80	-2.489649
223334444.60	-2.779375	223334445.85	-1.777524
223334444.65	-4.198187	223334445.90	-.961360
223334444.70	-5.522806	223334445.95	-.204253

We see that on this interval Z(t) has a very clear cut behaviour. It does not raise any suspicion about the truth of the Riemann hypothesis.

In contrast to the previous table we give the following example.

Table IV

t	$Z(t) \cong$
18136022013.30	1.291523
18136022013.35	1.360911
18136022013.40	1.099648
18136022013.45	.684036
18136022013.50	.287581
18136022013.55	.044555
18136022013.60	.003230
18136022013.65	.123447
18136022013.70	.283312
18136022013.75	.361195
18136022013.80	.238762
18136022013.85	-.106897
18136022013.90	-.630690
18136022013.95	-1.193526
18136022014.00	-1.605974
18136022014.05	-1.716188

From this table it is not clear what happens at $t \cong 18136022013.6$. Actually, I had to invoke a double precision program for Z(t) in order to decide whether I had a "counterexample" or not.

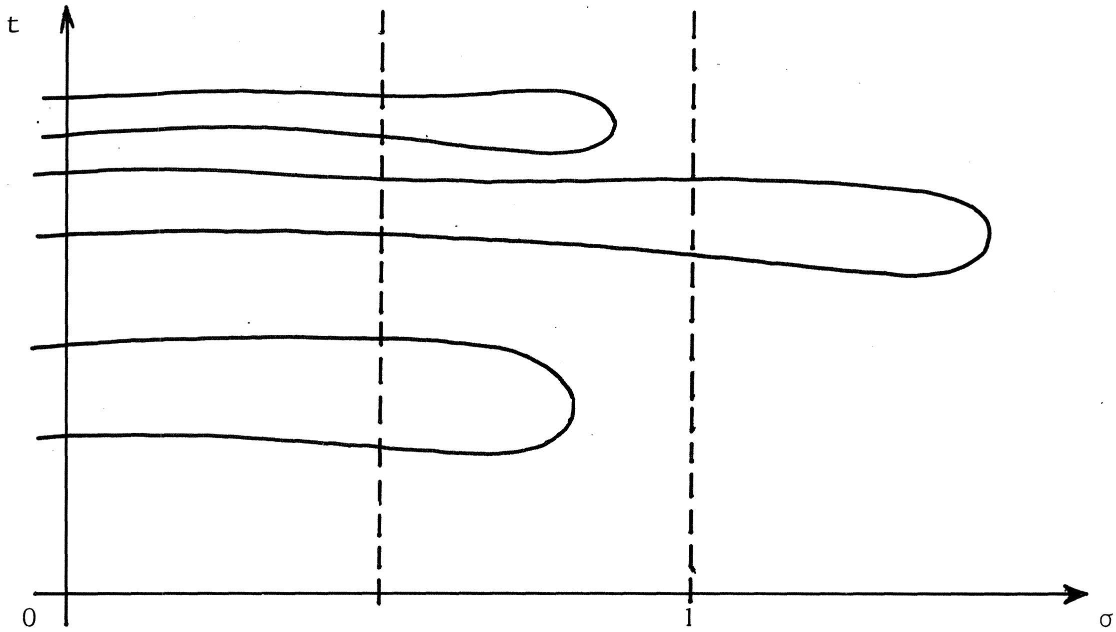
Here is the result of the double precision evaluation:

Table V

t	$Z(t) \cong$
18136022013.55	.0452
18136022013.56	.0211
18136022013.57	.0053
18136022013.58	-.0023
18136022013.59	-.0023
18136022013.60	.0049

3.2. How to force a Lehmer effect to occur?

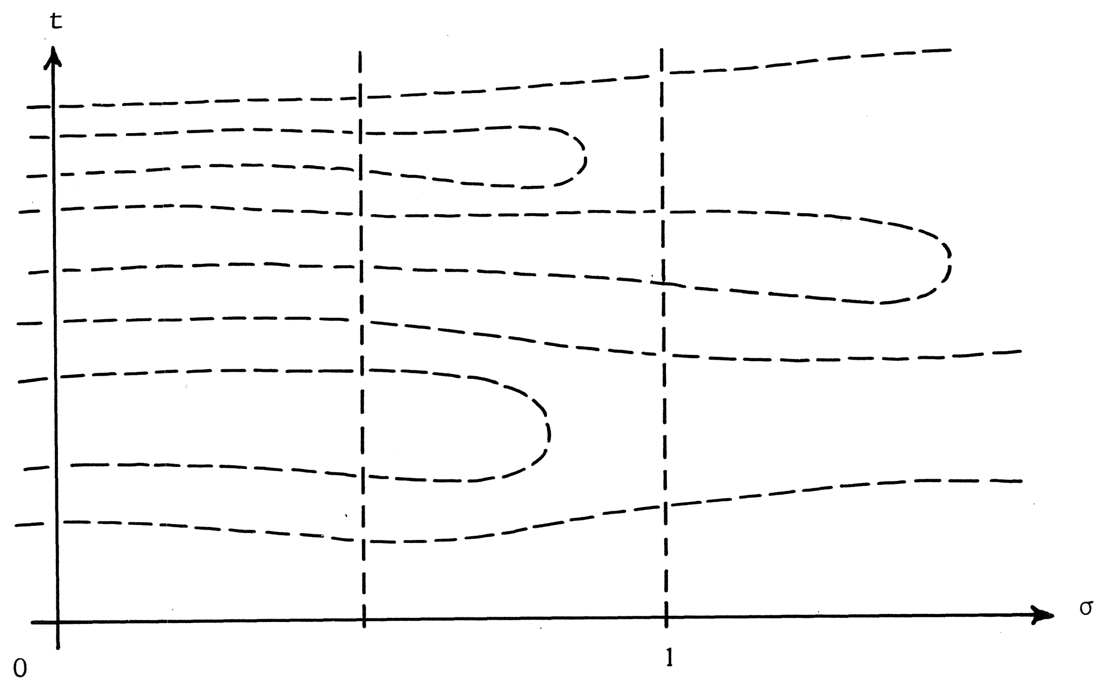
Numerical experiments indicate that the zero lines of Re $\zeta(s)$ traverse the complex plane as suggested by the following picture.



Zero-curves of Re $\zeta(s)$

More precisely, I only observed disjoint simple "loop lines" (as depicted above).

As to the zero-curves of Im $\zeta(s)$ the situation is quite different. I observed two kinds of zero lines of Im $\zeta(s)$ as suggested by the following picture.



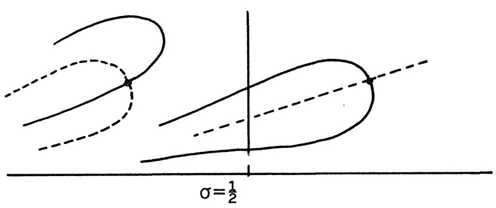
Zero-curves of Im $\zeta(s)$

One kind is crossing the halfplane $\sigma > 0$ more or less horizontally whereas the other kind has the form of a loop (similarly as the zero-curves of Re $\zeta(s)$). It can be proved that *these loops* do not stick out arbitrarily far to the right. I shall call such a *loop* an I_2-line and a zero-line of the other type an I_1-line. Zero lines of Re $\zeta(s)$ will be called R-lines. I find it somewhat remarkable that the zeros of $\zeta(s)$ are produced by I_1-lines as well as by I_2-lines.

For example: the first zero of $\zeta(s)$, to wit $s = \frac{1}{2} + i * 14.13...$ is a zero produced by an I_1-line. Such a zero will be called a Z_1. Similarly $s = \frac{1}{2} + i * 25.01...$ is a Z_1. On the other hand $s = \frac{1}{2} + i * 21.02...$ is a Z_2 (with the obvious meaning), similarly as $s = \frac{1}{2} + i * 30.42...$. Are there infinitely many zeros of both kinds and are the Z_2's eventually in the majority? I think so. I do not know what kind of *combinatorial interplay* there is between the zero-curves of Im $\zeta(s)$ and Re $\zeta(s)$.

From the above discussion I draw the *conjectural* conclusion that all non-trivial zeros of $\zeta(s)$ are *simple* (irrespective of the truth of the Riemann hypothesis).

If the Riemann hypothesis would be false I expect to have a situation as depicted below.



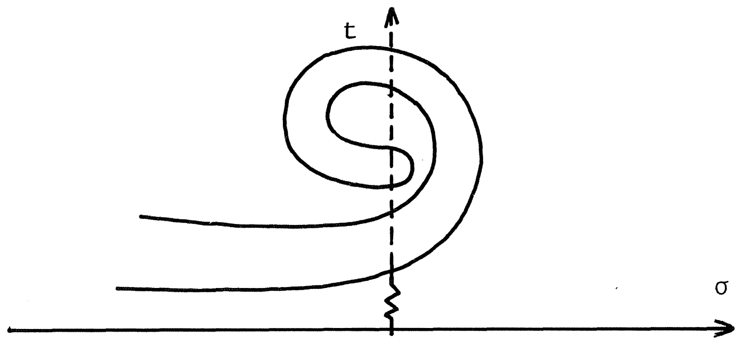
What I suggest here is that in case the Riemann hypothesis would be false I expect that there must be an R-line sticking out quite far to the right (this was my motivation for Section 1) "pushing" the neighbouring R-lines considerably to the left.

Question. Has it ever been shown that all R-lines and all I_2-lines (the loops thus) intersect the vertical $\sigma = \frac{1}{2}$?

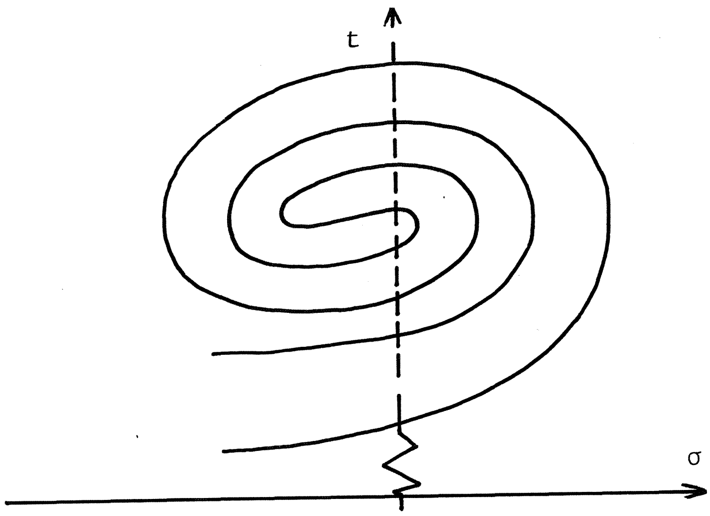
I expect that this is a consequence of the Riemann hypothesis but I do not see why it should imply the RH. Hence, it seems a less drastical assertion than the Riemann hypothesis.

My *conjectural* conclusion is that when searching for a counterexample (to the Riemann hypothesis) one might systematically search for $t > 0$ such that Re $\zeta(1+it)$ is negative, *preferably as strongly negative as possible*. A procedure for finding such t has been described in Section 2 and for such t I almost always found a "nice" Lehmer phenomenon.

The techniques of Section 1 indicate that there are R-lines of the following form



Indeed, taking $\sigma(>1)$ very close to 1 such that $\Sigma_p\ p^{-\sigma}$ assumes a very large value, I expect (for some $t > 0$) to have an R-line pattern as depicted above. By taking σ closer and closer to 1 we may even encounter an R-line pattern as depicted below



(the number of "semi windings" being finite, though as large as we please) and I consider it as possible that these R-lines might even facilitate the production of a counterexample to the Riemann hypothesis. It is to be expected that this type of R-lines will occur only when a hughe number of primes will "cooperate" in the sense implicitly described in Section 1. Needless to say, the corresponding t must be extremely large and we may, most probably, never be able to perform actual numerical computations for such large values of t.

After reading Section 1 it seems plausible that, when t_0 is such that $t_0 \log p \cong 0 \pmod{2\pi}$ for a considerable number of small primes, we may expect $\operatorname{Re} \zeta(1+it)$ to be large. Exploiting this suggestion numerically, and switching my attention to $\sigma = \frac{1}{2}$, I found

$$Z(t) < -453.9$$

for

$$t \cong 725,\ 177,\ 880,\ 629,\ 981.\ 914,\ 597$$

which is (to my knowledge) the largest value of $|Z(t)|$ ever observed till now (1983). In BRENT [1] the largest observed value of $Z(t)$ is $\cong 79.6$.

We list some more t's for which $|Z(t)|$ is large.

t	$Z(t) \cong$
18 132 299 244.660	-133.150
18 139 553 794.750	142.190
907 663 606 940.329 231	229.264
9 065 450 718 497.579	-253.501
45 323 986 866 893.743 300	-320.745
67 259 991 040 786.806 077	261.793
67 260 306 646 745.009 658	310.000
67 261 269 472 435.211 161	369.052
67 263 231 798 214.25	441.371
129 961 440 006 586.035 352	276.125
129 961 726 617 912.386 350	177.448
129 961 745 195 304.235 916	222.184

Choosing t_0 such that $t_0 \log p \cong \pi \pmod{2\pi}$ for a considerable number of small primes, one may expect Re $\zeta(1+it)$ to be close to zero and indeed this prediction has never failed. However, on intervals around such t_0 I never observed any hartbeating Lehmer phenomenon on $\sigma = \frac{1}{2}$, although $Z(t)$ is very flat on quite a long stretch (containing unusually many zeros).

In contrast to this I almost always found a nice Lehmer phenomenon close to points t_0 for which $|\mathrm{Re}\ \zeta(1+it_0)|$ is large.

Final remark. Quite frequently one will observe that Gram's Law (cf. EDWARDS [3; p. 127]) is violated in the neighbourhood of t-values for which $\zeta(1+it)$ behaves "unusually" in the sense described above. (For similar work on this topic we refer to KARKOSCHKA & WERNER [6].) We illustrate this observation by presenting the following tables of $Z(t)$ where t runs through a number of successive Gram points. In particular, note the *exceptions to Gram's "law"* in the vicinity of the large values of $|Z(t)|$. Quite frequently also Rosser's "rule" is violated here (cf. EDWARDS [3; Section 8.4]).

t	$Z(t) \cong$	Index of Gram point
18132299243.230435769999	.1523996748	59976759038
18132299243.518879106145	.4158883758	59976759039*
18132299243.807322442292	-.9886107034	59976759040*
18132299244.095765778438	5.9268844574	59976759041*
18132299244.384209114584	-47.8856001524	59976759042*
18132299244.672652450730	-132.7920625756	59976759043
18132299244.961095786877	-32.7853165793	59976759044*
18132299245.249539123023	8.4003406927	59976759045*
18132299245.537982459169	-.0368867287	59976759046*
18132299245.826425795315	.8297308165	59976759047*
18132299246.114869131461	1.0507797710	59976759048
18132299246.403312467608	-.1628780495	59976759049
18132299246.691755803754	1.2200077739	59976759050
18132299246.980199139900	-3.1675734967	59976759051
18132299247.268642476046	-.5140561072	59976759052*

In this table as well as in the next one a * indicates a bad Gram point.

t	Z(t) $\simeq$	Index of Gram point
18139553793.331320656541	-.0351333025	60001909963
18139553793.619758696005	.2219045841	60001909964
18139553793.908196735469	.4486144477	60001909965*
18139553794.196634774933	-7.8571110411	60001909966*
18139553794.485072814397	47.1504890495	60001909967*
18139553794.773510853861	142.9341035343	60001909968
18139553795.061948893325	45.6732738639	60001909969*
18139553795.350386932789	-4.9342609641	60001909970*
18139553795.638824972253	1.9837520958	60001909971*
18139553795.927263011717	.0974594703	60001909972
18139553796.215701051181	-.1460865584	60001909973
18139553796.504139090645	4.2181689790	60001909974
18139553796.792577130109	.3015076295	60001909975*
18139553797.081015169573	-.1066492127	60001909976*
18139553797.369453209037	-1.3583023258	60001909977
18139553797.657891248501	.0084506538	60001909978

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APPENDIX. THE PROGRAM REFERRED TO ON PAGE 6.

```
       PROGRAM SIGMAOB (OUTPUT,TAPE1=OUTPUT)
       INTEGER PRIME(3402),LAST(3402),BLOCK(31627)
       LENGTH=3402 $ SIGMA=1.19075048976742
       PISL2=2.*ATAN(1.0) $ KNTPR=0 $ BLOCK(1)=0
       DO 10 I10=2,LENGTH
       BLOCK(I10)=1
 10    CONTINUE
       DO 30 I30=2,LENGTH
       IF(BLOCK(I30).EQ.0)GOTO 30
       KNTPR=KNTPR+1 $ PRIME(KNTPR)=LAST(KNTPR)=INDEX=I30
 20    INDEX=INDEX+I30
       IF(INDEX.GT.LENGTH)GOTO 30
       BLOCK(INDEX)=0 $ LAST(KNTPR)=INDEX
       GOTO 20
 30    CONTINUE
       INCR=0
 40    INCR=INCR+LENGTH
       DO 50 I50=1,LENGTH
       BLOCK(I50)=1
 50    CONTINUE
       DO 70 I70=1,KNTPR
       INDEX=LAST(I70)+PRIME(I70)-INCR
       IF(INDEX.GT.LENGTH)GOTO 70
 60    BLOCK(INDEX)=0
       LAST(I70)=INDEX+INCR $ INDEX=INDEX+PRIME(I70)
       IF(INDEX.LE.LENGTH)GOTO 60
 70    CONTINUE
       DO 80 I80=1,LENGTH
       IF(BLOCK(I80).EQ.0)GOTO 80
       KNTPR=KNTPR+1 $ PRIME(KNTPR)=LAST(KNTPR)=I80+INCR
       IF(KNTPR.GE.LENGTH)GOTO 90
 80    CONTINUE
       GOTO 40
C.....PRIMES READY
 90    MAXNPR=PRIME(LENGTH)
       SUM=PISL2 $ DERIV=0.
       DO 100 I100=1,LENGTH
       RLNI100=ALOG(FLOAT(PRIME(I100)))
       POWER=EXP(-SIGMA*RLNI100) $ SUM=SUM-ASIN(POWER)
       DERIV=DERIV+RLNI100*POWER/SQRT(1.-POWER*POWER)
 100   CONTINUE
       DO 110 I110=1,LENGTH
       LAST(I110)=(MAXNPR/PRIME(I110))*PRIME(I110)
 110   CONTINUE
       WRITE(1,120)KNTPR,PRIME(KNTPR),PRIME(KNTPR)*PRIME(KNTPR)
 120   FORMAT(* #PR=*,I6,*   LARGEST PRIME=*,I7,*   ITS SQUARE=*,I14)
       INCR=0
       DO 170 I170=2,MAXNPR
       DO 130 I130=1,MAXNPR
       BLOCK(I130)=1
 130   CONTINUE
       INCR=INCR+MAXNPR
       DO 150 I150=1,LENGTH
       INDEX=LAST(I150)+PRIME(I150)-INCR
 140   BLOCK(INDEX)=0
       LAST(I150)=INDEX+INCR $ INDEX=INDEX+PRIME(I150)
       IF(INDEX.LE.MAXNPR) GOTO 140
 150   CONTINUE
       DO 160 I160=1,MAXNPR
       IF(BLOCK(I160).EQ.0) GOTO 160
       IPRIME=INCR+I160 $ RLNI=ALOG(FLOAT(IPRIME))
       POWER=EXP(-SIGMA*RLNI) $ SUM=SUM-ASIN(POWER)
       DERIV=DERIV+RLNI*POWER/SQRT(1.-POWER*POWER) $ KNTPR=KNTPR+1
 160   CONTINUE
 170   CONTINUE
       SIGMA=SIGMA-SUM/DERIV
       WRITE(1,180)KNTPR,IPRIME,SIGMA,SUM,DERIV
 180   FORMAT(* TOTAL # PR=*,I8,* LASTPR=*,I10,* NEW SIGMA=*,F20.14,/,
      $* SUM=*,F20.14,* DERIVATIVE=*,F20.14)
       END
```