Centrum voor Wiskunde en Informatica
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Department of Applied Mathematics Report AM-R8405 March

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LINEAR AUTONOMOUS RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS; A SHARP VERSION OF HENRY's THEOREM

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Small solutions play a crucial role in the theory of completeness of the generalized eigenfunctions of the infinitesimal generator of the C_0-semigroup $\{T(t)\}$ associated with a linear autonomous retarded functional differential equation. In this paper we shall prove a sharp version of Henry's Theorem on small solutions and, as a corollary, that the ascent α of $\{T(t)\}$ is equal to the ascent δ of the adjoint C_0-semigroup $\{T(t)^*\}$.

1980 MATHEMATICS SUBJECT CLASSIFICATION: 34K05, 45E10.

KEY WORDS & PHRASES: completeness, functional differential equation, small solution, Volterra convolution integral equation.

Report AM-R8405

Centre for Mathematics and Computer Science

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INTRODUCTION

Classically a linear autonomous retarded functional differential equation (rfde) is an equation of the following form

$$\frac{dx}{dt}(t) = Lx_t, \qquad t \geq 0,$$
(0.1)
$$x_0 = \phi \qquad ,$$

where $\phi \in C[-h,0]$ and L is a continuous functional on $C[-h,0]$. The condition $x_0 = \phi$ is called the initial condition of (0.1) and the space $C[-h,0]$ is called the state space of (0.1). The equation (0.1) has a unique solution $x(t;\phi)$ and the semigroup defined by translation along the solution is a C_0-semigroup. A good reference for the general study of this equation (0.1) is the book of HALE [11].

The approach of the equation (0.1) has some disadvantages; for example the state space $C[-h,0]$ is nonreflexive - which makes the theory of adjoint operators more complicated - and also is too small for several applications, for example in control theory cf. [18]. So in recent years one has tried to extend the state space $C[-h,0]$ of (0.1) to a larger reflexive space without losing the C_0-semigroup property. First BORSOVIC and TURBABIN [5] obtained that under three extra conditions on the functional L one can extend the state space of (0.1) to $M_2 = \mathbb{C}^n \times L^2[-h,0]$. Later VINTER proved in [20] that the three extra conditions on L are in fact redundant. From that time on the equation (0.1) with state space M_2 has been studied intensively; notably by DELFOUR and MANITIUS [8],[16].

In the generalization to the larger reflexive state space M_2, the definition of L was at first unchanged. It was Vinter who claimed that the minimal properties on L, such that the equation (0.1) still has the C_0-semigroup property, is just that L is a continuous functional on $H^1[-h.0]$. Vinter's condition on L is necessary and Delfour proved in [7] that it is sufficient as well.

Consider

$$\frac{dx}{dt}(t) = Lx_t, \qquad t \geq 0,$$
(0.2)
$$x_0 = \phi,$$

where $\phi \in M_2$ and L is a continuous functional on $H^1[-h,0]$. In this paper we shall, following the ideas of DIEKMANN [9],[10], associate with the equation (0.2) a Volterra convolution integral equation. It is indeed this Volterra convolution integral equation which makes it possible to extend the state space of (0.1) to the larger reflexive space M_2.

In the sections 1, 2 and 3 we shall study the equation (0.2) thoroughly starting from the Volterra convolution integral equation point of view with minimal assumptions on the kernel ζ. In our view this is a natural and easy approach, we get rid of the unpleasant bilinear forms in [8], [16] and obtain a more natural interpretation of the structural operators F and G, which will be defined in a way which differs slightly from the one of Delfour and Manitius. In section 4 we shall study the small solutions of the equation (0.2). Our main result will be a sharp version of Henry's theorem which has important applications in the theory of completeness of the generalized eigenfunctions of the infinitesimal generator.

Our work extends and simplifies many results of DELFOUR and MANITIUS [7],[8],[16]. We emphasize, however, that we derived much inspiration from reading their papers.

Notation and Terminology

Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively; the complex conjugate of an element $z \in \mathbb{C}$ will be denoted by $\bar{z}$. Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. The space of Lebesgue measurable maps $[a,b] \to \mathbb{C}^n$ which are square integrable will be denoted by $L^2[a,b]$.

Given $h \in \mathbb{R}_+$, let M_2 denote the product space $\mathbb{C}^n \times L^2[-h,0]$ endowed with the inner product

$$(\phi,\psi) = \phi^0 \, \overline{\psi^0} + \int_0^h \phi^1(-t)\overline{\psi^1(-1)}\,dt,$$

where each element ϕ of M_2 is identified with a pair (ϕ^0,ϕ^1), $\phi^0 \in \mathbb{C}^n$, $\phi^1 \in L^2[-h,0]$. We shall associate with an element ϕ of M_2 a function ϕ defined on $[-h,0]$ such that

$$\begin{aligned} \phi(0) &= \phi^0, \\ \phi(t) &= \phi^1(t), \qquad -h \le t < 0. \end{aligned}$$

Let $H^1[-h,0]$ denote the Sobolev space of functions $f:[-h,0] \to \mathbb{C}^n$ with $\frac{df}{dt} \in L^2[-h,0]$ provided with the norm

$$\|f\|_{H^1[-h,0]} = (|f(-h)|^2 + \int_0^h |\frac{df}{dt}(-t)|^2 dt)^{\frac{1}{2}}.$$

Since this norm is equivalent to the usual Sobolev norm on $H^1[-h,0]$, we have by the Sobolev embedding theorem that the embedding $j:H^1[-h,0] \to M_2$ is continuous.

Analogously, let F_2 denote the product space $\mathbb{C}^n \times L^2[0,h]$ endowed with the innerproduct

$$(f,g) = f^0\,\overline{g^0} + \int_0^h f^1(t)\overline{g^1(t)}dt.$$

We shall associate with an element f of F_2 a function f defined on $\mathbb{R}_+$ such that

$$f(t) = f^1(t), \qquad 0 \le t < h,$$
$$f(t) = f^0, \qquad t \ge h.$$

Let H_1 denote the Sobolev space of functions $f: \mathbb{R}_+ \to \mathbb{C}^n$ with essential support of $\frac{df}{dt}$ contained in $[0,h]$ and $\frac{df}{dt} \in L^2[0,h]$ provided with the norm

$$\|f\|_{H^1} = (|f(0)|^2 + \int_0^h |\frac{df}{dt}(t)|^2 dt)^{\frac{1}{2}}.$$

And let $i : H_1 \to F_2$ denote the continuous embedding of H_1 into F_2.

Since $(\phi^0,\phi^1) \to \phi^0 + \int_0^t \phi^1(-s)ds$ is an isometric isomorphism between M_2 and H_1 we take H_1 as the dual space of M_2 with the natural pairing

$$<\phi,f> = \phi^0\,\overline{f(0)} + \int_0^h \phi^1(-t)\,\overline{\frac{df}{dt}}(t)dt, \qquad \phi \in M_2,\ f \in H_1.$$

Analogously, we take $H^1[-h,0]$ as the dual space of F_2 with the natural pairing

$$<<f,\psi>> = f^0\,\overline{\psi(-h)} + \int_0^h f^1(t)\,\overline{\frac{d\psi}{dt}}(-t)dt, \qquad f \in F_2,\ \psi \in H^1[-h,0].$$

Let $M(\mathbb{C})$ denote the space of $n\times n$-matrices with elements in $\mathbb{C}$. The space of Lebesgue measurable matrix valued functions $A:[a,b] \to M(\mathbb{C})$ which are square integrable, i.e. with $L^2[a,b]$ elements, will be denoted by $ML^2[a,b]$ provided with the norm the sum of the L^2-norms of the elements A_{ij} of the matrix function A. This norm makes $ML^2[a,b]$ into a Banach space.

Let V_2 denote the product space $M(\mathbb{C}) \times ML^2[0,h]$, we shall associate with an element $\zeta = (\zeta^0,\zeta^1) \in V_2$ a matrix valued function ζ such that

$$
\begin{aligned}
\zeta(t) &= 0, && t \le 0,\\
\zeta(t) &= \zeta^1(t), && 0 \le t < h,\\
\zeta(t) &= \zeta^0, && t \ge h.
\end{aligned}
$$

Let $|\cdot|$ denote the Euclidian norm on $\mathbb{R}^n$, $\mathbb{C}^n$ or $M(\mathbb{C})$, in the last case this norm is defined by the sum of the Euclidian norms of the elements of the matrix.

If x is a function defined on $[-h,\infty)$ then x_t will denote the translate of x over t considered as a function defined on $[-h,0]$ i.e. $x_t(s) = x(t+s)$, $-h \le s \le 0$. By abuse of notation we shall write $x_t = (x(t),x_t)$ if x_t is considered as an element of M_2.

Finally, we shall choose $h \in \mathbb{R}_+$ in such a way that at least one of the elements of ζ does not vanish almost everywhere in any neighbourhood of h.

1. THE VOLTERRA CONVOLUTION INTEGRAL EQUATION

We shall start to study a general class of Volterra convolution integral equations.

DEFINITION 1.1. A *Volterra convolution integral equation* is an equation of the following form

$$x - \zeta * x = f$$

where the kernel ζ is an element of V_2 and the forcing function f is an element of F_2.

The solution $x = x(\cdot;f)$ of (1.1) is defined on $\mathbb{R}_+$ with values in $\mathbb{C}^n$.

PROPOSITION 1.2. *The Volterra convolution integral equation* (1.1) *has a unique solution*

$$x = x(\cdot;f) = f - R * f$$

where R *is defined to be the solution of the equation*

$$R = R * \zeta - \zeta.$$

PROOF. Since ζ is L^1 on $[0,h]$ and constant on $[h,\infty)$ we can choose a $\mu \in \mathbb{R}$ such that

$$\int_0^\infty |\zeta(t)|e^{-\mu t}dt < 1.$$

Define $\tilde{R}(t) = R(t)e^{-\mu t}$ and $\tilde{\zeta}(t) = \zeta(t)e^{-\mu t}$ then the equation

$$R = R * \zeta - \zeta$$

transforms to the equation

$$\tilde{R} = \tilde{R} * \tilde{\zeta} - \tilde{\zeta}.$$

Consider the linear operator $T\colon ML^2(\mathbb{R}_+) \to ML^2(\mathbb{R}_+)$ defined by

$$TR \to R * \tilde{\zeta} - \tilde{\zeta}.$$

By Theorem (21.32) of HEWITT and STROMBERG [13] and the definition of the norm on $ML^2(\mathbb{R}_+)$, we obtain

$$\| TR_1 - TR_2 \|_2 = \| (R_1-R_2)*\tilde{\zeta} \|_2 \le \| R_1-R_2 \|_2 \| \tilde{\zeta} \|_1 < \| R_1-R_2 \|_2.$$

Hence, T is a contraction and the Banach Fixed Point Theorem implies that the equation $R = R * \zeta - \zeta$ has a unique solution R. Since $x = \zeta * x + f$ and $R = R * \zeta - \zeta$ we find $R * x = R * \zeta * x + R * f$ and $R * x = R * \zeta * x - \zeta * x$. Hence, $R * f = - \zeta * x$ and so the solution x is unique and the representation

$x = f - R * f$ holds. □

COROLLARY 1.3. *The unique solution* $x = x(\cdot;f)$ *of* (1.1) *is locally* L^2. *Moreover,* $x(t)$ *is continuous for* $t \geq h$ *and the inequality*

$$\|x\|_{L^2[0,T]} \leq C(T)\|f\|_{F^2}$$

holds.

PROOF. We use Theorem (21.33) of Hewitt and Stromberg: If $f,g \in L^2$ then $f*g$ is continuous vanish at infinity and the supnorm of $f*g$ satisfies

$$\|f*g\|_u \leq \|f\|_2 \|g\|_2.$$

Consequently, x is locally L^2 and since $f(t)$ is constant for $t \geq h$ $x(t)$ is continuous for $t \geq h$.
And

$$\|x\|^2_{L^2[0,T]} = \|f-R*f\|^2_{L^2[0,T]}$$

$$\leq \|f\|^2_{L^2[0,T]} + \|R*f\|^2_{L^2[0,T]}$$

$$\leq \|f\|^2_{L^2[0,T]} + T\|R\|^2_{ML^2[0,T]} \|f\|_{L^2[0,T]}$$

$$\leq C_1(T)\|f\|^2_{F_2} + C_2(T)\|R\|^2_{ML^2[0,T]} \|f\|^2_{F_2}$$

$$\leq C(T)\|f\|^2_{F_2}. \quad \square$$

We shall now associate with the equation (1.1) a C_0-semigroup $\{S(t)\}$.

DEFINITION 1.4. Let $(X,\|\cdot\|)$ be a Banach space and suppose that to every $s \in \mathbb{R}_+$ is associated a bounded operator $T(s): X \to X$, in such a way that

(i) $T(0) = I$;

(ii) $T(s_1+s_2) = T(s_1)T(s_2)$ for all $s_1,s_2 \in \mathbb{R}_+$;

(iii) $\lim_{s\downarrow 0} \|T(s)\phi-\phi\| = 0$ for every $\phi \in X$.

Then $\{T(s)\}$ is called a C_0-*semigroup*.

To every c_0-semigroup $\{T(s)\}$ we can associate an *infinitesimal generator* A defined by

$$A\phi = \lim_{s\downarrow 0} \frac{1}{s} [T(s)\phi-\phi]$$

for all $\phi \in D(A)$, that is, for all $\phi \in X$ for which the limit exists in the norm topology of X. The following theorem can be found in RUDIN [17].

THEOREM 1.5. *Let* $\{T(s)\}$ *be a* c_0*-semigroup then*

(a) $s \to T(s)x$ *is a continuous mapping from* $\mathbb{R}_+$ *into* X, *for every* $x \in X$;

(b) A *is a closed densely defined operator on* X;

(c) *For every* $x \in D(A)$, $T(s)x$ *satisfies the differential equation*

$$\frac{d}{ds} T(s)x = AT(s)x = T(s)Ax.$$

We shall associate with (1.1) a c_0-semigroup $\{S(t)\}$ acting on F_2 such that

$$x(t+\cdot) = \zeta * x_t + S(t)f$$

Since

$$x(t+s) = \int_0^{t+s} \zeta(\theta)x_t(s-\theta)d\theta+f(t+s)$$

we obtain

$$(1.6) \qquad (S(t)f)(s) = (f^0+\zeta^0 \int_0^t x(\tau)d\tau,\ f^1(t+s)+\int_0^t \zeta^1(t+s-\tau)x(\tau)d\tau).$$

This motivates the following definition.

DEFINITION 1.7. For every $t \in \mathbb{R}_+$ define the linear operator $S(t): F_2 \to F_2$ by

$$(S(t)f)(s) = (f^0+\zeta^0 \int_0^t (f^1(\tau)-R*f^1(\tau))d\tau, f^1(t+s)+(\zeta_s^1-\zeta_s^1*R)*f^1(t)).$$

THEOREM 1.8. *The family of operators* $\{S(t)\}$ *is a* c_0*-semigroup.*

PROOF. That $\{S(t)\}$ is a family of bounded operators satisfying the semigroup properties (1.4)(i),(ii) is clear from the representation (1.6) and Corollary (1.3). So, only the C_0-property (1.4(iii) remains to be proved.

$$\|S(t)f-f\|^2_{F_2} = |\zeta^0 \int_0^t (f^1(\tau)-R*f^1(\tau))d\tau|^2 + \int_0^h |f^1(t+s)-f^1(s) + (\zeta^1_s-\zeta^1_s*R)*f^1(t)|^2 ds.$$

Since $\zeta^0 \int_0^t (f^1(\tau)-R*f^1(\tau))d\tau$ is continuous we have

$$\lim_{t\downarrow 0} |\zeta^0 \int_0^t (f^1(\tau)-R*f^1(\tau))d\tau|^2 = 0.$$

Hence, it is enough to prove

$$\int_0^h |f(t+s)-f(s)|^2 ds \to 0 \quad \text{as} \quad t \downarrow 0$$

and

$$\int_0^h |(\zeta^1_s-\zeta^1_s*R)*f(t)|^2 ds \to 0 \quad \text{as} \quad t \downarrow 0.$$

But this is clear from Lebesgue's Dominated Convergence Theorem and the fact that translation is continuous in the L^2-normtopology. □

THEOREM 1.9. *The infinitesimal generator* B *of* $\{S(t)\}$ *is defined by*

$$Bg = (\zeta^0 g^1(0), \ \frac{dg^1}{dt} + \zeta^1(\cdot)\ g^1(0))$$

with

$$D(B) = \{g \in F_2 \mid \frac{dg^1}{dt} \in L^2[0,h]\}.$$

PROOF. By property (1.4)(iii) and Theorem (1.5)(c) we have for every $g \in D(B)$

$$Bg = (\zeta^0 g^1(0), \ \frac{dg^1}{dt} + \zeta^1(\cdot)g^1(0)).$$

Hence

$$D(B) \subset \{g \in F_2 \mid \frac{dg^1}{dt} + \zeta^1(\cdot)g^1(0) \in L^2[0,h]\}.$$

To prove the remaining inclusion

$$\{g \in F_2 \mid \frac{dg^1}{dt} \in L^2[0,h]\} \subset D(B)$$

let $g \in \{g \in F_2 \mid \frac{dg^1}{dt} \in L^2[0,h]\}$ and choose a $f \in F_2$, $\lambda \in \rho(B)$ such that

$$f^1 = \lambda g^1 - \frac{dg^1}{dt} - \zeta(\cdot)g^1(0).$$

Define $h = (\lambda I - B)^{-1}f$ then $h \in D(B)$ and we shall prove that $g^1 = h^1$ which implies that $g \in D(B)$.
Define $z^1 = g^1 - h^1$ then z^1 satisfies the differential equation

$$\frac{dz^1}{dt} - \lambda z^1 + \zeta^1(\cdot)z^1(0) = 0.$$

Hence

$$z^1(t) = e^{\lambda t}(I - \int_0^t e^{-\lambda s}\zeta(s)\,ds)z^1(0).$$

But for $t \geq h$ we must have

$$\frac{dz^1}{dt} = \frac{dg^1}{dt} - \frac{dh^1}{dt} = 0.$$

Since we can choose $\lambda \in \rho(B)$ with $\mathrm{Re}(\lambda)$ arbitrary large, we obtain $z^1(0) = 0$.
Hence, $z^1 \equiv 0$ and $g^1 \equiv h^1$. □

Let $\lambda \in \rho(B)$ then it follows from the Closed Graph Theorem that the operator $S_\lambda : H_1 \to F_2$ defined by

$$S_\lambda = (\lambda I - B)i$$

is a bounded invertible operator.

PROPOSITION 1.10. *The norm* $\|\cdot\|_{H_1}$ *on* H_1 *is equivalent to the graph norm*

$$\|f\|_B = \|if\|_{F_2} + \|Bif\|_{F_2}, \qquad f \in H_1$$

of B.

PROOF. For every $x \in H_1$ we have

$$\|x\|_B \leq [\|i\|_{L(H_1,F_2)} + \|Bi\|]\ \|x\|_{H_1}$$

and, conversely, for every $\lambda \in \rho(B)$ with $|\lambda| > 1$

$$\|x\|_{H_1} \leq \|[(\lambda I-B)i]^{-1}\|_{L(F_2,H_1)}\ \|(\lambda I-B)ix\|_{F_2}$$

$$\leq \|[(\lambda I-B)i]^{-1}\|_{L(F_2,H_1)} |\lambda|\ \|x\|_B. \quad \square$$

From the commutativity of $S(t)$ and B it follows that the restriction $\tilde{S}(t)$ of $S(t)$ to $D(B)$ is a C_0-semigroup with respect to the graph norm. So by Proposition (1.10) we have the following theorem.

THEOREM 1.11. *The family* $\{\tilde{S}(t)\}$ *defined on* H_1 *is a* C_0*-semigroup.*

THEOREM 1.12. *The infinitesimal generator* $\tilde{B}$ *of* $\{\tilde{S}(t)\}$ *is given by*

$$\tilde{B}g = \frac{dg}{dt} + \zeta^1(\cdot)g(0)$$

with

$$D(\tilde{B}) = \{g \in H_1 \mid \frac{dg}{dt} + \zeta^1(\cdot)g(0) \in H_1\}.$$

PROOF. By Proposition (1.10) the limit

$$h = \lim_{t\downarrow 0} \frac{1}{t} [\tilde{S}(t)f-f]$$

exists in H_1 if and only if

$$\lim_{t \downarrow 0} \left\| \frac{S(t)if-if}{t} - ih \right\|_{F_2} = 0$$

and

$$\lim_{t \downarrow 0} \left\| \frac{S(t)Bif-Bif}{t} - Bih \right\|_{F_2} = 0.$$

The first expression is equivalent to $Bif = ih$ and the second to $Bif \in D(B)$. Consequently, $f \in D(\tilde{B})$ and $\tilde{B}f = h$. □

COROLLARY 1.13. *The* c_0*-semigroups* $\{S(t)\}$ *and* $\{\tilde{S}(t)\}$ *are intertwined. i.e. there is a bounded invertible operator* Ω *such that* $\Omega^{-1}\tilde{S}(t)\Omega = S(t)$.

PROOF. Define, as above for some $\lambda \in \rho(B)$ $S_\lambda = (\lambda I-B)i\colon H_1 \to F_2$. Then for every $f \in H_1$

$$S(t)S_\lambda f = (\lambda I-B)S(t)if = (\lambda I-B)i\tilde{S}(t)f = S_\lambda\tilde{S}(t)f. \quad \square$$

2. THE LINEAR AUTONOMOUS RFDE

In this section we shall study the linear autonomous rfde (2.1) and discuss how this equation is related to the Volterra equation of section 1.

DEFINITION 2.1. A linear autonomous rfde is an equation of the following form

$$\frac{dx}{dt}(t) = Lx_t, \qquad t \geq 0,$$
$$x_0 = \phi,$$

where $\phi \in M_2$ and L is a continuous mapping from $H^1[-0,h]$ into $\mathbb{C}^n$.

Since F_2 is dualspace of $H^1[-h,0]$ there is a $\zeta \in V_2$ such that

$$L\phi = \zeta^0\, \phi(-h) + \int_0^h \zeta^1(s) \frac{d\phi}{dt}(-s)ds.$$

A priori the equation (2.1) is only defined for $\phi \in H^1[-h,0]$. We shall first show that in this case the equation (2.1) is equivalent to the

Volterra equation (1.1) with kernel ζ and forcingspace H_1.

Choose a $\phi \in H^1[-h,0]$ and define $x_0 = \phi$. If we differentiate (1.1) we obtain for $0 \le t \le h$

$$\frac{dx}{dt}(t) - \frac{df}{dt}(t) = \frac{d}{dt}\left(\int_0^h \zeta(s)x(t-s)\,ds - \int_t^h \zeta(s)\phi(t-s)\,ds\right),\ f \in H_1.$$

Hence

$$\frac{dx}{dt}(t) - \zeta^0.x(t-h) - \int_0^h \zeta(s)\frac{dx}{dt}(t-s)\,ds = -\zeta^0\phi(t-h) - \frac{d}{dt}\int_t^h \zeta(s)\phi(t-s)\,ds + \frac{df}{dt}(t).$$

Thus, if we choose the forcing function f of (1.1) such that

$$\frac{df}{dt}(t) = \zeta^0\phi(t-h) + \frac{d}{dt}\int_t^h \zeta(s)\phi(t-s)\,ds \text{ a.e for } 0 \le t \le h$$

and

$$\frac{df}{dt}(t) = 0 \quad \text{a.e} \quad \text{for } t \ge h$$

and

$$f(0) = \phi(0).$$

Then f is a uniquely determined element of H_1 and we obtain that for $t \ge 0$ the solution $x(t;\phi)$ of (2.1) is equal to the solution $x(t;f)$ of (1.1).

DEFINITION 2.2. Let the linear operator $\tilde{F}: H^1[-h,0] \to H_1$ defined by

$$\begin{aligned}(\tilde{F}\phi)(t) &= \phi(0) + \zeta^0\int_0^t \phi(s-h)\,ds + \int_t^h \zeta(s)\phi(t-s)\,ds - \int_0^h \zeta(s)\phi(-s)\,ds \\ &\qquad \text{for } t \le h, \\ &= (\tilde{F}\phi)(h) \qquad \text{for } t \ge h\end{aligned}$$

be the linear operator which maps the initial condition $\phi \in H^1[-h,0]$ of (2.1) to the corresponding forcing function $f = \tilde{F}\phi$ of (1.1).

THEOREM 2.3. *The linear operator* $\tilde{F}$ *is bounded.*

PROOF. It is enough to prove

$$\left\| \frac{d}{dt} \tilde{F}\phi \right\|^2_{L^2[0,h]} \leq c^2 \|\phi\|^2_{H^1[-h,0]}$$

$$\left\| \frac{d}{dt} \tilde{F}\phi \right\|^2_{L^2[0,h]} \leq |\zeta^0|^2 \|\phi\|^2_{L^2[-h,0]} + \|\zeta\|^2_{ML^1[0,h]} \cdot \left\| \frac{d\phi}{dt} \right\|^2_{L^2[-h,0]}$$

$$+\|\zeta\|^2_{ML^2[0,h]} |\phi(0)|^2 \leq c^2 \|\phi\|^2_{H^2[-h,0]} . \quad \square$$

REMARK 2.4. If ζ is of bounded variation then an adaption of Theorem (2.1) of DELFOUR and MANITIUS [8] shows that one can extend $\tilde{F}$ to a bounded linear mapping from M_2 into H_1. However, for arbitrary $\zeta \in V_2$ this is not true anymore, as is easily seen from the fact that expression for $\tilde{F}$ makes sense for arbitrary $\phi \in M_2$ but that, in general, the function thus defined is only continuous and not contained in H_1. This is one of the reasons why we are studying Volterra equations with forcing space F_2.

THEOREM 2.5. *The linear operator* $F\colon M_2 \to F_2$ *defined by*

$$(F\phi)(t) = (\phi^0 + \zeta^0 \int_0^h \phi^1(s-h)\,ds - \int_0^h \zeta(s)\phi(-s)\,ds, \phi^0 + \zeta^0 \int_0^t \phi^1(s-h)\,ds +$$

$$\int_t^h \zeta(s)\phi^1(t-s)\,ds - \int_0^h \zeta(s)\phi^1(-s)\,ds)$$

is well defined and bounded.

PROOF. It is clear that the operator F is well defined. To prove

$$\|F\phi\|^2_{F_2} \leq c^2 \|\phi\|^2_{M_2}$$

it is enough to prove

$$\|(F\phi)^1\|^2_{L^2[0,h]} \leq c^2 \|\phi^1\|^2_{L^2[-h,0]} .$$

And

$$\int_0^h |(F\phi)^1|^2 \le \int_0^h |\zeta^0 \int_0^t \phi^1(s-h)ds|^2 dt + \int_0^h |\int_t^h \zeta(s)\phi^1(t-s)ds|^2 dt +$$

$$\int_0^h |\int_0^h \zeta(s)\phi^1(-s)ds|^2 dt$$

$$\le C_1|\zeta^0|^2 \|\phi^1\|^2_{L^2[-h,0]} + C_2 \|\zeta\|^2_{ML^2[0,h]} \|\phi^1\|^2_{L^2[-h,0]} +$$

$$C_3 \|\zeta\|^2_{ML^2[0,h]} \|\phi^1\|^2_{L^2[-h,0]} \le C^2 \|\phi^1\|^2_{L^2[-h,0]}. \quad \square$$

COROLLARY 2.6. *The linear autonomous* rfde (2.1) *has a unique continuous solution* $x = x(\cdot;\phi) = x(\cdot;F\phi)$ *and* $\|x_t\|_{M_2} \le C(t)\|x_0\|_{M_2} = C(t)\|\phi\|_{M_2}$.

PROOF. We can write the solution $x(\cdot;\phi)$ as

$$x(\cdot;\phi) = f - R*f$$

with $f = F\phi$ and $F\phi$ is continuous. $\square$

REMARK 2.7. Theorem (2.5) shows that it is the existence of the operator F which makes it possible to extend the state space of (2.1) to M_2. Following Delfour and Manitius, we shall call the operators F and $\tilde{F}$ *structural operators*.

We shall now associate a c_0-semigroup to the linear autonomous rfde (2.1).

DEFINITION 2.8. For every $t \in \mathbb{R}_+$ define the following operator $T(t): M_2 \to M_2$ by

$$T(t)\phi = x_t(\cdot;\phi) = (x(t;\phi), x_t(\cdot;\phi)).$$

THEOREM 2.9. *The family of linear operators* $\{T(t)\}$ *is a* c_0*-semigroup.*

PROOF. That $\{T(t)\}$ is a family of bounded operators satisfying the semigroup properties (1.4)(i) and (ii) is clear from Corollary (2.6). The remaining C_0-semigroup property (1.4)(iii) is clear from the fact that translation is continuous in the L^2-normtopology. □

THEOREM 2.10. *The infinitesimal generator* A *of* $\{T(t)\}$ *is defined by*

$$A\phi = (L\phi^1, \frac{d\phi^1}{dt})$$

with

$$D(A) = \{\phi \in M_2 \mid \frac{d\phi^1}{dt} \in L^2[-h,0],\ \phi^0 = \phi^1(0)\}.$$

PROOF. Because of Theorem (1.5) and property (1.4)(iii) it follows that for every $\phi \in D(A)$

$$A\phi = (L\phi^1, \frac{d\phi^1}{dt}).$$

Hence

$$D(A) \subset \{\phi \in M_2 \mid \frac{d\phi^1}{dt} \in L^2[-h,0],\ \phi^0 = \phi^1(0)\}.$$

To prove the remaining inclusion

$$\{\phi \in M_2 \mid \frac{d\phi^1}{dt} \in L^2[-h,0],\ \phi^0 = \phi^1(0)\} \subset D(A)$$

Let ϕ be an element of $\{\phi \in M_2 \mid \frac{d\phi^1}{dt} \in L^2[-h,0],\ \phi^0 = \phi^1(0)\}$ then for all $T \in \mathbb{R}_+$ $x(\cdot;\phi) \in H^1[-h,T]$.
By Theorem (1.5)(a) we have

$$s \mapsto Lx_s \text{ is continuous.}$$

Hence

$$\frac{x(t)-x(0)}{t} = \frac{1}{t}\int_0^t \frac{dx}{dt}(s)\,ds = \frac{1}{t}\int_0^t Lx_s\,ds \to Lx_0 = L\phi^1 \text{ as } t \downarrow 0.$$

Since translation is continuous in the L^2-normtopology we obtain

$$\int_0^h \left| \frac{x(t-\theta)-x(-\theta)}{t} - \frac{dx}{dt}(-\theta)\right|^2 d\theta \to 0 \quad \text{as } t \downarrow 0.$$

Consequently, $\lim_{t\downarrow 0} \frac{1}{t}[T(t)\phi-\phi]$ converges in the M_2-normtopology to $(L\phi^1, \frac{d\phi^1}{dt})$ and $\phi \in D(A)$. □

REMARK 2.11. Theorem (2.9) and Theorem (2.10) show that the class (2.1) of rfde is the largest possible class of rfde with the C_0-semigroup property on the state space M_2.

In the same way as done above, the restriction of $\{T(t)\}$ to $D(A)$ induces a C_0-semigroup $\{\tilde{T}(t)\}$ on $H^1[-h,0]$ such that

$$j\tilde{T}(t) = T(t)j$$

where $j: H^1[-h,0] \to M_2$ is the embedding of $H^1[-h,0]$ into M_2. And we have the following theorems.

THEOREM 2.12. *The family of operators* $\{\tilde{T}(t)\}$ *on* $H^1[-h,0]$ *is a* C_0*-semigroup with infinitesimal generator*

$$\tilde{A}\phi = \frac{d\phi}{dt}$$

with

$$D(\tilde{A}) = \{\phi \in H^1[-h,0] \mid \frac{d\phi}{dt} \in H^1[-h,0]\}.$$

THEOREM 2.13. *The* C_0*-semigroups* $\{T(t)\}$ *and* $\{\tilde{T}(t)\}$ *are intertwined.*

We shall now define another structural operator $G: F_2 \to M_2$ - a generalization of the one first introduced by MANITIUS in [16] - which maps a forcing function f of (1.1) to an initial condition ϕ of (2.1).

DEFINITION 2.14. The linear operator $G: F_2 \to M_2$ is defined by

$$(Gf)^1 = f^1 - R*f^1(h),$$
$$(Gf)^1 = (f^1-R*f^1)(\cdot+h).$$

Note, that G translates the solution of (1.1) corresponding to f backwards over a distance h. The restriction of G to D(B) induces a mapping $\tilde{G}$ from H_1 into $H^1[-h,0]$ such that $i\tilde{G} = Gi$, where $i:H^1[-h,0] \to M_2$ is the embedding from $H^1[-h,0]$ into M_2.

PROPOSITION 2.15.

(i) G *is a bounded linear operator*;

(ii) $\tilde{G}$ *is a bounded linear operator*;

(iii) $N(G) = \{0\}$;

(iv) $R(G) = M_2$;

(v) $R(\tilde{G}) = H^1[-h,0]$.

PROOF. The properties (i), (ii) and (iii) are clear from the definitions. Since the proofs of properties (iv) and (v) are equivalent, we shall only prove (v). Let $\phi \in H^1[-h,0]$ and define f by

$$f(t) = \phi(t-h) - \int_0^t \zeta(t-s)\phi(-h+s)\,ds$$

for $0 \le t < h$ and constant for $t \ge h$. Then $f \in H_1$ and for $0 \le t \le h$

$$x(t;f) = \phi(-h+t)$$

which implies that $\tilde{G}f = \phi$. □

The following proposition shows the useful interplay between the structural operators and the C_0-semigroups.

PROPOSITION 2.16.

(i) $T(t)G = GS(t)$ *for all* $t \in \mathbb{R}_+$;

(ii) $\tilde{T}(t)\tilde{G} = \tilde{G}\tilde{S}(t)$ *for all* $t \in \mathbb{R}_+$;

(iii) $FT(t) = S(t)F$ *for all* $t \in \mathbb{R}_+$;

(iv) $\tilde{F}\tilde{T}(t) = \tilde{S}(t)\tilde{F}$ *for all* $t \in \mathbb{R}_+$;

(v) $GF = T(h)$;

(vi) $\tilde{G}\tilde{F} = \tilde{T}(h)$;

(vii) $FG = S(h)$;

(viii) $\tilde{F}\tilde{G} = \tilde{S}(h)$.

PROOF. Clear from the definitions. □

THEOREM 2.17. (The Intertwining Property). *The C_0-semigroups* $\{T(t)\}$ *and* $\{S(t)\}$ *are intertwined.*

PROOF. Because of Proposition (2.16) G is a bounded invertible mapping from F_2 onto M_2 such that $T(t)G = GS(t)$. □

3. THE ADJOINT EQUATIONS

Let $\zeta \in V_2$ be fixed. Apart from the equations (1.1) and (2.1) we can also consider the equations (1.1) and (2.1) with the transposed conjugate kernel ζ^*:

$$(3.1) \qquad x = \zeta^* * x + f, \qquad f \in F_2;$$

$$(3.2) \qquad \frac{dx}{dt}(t) = \zeta^{0*}x(t-h) + \int_0^h \zeta^*(s)\frac{dx}{dt}(t-s)ds, \qquad t \geq 0$$

$$x_0 = \phi, \ \phi \in M_2.$$

For reasons which will become more clear later, we call these equations the adjoint equations of (1.1) and (2.1). In the same way as done above, we can define for the adjoint equations C_0-semigroups and structural operators: $\{S(t;\zeta^*)\}$,$\{T(t;\zeta^*)\}$, $F(\zeta^*)$, $G(\zeta^*)$ ect.

In this section we shall show that there are important duality relations between the introduced C_0-semigroups and structural operators. Before we start to describe the adjoint operator A^* of A we shall give some elementary spectral properties of A.

Let $\Delta(z)$ denote the complex matrix function

$$(3.3) \qquad \Delta(z) = z\,I - e^{-zh}\zeta^0 - z\int_0^h e^{-zt}\zeta(t)dt.$$

The complex matrix function $\Delta(z)$ is called the *characteristic matrix* of (1.1) or (2.1) and appears in a natural way if one Laplace transforms the equation (1.1) or (2.1).

LEMMA 3.4. *Let* $\psi \in M_2$ *and* $\lambda \in \mathbb{C}$ *be such that* $\det \Delta(\lambda) \neq 0$. *Then* $\lambda \in \rho(A)$ *and* $\phi = (\lambda I - A)^{-1}\psi$ *is given explicitly by*

$$\phi(t) = (\Delta^{-1}(\lambda)K(\psi),\ e^{\lambda t}\{\Delta^{-1}(\lambda)K(\psi) - \int_0^t e^{-\lambda\tau}\psi(\tau)d\tau\})$$

where

$$K(\psi) = \lambda \int_0^\infty e^{-\lambda t}(F\psi)(t)dt.$$

PROOF. Since $(\lambda I - A)\phi = \psi$, ϕ satisfies the following conditions:

(i) $\lambda\phi^1 - \frac{d\phi^1}{dt} = \psi^1$;

(ii) $\lambda\phi^0 - L\phi^1 = \psi^0$;

(iii) $\phi \in D(A)$.

Define

$$\phi^1(t) = e^{\lambda t}\phi^0 - \int_0^t e^{\lambda(t-\tau)}\psi^1(\tau)d\tau, \qquad -h \leq t \leq 0.$$

Then ϕ satisfies the conditions (i) and (iii). Moreover, the condition (ii) becomes

$$\Delta(z)\phi^0 = K(\psi).$$

Since $\det \Delta(\lambda) \neq 0$, we can solve $\phi^0 = \Delta^{-1}(\lambda)K(\psi)$. □

COROLLARY 3.5. *The spectrum of* A *satisfies*

$$\sigma(A) = P\sigma(A) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}.$$

PROOF. Because of the proof of Lemma (3.4) we have

$$\{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) \neq 0\} \subset \rho(A).$$

To prove the remaining inclusion choose a $\lambda \in \mathbb{C}$ such that $\det \Delta(\lambda) = 0$ and define

$$\phi^1(t) = e^{\lambda t}\phi^0$$

for $-h \le t \le 0$, where ϕ^0, $\phi^0 \ne 0$, is an element of the nullspace of $\Delta(\lambda)$. Then

$$A\phi = (L\phi^1, \frac{d\phi^1}{}) = (\lambda\phi^0, \lambda\phi^1) = \lambda\phi.$$

Hence, $\lambda \in P\sigma(A)$. □

COROLLARY 3.6. $N((\lambda I-A)) = \{\phi \in M_2 \mid \phi^1(t) = e^{\lambda t}\phi^0,\ -h \le t \le 0,$ $\phi^0 \in N(\Delta(\lambda))\}$.

LEMMA 3.7. *Let* $f \in H_1$ *and* $\lambda \in \mathbb{C}$ *be such that* $\det \Delta(\lambda) \ne 0$. *Then* $q = (\lambda I-A^*)^{-1}f$ *satisfies the differential equation*

$$\lambda q - \zeta^*(\cdot)q(0) - \frac{dq}{dt} = f.$$

PROOF. Since $(\lambda I-A^*)^{-1} = (\bar{\lambda}I-A)^{-1*}$, choose $\phi, \psi \in M_2$ such that $\phi = (\bar{\lambda}I-A)^{-1}\psi$. Then q satisfies the equation

$$\langle\phi, f\rangle = \langle\psi, q\rangle.$$

And

$$\begin{aligned}
\langle\psi, q\rangle &= q(0)\overline{\psi^0} + \int_0^h \frac{dq}{dt}(s)\overline{\psi^1(-s)}ds \\
&= q(0)\overline{(\bar{\lambda}\phi^0-L\phi^1)} + \int_0^h \frac{dq}{dt}(s)\overline{\{\bar{\lambda}\phi^1(-s) - \frac{d\phi^1}{dt}(-s)\}}ds \\
&= \lambda q(h)\overline{\phi^1(-h)} - q(0)\overline{L\phi^1} + \int_0^h \{-\frac{dq}{dt}(s) + \lambda q(s)\}\overline{\frac{d\phi^1}{dt}}(-s)ds \\
&= (\lambda q(h) - \bar{\zeta}^{0T}q(0))\overline{\phi^1(-h)} + \int_0^h \{\lambda q(s) - \bar{\zeta}^T(s)q(0) - \frac{dq}{dt}(s)\}\overline{\frac{d\phi^1}{dt}(-s)}ds \\
&= \langle\phi, f\rangle
\end{aligned}$$

$$= f(h)\overline{\phi^1(-h)} + \int_0^h f(s)\,\overline{\frac{d\phi^1}{dt}}(-s)\,ds.$$

Hence, $f = \lambda q - \zeta^*(\cdot)q(0) - \frac{dq}{dt}$. □

THEOREM 3.8. *The adjoint operator* $A^*: H_1 \to H_1$ *is given by*

$$A^*q = \frac{dq}{dt} + \zeta^*(\cdot)q(0)$$

with

$$D(A^*) = \{q \in H^1 \mid \frac{dq}{dt} + \zeta^*(\cdot)q(0) \in H_1\}.$$

PROOF. By Lemma (3.7) we have

$$A^*q = \frac{dq}{dt} + \zeta^*(\cdot)q(0)$$

and

$$D(A^*) \subset \{q \in H^1 \mid \frac{dq}{dt} + \zeta^*(\cdot)q(0) \in H_1\}.$$

The remaining inclusion can be proved in the same way as done in the proof of Theorem (1.9). □

PROPOSITION 3.9. *A* C_0*-semigroup is uniquely determined by its infinitesimal generator.*

PROOF. Let $\{T_1(t)\}$ and $\{T_2(t)\}$ be two C_0-semigroups with the same infinitesimal generator A. Choose a $t \in \mathbb{R}_+$ and define

$$T_3(s) = T_1(t-s)T_2(s).$$

Then Theorem (1.5)(c) yields

$$\frac{d}{ds}T_3(s)f = \frac{d}{ds}T_1(t-s)T_2(s) = T_1(t-s)AT_2(s)f - T_1(t-s)AT_2(s)f = 0$$

for every $f \in D(A)$. Hence, $T_3(t) = T_3(0)$ and so for all $f \in D(A)$

$$T_1(t)f = T_2(t)f.$$

Since t was chosen arbitrary and D(A) is dense, the proof is complete. □

THEOREM 3.10 (The duality principle).
(i) *The* c_0*-semigroups* $\{T(t)^*\}$ *and* $\{\tilde{S}(t;\zeta^*)\}$ *are identical;*
(ii) *The* c_0*-semigroups* $\{\tilde{T}(t)^*\}$ *and* $\{S(t;\zeta^*)\}$ *are identical.*

PROOF. Since M_2 and $H^1[-h,0]$ are reflexive spaces, we have that the adjoint semigroups $\{T(t)^*\}$ and $\{\tilde{T}(t)^*\}$ are c_0-semigroups. Now (i) is clear from Theorem (3.8) and Proposition (3.9). To prove (ii), in the same way as done above, we can describe $\tilde{A}^*$:

$$\tilde{A}q^* = (\zeta^{0*}q^1(0),\ \frac{dq^1}{dt} + \zeta^*(\cdot)q^1(0))$$

with

$$D(\tilde{A}) = \{q \in F_2 \mid \frac{dq^1}{dt} \in L^2[0,h]\}.$$

And Proposition (3.9) completes the proof. □

One of the great advantages of the Duality Principle is that it associates equations to the adjoint c_0-semigroups.

The structural operators are also related by duality:

THEOREM 3.11.
(i) $\tilde{F}(\zeta^*) = F^*$;
(ii) $F(\zeta^*) = \tilde{F}^*$;
(iii) $\tilde{G}(\zeta^*) = G^*$;
(iv) $G(\zeta^*) = \tilde{G}^*$.

PROOF. By Proposition (2.15), Proposition (2.16) and the Duality Principle it is enough to prove (iii) and (iv). Since the proofs of (iii) and (iv) are equivalent, we shall only prove relation (iv). We have to prove

$$<G(\zeta^*)f,q> = <<f,\tilde{G}q>>$$

for all $f \in F_2$ and $q \in H_1$. Note, that in the notation of Proposition (1.2) $R(\zeta^*) = R^*$. Hence

$$<G(\zeta^*)f,g> = (f^0 - R^* * f^1(h))\overline{g(0)} + \int_0^h (f^1 - R^* * f^1)(h-t)\overline{\frac{dg}{dt}(t)}dt$$

$$= (f^0 - R^* * f^1(h))\overline{g(0)} + \int_0^h f^1(t)\overline{\frac{dg}{dt}}(h-t)dt$$

$$- \int_0^h f^1(t)\overline{(R*\frac{dg}{dt})}(h-t)dt$$

$$= f^0\,\overline{g(0)} + \int_0^h f^1(t)\,\frac{d}{dt}\overline{(g-R*g)}(h-t)dt$$

$$= <<f,\tilde{G}g>>. \quad \square$$

<u>COROLLARY 3.12</u>. *The following commutative diagrams*

$$\frac{dx}{dt}(t) = \zeta^0\, x(t-h) + \int_0^h \zeta(s)\,\frac{dx}{dt}(t-s)ds \qquad\qquad x - \zeta * x = f$$

$$\begin{array}{ccc} M_2 & \underset{G}{\overset{F}{\rightleftarrows}} & F_2 \\ \{T(t)\},A \downarrow & & \downarrow \{S(t)\},B \\ M_2 & \underset{G}{\overset{F}{\rightleftarrows}} & F_2 \end{array}$$

$$\frac{dx}{dt}(t) = \zeta^{0*}\, x(t-h) + \int_0^h \zeta^*(s)\,\frac{dx}{dt}(t-s)ds \qquad\qquad x - \zeta^* * x = f$$

$$\begin{array}{ccc} H^1[-h,0] & \underset{\tilde{G}(\xi^*)}{\overset{\tilde{F}(\zeta^*)}{\rightleftarrows}} & H_1 \\ \{\tilde{T}(t;\zeta^*)\},\tilde{A}(\zeta^*) \downarrow & & \downarrow \{\tilde{S}(t;\zeta^*)\},\tilde{B}(\zeta^*) \\ H^1[-h,0] & \underset{\tilde{G}(\zeta^*)}{\overset{\tilde{F}(\zeta^*)}{\rightleftarrows}} & H_1 \end{array}$$

are related by duality.

REMARK 3.13. The strange looking choice of H_1 as the realisation of the dual space of M_2 avoids a lot of intertwining operators. For example, it makes it possible to state the Duality Principle in such a clear way as above.

REMARK 3.14. We could obtain a proof of Theorem (3.10) by calculating the adjoint semigroup $\{T(t)^*\}$ explicitly, but this involves a lot of calculations. See DIEKMANN [9],[10], who proves a variant of Theorem (3.10) in the case that $C[-h,0]$ is the state space of (0.1). Since in that case the adjoint semigroup $\{T(t)^*\}$ is not a c_0-semigroup, property (1.4)(iii) does not hold, our proof will not work in that case.

REMARK 3.15. By considering a slightly more general Volterra convolution integral equation involving measures, the set up given here can be used to treat the state space theory for neutral equations in general. Compare SALAMON [18] for analogous results.

4. SMALL SOLUTIONS AND COMPLETENESS; A REFINEMENT OF HENRY's THEOREM

In this section we shall study the small solutions of the equations (1.1), (2.1), (3.1) and (3.2). Since Henry's paper [12] it is well known that small solutions play an important role in completeness of the generalized eigenfunctions. And, moreover, that they indicate a certain redundancy in the state space. Our main result will be a sharp version of Henry's Theorem (4.23) and, as a corollary, that the ascent α of $N(T(t))$ is equal to the ascent δ of $N(T(t)^*)$. Moreover, we shall give an easy to verify necessary and sufficient condition for completeness of the generalized eigenfunctions (4.31).

The concept of completeness has been previously considered by LEVINSON and McCALLA [15] for scalar equations only; by BARTOSIEWICZ [2] and DELFOUR and MANITIUS [8], [16] in the case that ζ has finitely many jumps and an absolutely continuous part.

The definitions and proofs in this section are based on some results from complex analysis which we have collected in an Appendix to this section.

Define

$$R(z,A) = (zI-A)^{-1}$$

the resolvent of A. Let $\psi \in M_2$ be fixed and consider the function $R(z,A)\psi$ as a function of z. By Lemma (3.4) we have that $R(z,A)\psi$ is a meromorphic function with poles λ satisfying the equation

$$\det \Delta(z) = 0.$$

This property of R(z,A) makes it possible to use Theorem V 10.1 of TAYLOR [19] to obtain in our case:

THEOREM 4.1. *Let* λ *be a pole of* R(z,A) *of order* m. *Then*

(i) $N((\lambda I-A)^m) = N((\lambda I-A)^{m+1})$;

(ii) $R((\lambda I-A)^m) = R((\lambda I-A)^{m+1})$;

(iii) $R((\lambda I-A)^m)$ *is closed;*

(iv) $M_2 = N((\lambda I-A)^m) \oplus R((\lambda I-A)^m)$;

(v) *The corresponding spectral projection* P_λ *on* $N((\lambda I-A)^m)$ *can be represented by*

$$P_\lambda \phi = \frac{1}{2\pi i} \int_{\Gamma_\lambda} R(z,A)\phi dz$$

where Γ_λ *is a closed rectifiable curve surrounding only* λ *of the discrete set* $\sigma(A)$.

Let M_λ denote the generalized eigenspace $N((\lambda I-A)^m)$ corresponding to an eigenvalue λ of A. By Lemma (3.4) and the definition of A we have that the elements of M_λ invovle combinations of

$$t^k e^{\lambda t} d_k$$

where $k = 1,2,\ldots,m$ and $d_k \in \mathbb{C}^n$ satisfy a system of linear equations. So M_λ is finite dimensional and in fact it is not difficult to construct an explicit base for M_λ, see DELFOUR and MANITIUS [8], HALE [11].

Let Q_λ denote $R((\lambda I-A)^m)$. The proof of the following lemma is clear from the fact that A and $\{T(t)\}$ commute.

<u>LEMMA 4.2</u>. *The linear subspaces* M_λ *and* Q_λ *are* $\{T(t)\}$*-invariant.*

<u>COROLLARY 4.3</u>. *Let* Λ *be a finite set of eigenvalues. Then* M_2 *can be decomposed into two closed* $\{T(t)\}$*-invariant subspaces* M_Λ *and* Q_Λ

$$M_2 = M_\Lambda \oplus Q_\Lambda$$

where $M_\Lambda = \oplus_{\lambda\in\Lambda} M_\lambda$, $Q_\Lambda = \cap_{\lambda\in\Lambda} Q_\lambda$ *and the projection* P_Λ *on* M_Λ *is given by*

$$P_\Lambda = \sum_{\lambda\in\Lambda} P_\lambda .$$

<u>PROOF</u>. It is enough to note that generalized eigenfunctions belonging to different eigenvalues are linearly independent. □

<u>REMARK 4.4</u>. Since the projections P_λ are in general not orthogonal, the set Λ of eigenvalues must indeed be finite. However, as we shall see shortly it is an important problem to determine conditions which allows us to extend the above decomposition to the set $\Lambda = \sigma(A)$.

<u>DEFINITION 4.5</u>. The linear subspace M_1 generated by $\{M_\lambda \mid \lambda\in\sigma(A)\}$ is called the *generalized eigenspace* of A. The generalized eigenspace M_1 of A is called *complete* if the closure $\bar{M}_1$ of M_1 is the whole space.
Analogously, the generalized eigenspaces of B, A^*, B^* ect will be denoted by $M_2, \tilde{M}_1(\zeta^*), \tilde{M}_2(\zeta^*)$ etc.

<u>PROPOSITION 4.6</u>. *Let* λ *be a zero of* $\det\Delta(z)$ *of order* m. *Then for all* ℓ *with* $1 \le \ell \le m$ *the following relations hold*

(i) $FN((\lambda I-A)^\ell) = N((\lambda I-B)^\ell)$;
(ii) $GN((\lambda I-B)^\ell) = N((\lambda I-A)^\ell)$;
(iii) $F^*N((\lambda I-A^*)^\ell) = N((\lambda I-B^*)^\ell)$;
(iv) $G^*N((\lambda I-B^*)^\ell) = N((\lambda I-A^*)^\ell)$.

<u>PROOF</u>. By the Duality Principle it is enough to prove (i) and (ii). Since the proofs of (i) and (ii) are equivalent we shall only prove (ii). The

nullspace $N((\lambda I-A)^{\ell})$ is finite dimensional, $\{T(t)\}$-invariant and

$$N((\lambda I-A)^{\ell}) \cap N(T(t)) = \{0\}.$$

Hence, by Proposition (2.16)

$$N((\lambda I-A)^{\ell}) = T(h)N((\lambda I-A)^{\ell}) = G.FN((\lambda I-A)^{\ell}) \subset GN((\lambda I-B)^{\ell}) \subset N((\lambda I-A)^{\ell}). \square$$

DEFINITION 4.7. A *small solution* x of (1.1) is a non almost everywhere zero solution of (1.1) such that

$$\lim_{t\to\infty} e^{kt} x(t) = 0$$

for all $k \in \mathbb{R}$.

Note, that a small solution has an entire Laplace transform.

The determinant of the matrixfunction $\Delta(z)$ defined by (3.3), $\det \Delta(z)$, is an entire function of exponential type. Let τ denote the exponential type of $\det \Delta(z)$, by the Paley-Wiener Theorem (A.3) we have that τ is less than or equal to nh.

THEOREM 4.8. *Let* x *be a small solution of* (1.1) *then*

$$x(t) = 0 \quad \text{a.e.}$$

for all $t \geq nh - \tau$.

PROOF. Let x be a small solution of the equation

$$x = \zeta * x + f$$

$f \in F_2$. Then x is L^2-integrable along the positive real axis. By the Plancherel Theorem we have

$$\int_0^\infty e^{-zt} x(t)\,dt$$

is L^2-integrable along the imaginary axis. Laplace transformation of the equation (1.1) yields

$$\Delta(z)\int_0^\infty e^{-zt}x(t)dt = z\int_0^\infty e^{-zt}f(t)dt$$

or

$$\Delta(z)\int_0^\infty e^{-zt}x(t)dt = z\int_0^h e^{-zt}f^1(t)dt + e^{-zh}\, f^0.$$

Let $\mathrm{adj}\Delta(z)$ be the matrixfunction of cofactors of $\Delta(z)$, i.e. $\mathrm{adj}\Delta(z)$ satisfies the equation

$$\mathrm{adj}\Delta(z).\Delta(z) = \det\Delta(z).I.$$

Multiply the above equation by $\mathrm{adj}\Delta(z)$ then we obtain

$$\det\Delta(z)\int_0^\infty e^{-zt}x(t)dt = \mathrm{adj}\Delta(z)\{z\int_0^h e^{-zt}f^1(t)dt + e^{-zh}\, f^0\}.$$

Since the quotient of two functions of exponential type is again of exponential type we obtain that the entire function $\int_0^\infty e^{-zt}x(t)dt$ is of exponential type. Moreover, by Proposition (A.2) the right hand side has exponential type $\le nh$. And so again by Proposition (A.2) $\int_0^\infty e^{-zt}x(t)dt$ has finite exponential type σ and $\sigma \le nh-\tau$.
Hence, by the Paley-Wiener Theorem (A.3)

$$\int_0^\infty e^{-zt}x(t)dt = \int_0^\sigma e^{-zt}x(t)dt$$

and $x(t) = 0$ a.e. for all $t \ge 0$. Since $\sigma \le nh-\tau$, $x(t) = 0$ a.e. for all $t \ge nh-\tau$. □

<u>DEFINITION 4.9</u>. Let α denote the ascent of $N(S(t))$,

$$\text{i.e. } \alpha = \inf\{\eta \in \mathbb{R}_+ : \forall\varepsilon > 0\ N(S(\eta+\varepsilon)) = N(S(\eta))\}.$$

Let δ denote the ascent of $N(S(t)^*)$, i.e. $\delta = \inf\{\eta \in \mathbb{R}_+ : \forall\varepsilon > 0\ N(S(\eta+\varepsilon)^*) = N(S(\eta)^*)\}$.

PROPOSITION 4.10.
(i) α *is the ascent of* $N(S(t)), N(\tilde{S}(t)), N(T(t))$ *and* $N(\tilde{T}(t))$;
(ii) δ *is the ascent of* $N(S(t)^*), N(S(t)^*), N(T(t)^*)$ *and* $N(T(t)^*)$.

PROOF. All the C_0-semigroups in (i) respectively (ii) are intertwined. □

COROLLARY 4.11.
(i) $\alpha \le nh-\tau$;
(ii) $\delta \le nh-\tau$.

PROOF. By Theorem (4.8) and the Duality Principle. □

COROLLARY 4.12. $N(S(\alpha)) = \{f \in F_2 : \Delta^{-1}(z)(z\int_0^\infty e^{-zt}f(t)dt)$ *is entire.*$\}$

PROPOSITION 4.13. $N(T(\alpha)) = \{\psi \in M_2 : z \to R(z,A)\psi$ *is entire*$\}$.

PROOF. Because of Lemma (3.4) and Corollary (4.12) only the fact that

$$\psi \in N(T(\alpha)) \quad \text{if and only if} \quad F\psi \in N(S(\alpha))$$

remains to be proved. But this is clear from the definition of α. □

PROPOSITION 4.14. $\bigcap_{\lambda\in\sigma(A)} N(P_\lambda) = \{\psi \in M_2 : z \mapsto R(z,A)\psi$ *is entire*$\}$.

PROOF. This follows from the Laurent series of $R(z,A)$ in a pole λ of order m:

$$R(z,A) = \frac{D_\lambda^m}{(z-\lambda)^{m+1}} + \ldots + \frac{D_\lambda}{(z-\lambda)^2} + \frac{P_\lambda}{(z-\lambda)} + R_0(z,A)$$

where $R_0(z,A)$ is holomorphic and $D_\lambda = (\lambda I-A)P_\lambda$, see KATO [14]. □

COROLLARY 4.15. $\bigcap_{\lambda\in\sigma(A)} N(P_\lambda) = N(T(\alpha))$.

Although in general $R(S(t))$ becomes smaller with increasing time Corollary (4.11) states that the closure of the range $S(t)$, $\overline{R(S(t))}$, becomes constant for $t \ge \delta$. In fact we can say even more.

THEOREM 4.16.

(i) $\overline{M_1} = \overline{R(T(\delta))}$;

(ii) $\overline{M_2} = \overline{R(S(\delta))}$;

(iii) $\widetilde{M}_2(\zeta^*) = R(T(\alpha)^*)$;

(iv) $\widetilde{M}_1(\zeta^*) = R(S(\alpha)^*)$.

PROOF. Because of the Intertwining Property, Proposition (4.6) and the Duality Principle it is enough to prove (iii). Since $\sigma(A) = \overline{\sigma(A^*)}$ and order (λ) = order $(\bar{\lambda})$ we obtain

$$\begin{aligned}
\widetilde{M}_2(\zeta^*)^{\perp} &= \{\bigcup_{\lambda\in\sigma(A^*)} N((\lambda I-A^*)^{m_\lambda})\} \\
&= \bigcap_{\lambda\in\sigma(A^*)} R((\bar{\lambda}I-A)^{m_\lambda}) \\
&= \bigcap_{\lambda\in\sigma(A)} Q_\lambda \\
&= \bigcap_{\lambda\in\sigma(A)} N(P_\lambda) = N(T(\alpha)). \quad \square
\end{aligned}$$

REMARK 4.17. Theorem (4.8) and Theorem (4.16) are due to HENRY [12]. Henry's proof of Corollary (4.15) uses the estimate of HALE [11] on the complementary subspace Q_λ in the decomposition (4.3). Later Delfour and Manitius gave another proof of Theorem (4.16) using an explicit representation for R(z,A).

COROLLARY 4.18.

M_1 *is complete if and only if* $\delta = 0$.

$M_1(\zeta^*)$ *is complete if and only if* $\alpha = 0$.

The following corollary is due to MANITIUS [16].

COROLLARY 4.19. M_1 *is complete if and only if* $N(F^*) = \{0\}$.

PROOF. By Proposition (2.16) and the Duality Principle we have

$$N(F^*) = N(S(h)^*).$$

Hence

$$N(F^*) = \{0\} \text{ if and only if } \delta = 0. \quad \square$$

Although the above condition for completeness is necessary and sufficient, it is not satisfactory: first of all because the condition is related to the adjoint equation which is a little unnatural and secondly because the condition is not easy to verify. In fact the results given until now in this section show that it is important to know more about the relation between α and δ and to find ways to calculate these numbers explicitly.

Define ε by

$$\text{exponential type } \det\Delta(z) = nh-\varepsilon.$$

Since we noted already that the exponential type of $\det\Delta(z) \le nh$, we have $\varepsilon \ge 0$. Consider

$$\text{adj } \Delta(z) = (C_{ij})$$

the matrix function of cofactors of $\Delta(z)$. Since the cofactors are $(n-1)\times(n-1)$ subdeterminants the exponential type of the cofactors C_{ij} is less than or equal to $(n-1)h$. Moreover, since

$$\Delta(z).\text{adj } \Delta(z) = \det \Delta(z).I \tag{4.20}$$

there has to be a cofactor of exponential type $\ge (n-1)h-\varepsilon$. Define σ by

$$\max_{\substack{1\le i\le n\\ 1\le j\le n}} \text{ exponential type } C_{ij} = (n-1)h-\sigma.$$

Then $\sigma \le \varepsilon$ and in fact we can say even more.

THEOREM 4.21. *If* $\varepsilon > 0$ *then* $\sigma < \varepsilon$.

PROOF. Suppose $\sigma = \varepsilon$. We shall calculate exponential type det adj$\Delta(z)$ on two different ways. Since $\sigma = \varepsilon$ we have

$$\text{type det adj}\Delta(z) \leq n((n-1)h-\varepsilon) = (n-1)(nh-\varepsilon)-\varepsilon.$$

On the other hand by (4.20) and Proposition (A.2) we have

$$\text{type det adj}\Delta(z) = \text{type}(\det\Delta(z))^{n-1} = (n-1)(nh-\varepsilon)$$

Hence

$$(n-1)(nh-\varepsilon) \leq (n-1)(nh-\varepsilon)-\varepsilon$$

which is a contradiction if $\varepsilon > 0$. $\square$

THEOREM 4.22.
(i) $\alpha \leq \varepsilon-\sigma$;
(ii) $\delta \leq \varepsilon-\sigma$.

PROOF. This follows from the proof of Theorem (4.8) using the above terminology. $\square$

We can now state and prove our main result.

THEOREM 4.23. *The ascent* α *of* $\{S(t)\}$ *satisfies*

$$\alpha = \varepsilon-\sigma.$$

Or, equivalently, all small solutions of (1.1) *vanish almost everywhere for* $t \geq \varepsilon-\sigma$ *and* $\varepsilon-\sigma$ *is the smallest possible time with this property.*

PROOF. If $\varepsilon = 0$ we have $\sigma = 0$ and so by Theorem (4.22) $\alpha = 0$. Let $\varepsilon > 0$. Because of Proposition (4.10) it is enough to construct a small solution of the equation

$$x-\zeta*x = f$$

$f \in H_1$, such that $x(t) = 0$ a.e. for $t \geq \varepsilon-\sigma$ and $x \not\equiv 0$ a.e. in any neighborhood of $\varepsilon-\sigma$. Laplace transformation yields that this is equivalent to constructing a Paley-Wiener function F of exponential type $\varepsilon-\sigma$ such that

$$\Delta(z)F(z) = c + q(z)$$

where $c \in \mathbb{C}^n$ and q is a Paley-Wiener function of exponential type $\leq h$.

Choose a column of the matrix function $\operatorname{adj}\Delta(z)$ such that one of the elements of this column is the cofactor of maximal exponential type $(n-1)h-\sigma$. Since the arguments given below can be repeated for all other columns we may assume that we can choose the first column

$$\begin{pmatrix} C_{11} \\ \vdots \\ C_{n1} \end{pmatrix}$$

of $\operatorname{adj}\Delta(z)$. By (4.20) we have

$$(4.24) \qquad \Delta(z) \begin{pmatrix} C_{11} \\ \vdots \\ C_{n1} \end{pmatrix} = \begin{pmatrix} \det\Delta(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We have to consider two cases:

I. $\varepsilon \leq (n-1)h$;

II. $(n-1)h < \varepsilon \leq nh$.

Case I. Suppose $\varepsilon \leq (n-1)h$. Let for $1 \leq j \leq n$, c_j denote the Taylor expansion of C_{j1} of order $n-1$ in 0. Then the functions F_j defined by

$$F_j(z) = \frac{C_{j1}(z) - c_j(z)}{z^n},$$

$1 \leq j \leq n$, are entire. Let

$$\Delta(z) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

The functions d_j, $1 \le j \le n$, are polynomials of degree n with coefficients constants plus Paley-Wiener functions of exponential type $\le h$. Furthermore

$$(4.25)\qquad \Delta(z)\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} \frac{\det\Delta(z)-d_1}{z^n} \\ \frac{d_2}{z^n} \\ \vdots \\ \frac{-d_n}{z^n} \end{pmatrix}$$

Since $\det\Delta(z)$ is a polynomial of degree n with coefficients constants plus Paley-Wiener functions we have by the Paley-Wiener Theorem (A.3) that the right hand side of (4.25) can be written as follows

$$c + \int_0^{nh-\varepsilon} e^{-zt}h(t)\,dt$$

where $c \in \mathbb{C}^n$ and $h \in L^2([0,nh-\varepsilon];\mathbb{C}^n)$. Furthermore, the cofactors are polynomials of degree $(n-1)$ with coefficients constants plus Paley-Wiener functions. Hence, F is a Paley-Wiener function and by the Paley-Wiener Theorem (A.3) we have

$$F(z) = \int_0^{(n-1)h-\sigma} e^{-zt}\psi(t)\,dt$$

where $\psi \in L^2([0,(n-1)h-\sigma];\mathbb{C}^n)$. And so the equation (4.25) can be rewritten as follows

$$(4.26)\qquad \Delta(z)\int_0^{(n-1)h-\sigma} e^{-zt}\psi(t)\,dt = c + \int_0^{nh-\varepsilon} e^{-zt}h(t)\,dt.$$

Hence, the function ψ satisfies the equation

$$x-\zeta{*}x = g$$

where $g(t) = c+\int_0^t h(s)ds$ for $0 \le t \le nh-\varepsilon$ and constant for $t \ge nh-\varepsilon$.

Hence, by Proposition (1.2)

$$\frac{d\psi}{dt} \in L^2[0,(n-1)h-\sigma].$$

Rewrite the equation (4.26) as follows

$$\Delta(z)\left\{\int_0^{(n-1)h-\varepsilon} e^{-zt}\psi(t)\,dt + \int_{(n-1)h-\varepsilon}^{(n-1)h-\sigma} e^{-zt}\psi(t)\,dt\right\} = c + \int_0^{nh-\varepsilon} e^{-zt}h(t)\,dt$$

Hence.

$$\Delta(z)\int_{(n-1)h-\varepsilon}^{(n-1)h-\sigma} e^{-zt}\,\psi(t)\,dt = c + \int_0^{nh-\varepsilon} e^{-zt}h(t)\,dt - \Delta(z)\int_0^{(n-1)h-\varepsilon} e^{-zt}\psi(t)\,dt.$$

And so

$$e^{-((n-1)h-\varepsilon)z}\,\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\psi((n-1)h-\varepsilon+t)\,dt = \tag{4.27}$$

$$= c + \int_0^{nh-\varepsilon} e^{-zt}h(t)\,dt - \Delta(z)\int_0^{(n-1)h-\varepsilon} e^{-zt}\psi(t)\,dt.$$

Since the right hand side of (4.27) has exponential type less than or equal to $nh-\varepsilon$ we have by Proposition (A.2)

$$\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\psi((n-1)h-\varepsilon+t)\,dt \tag{4.28}$$

has exponential type less than or equal to h. Furthermore, since $\frac{d\psi}{dt} \in L^2[0,(n-1)h-\sigma]$ partial integration yields that (4.28) can be written as a constant plus a Paley-Wiener function. Hence

$$\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\psi((n-1)h-\varepsilon+t)\,dt = b + \int_0^h e^{-zt}\phi(t)\,dt$$

where $b \in C^n$ and $\phi \in L^2[0,h]$. Hence, $\psi((n-1)h-\varepsilon+\cdot)$ is a small solution such that

$$\psi((n-1)h-\varepsilon+\cdot) \not\equiv 0 \text{ a.e. in any neighborhood of } \varepsilon-\sigma.$$

This yields $\alpha = \varepsilon-\sigma$.

Case II. Suppose $(n-1)h<\varepsilon\le nh$. In this case τ = exponential type $\det\Delta(z) < h$. Multiply both sides of the equation (4.24) by

$$\int_0^{h-\tau} e^{-zt}dt$$

to obtain

$$\Delta(z)\begin{pmatrix}\tilde{C}_{11}\\ \vdots\\ \tilde{C}_{n1}\end{pmatrix}=\begin{pmatrix}G(z)\\ 0\\ \vdots\\ 0\end{pmatrix}$$

where $G(z) = \int_0^{h-\tau} e^{-zt}dt\ \det\Delta(z)$ has type h, $\tilde{C}_{j1} = \int_0^{h-\tau} e^{-zt}dt\ C_{j1}$, $1 \le j \le n$, and the function $\tilde{C}$ has type $\varepsilon-\sigma$. The same arguments as used in Case I applied to the function $\tilde{C}$ yields

$$\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\tilde{\psi}(t)dt = \tilde{c} + \int_0^{h} e^{-zt}\tilde{h}(t)dt.$$

Hence, $\tilde{\psi}$ is a small solution such that

$$\tilde{\psi} \not\equiv 0 \text{ a.e.}$$

in any neighborhood of $\varepsilon-\sigma$. This yields $\alpha = \varepsilon-\sigma$. □

The following corollary yields the answer to a well-known question in rfde theory.

COROLLARY 4.29. *The ascent α of the c_0-semigroup $\{S(t)\}$ and the ascent δ of the adjoint c_0-semigroup $\{S(t)^*\}$ are equal.*

PROOF. Since $\mathrm{adj}\Delta(z;\zeta^*) = \overline{(\mathrm{adj}\ (\bar{z}))}^T$ we have $\varepsilon(\zeta^*) = \varepsilon$ and $\sigma(\zeta^*) = \sigma$. □

The following special case of Corollary (4.29) yields the answer to a question of DELFOUR and MANITIUS [8] and extends a recently given result of BARTOSIEWICZ [2].

COROLLARY 4.30. $N(F) = \{0\}$ *if and only if* $N(F^*) = \{0\}$.

PROOF. By Proposition (2.16) the corollary is just a restatement of $\alpha = 0$ if and only if $\delta = 0$. □

The following corollary yields an easy to verify necessary and sufficient condition for completeness.

COROLLARY 4.31. $\bar{M}_1 = M_2$ *if and only if* $N(F) = \{0\}$
if and only if type $\det\Delta(z) = nh$.
Or, equivalently, completeness holds if and only if there are no small solutions.

Note, that Theorem (4.23) proves the existence of a small solution if type $\det\Delta(z) < nh$.

COROLLARY 4.32. *Consider a linear autonomous rfde with state space* $C[-h,0]$, *the space of continuous function on* $[-h,0]$. *Then all small solutions vanish identically for* $t \geq \varepsilon-\sigma-h$ *and* $\varepsilon-\sigma-h$ *is the smallest possible time with this property.*

PROOF. In the proof of Theorem (4.23) we constructed a small solution with maximal support of the Volterra convolution integral equation with forcing space H_1. Use the operator $\tilde{G}$ to map this small solution to a small solution with necessary maximal support of the rfde, since $R(\tilde{G}) = H^1[-h,0]$ this small solution is continuous. □

DELFOUR and MANITIUS also introduced in their papers [8], [16] the concept of F-completeness. We shall see that Theorem (4.23) also yields an easy to verify necessary and sufficient condition for F-completeness. The generalized eigenspace M_1 of A is called F-complete if

$$\overline{FM_1} = \overline{R(F)}.$$

By Proposition (4.6) $FM_1 = M_2$ and because of Proposition (2.10) and Theorem (4.16) we have M_1 is F-complete if and only if $\delta \leq h$. And so Theorem (4.23) yields.

COROLLARY 4.33. M_1 *is F-complete if and only if* $\alpha \leq h$.

Or, equivalently, F-*completeness holds if and only if all small solutions are in the kernel of* F.

CORROLLARY 4.34. *For* n = 2 F-*completeness holds if and only if type* $\det\Delta(z) \geq h$.

EXAMPLES 4.35.

a)
$$\frac{dx_1}{dt}(t) = -x_2(t) + x_3(t-1)$$
$$\frac{dx_2}{dt}(t) = x_1(t-1) + x_3(t-\tfrac{1}{2})$$
$$\frac{dx_3}{dt}(t) = x_3(t).$$

Then
$$\Delta(z) = \begin{pmatrix} z & 1 & -e^{-z} \\ -e^{-z} & z & -e^{-\frac{1}{2}z} \\ 0 & 0 & z+1 \end{pmatrix}$$

and
$$\det\Delta(z) = (z+1)(z^2+e^{-z}).$$

So $\varepsilon = 2$.

And,
$$C_{23} = -\begin{vmatrix} z & -e^{-z} \\ -e^{-z} & -e^{-\frac{1}{2}z} \end{vmatrix} = ze^{-\frac{1}{2}z} + e^{-2z}$$

So $\sigma = 0$ and $\alpha = \delta = \varepsilon - \sigma = 2$, and F-completeness fails.

b)
$$\frac{dx_1}{dt}(t) = -x_2(t) - x_3(t)$$
$$\frac{dx_2}{dt}(t) = x_1(t-1) + x_3(t-\tfrac{1}{2})$$
$$\frac{dx_3}{dt}(t) = x_3(t).$$

Then
$$\Delta(z) = \begin{pmatrix} z & 1 & 1 \\ -e^{-z} & z & -e^{-\frac{1}{2}z} \\ 0 & 0 & z+1 \end{pmatrix}$$

and

$$\det(z) = (z+1)(z^2+e^{-z}).$$

So $\varepsilon = 2$. Further $\sigma = 1$. Hence

$$\alpha = \delta = \varepsilon - \sigma = 1,$$

and F-completeness holds but completeness fails.

REMARK 4.36. As already noticed in section 1 we can associate with an arbitrary neutral equation a Volterra convolution integral equation where the kernel ζ involves measures. Using the more general Paley-Wiener Theorem involving distributions, given in RUDIN [17], the proof of Theorem (4.23) can be generalized such that Theorem (4.23) holds for neutral equations in general.

REMARK 4.37. By the results of this section we have that if the decomposition given in Corollary (4.3) holds for $\Lambda = \sigma(A)$ then

$$M_2 = \overline{R(T(\alpha))} \oplus N(T(\alpha)).$$

This decomposition for M_2 is important since it makes it possible to restrict (in the case of noncompleteness) the C_0-semigroup $\{T(t)\}$ to $\overline{R(T(\alpha))}$ such that the restricted C_0-semigroup $\{\hat{T}(t)\}$ is injective with dense range, like in the case of completeness. However, an easy example of Diekmann shows that $\overline{R(T(\alpha))} \oplus N(T(\alpha))$ need not to be closed and so $M_2 \neq \overline{R(T(\alpha))} \oplus N(T(\alpha))$. But in the example

$$M_2 = \overline{\overline{R(T(\alpha))} \oplus N(T(\alpha))}$$

holds. By the Duality Principle this is equivalent to

$$\overline{R(T(\alpha))} \cap N(T(\alpha)) = \{0\}. \tag{4.38}$$

The condition (4.38) seems plausible and is enough to treat the noncomplete case in a satisfactory way; instead of the restriction of $\{T(t)\}$ to $R(T(\alpha))$ we consider the by $\{T(t)\}$ induced C_0-semigroup on the quotient space

$$M_2/N(T(\alpha))$$

the obtained semigroup $\{\hat{T}(t)\}$ is an injective C_0-semigroup with dense range like in the case of completeness.

The study of necessary and sufficient conditions such that (4.38) holds, which is closely related to the study of convergence results for the projection series of generalized eigenfunctions, will be a subject for further research.

ACKNOWLEDGEMENT

I am grateful to O. Diekmann for many stimulating discussions during the preparation of this report.

APPENDIX

Let F be a vector valued function, we call F entire, of exponential type etc. if all the components of F are enitre, of exponential type etc. Furthermore, the exponential type of a vector valued function F is defined by the maximal exponential type of the components of F.

It is not difficult to see that all entire functions considered in this paper are entire functions of order 1.

$$\text{i.e.} \quad \limsup_{r\to\infty} \frac{\log \log M(r)}{\log r} = 1$$

where $M(r) = \max\{|F(z)| \mid z = re^{i\theta},\ 0 \le \theta \le 2\pi\}$.

DEFINITION A.1. The entire function F of order 1 is of exponential type τ if

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = \tau \qquad (0\le\tau\le\infty).$$

For entire functions of exponential type the rate of growth in different directions can be specified by the Phragmén-Lindelöf indicator function h

$$h(\theta) = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r} \quad .$$

PROPOSITION A.2. *Let* F *and* G *be entire functions of exponential type such that* F *and* G *are* $O(z^m)$, $m \in \mathbb{Z}$ *in the closed right half plane. Then*

$$\text{type }(FG) = \text{type }(F) + \text{type }(G),$$

PROOF. Because of the Ahlfors-Heins Theorem (7.2.6) of BOAS [4] we have for a dense set of $\theta \in [\pi/2, 3\pi/2]$

$$h_F(\theta) = -(\text{type}F)\cos\theta$$

$$h_G(\theta) = -(\text{type}G)\cos\theta.$$

Hence

$$h_{FG}(\theta) = \limsup_{r\to\infty} \{\frac{1}{r}\log|F(re^{i\theta})| + \frac{1}{r}\log|G(re^{i\theta})|$$

$$= -(\text{type}(F)+\text{type}(G))\cos\theta$$

$$= -(\text{type}(FG))\cos\theta \text{ for a dense set of } \theta \in [\pi/2, 3\pi/2].$$

And so

$$\text{type}(FG) = \text{type}(F)+\text{type}(G). \quad \square$$

An application of the Paley-Wiener Theorem 6.8.1 of BOAS [4] yields.

THEOREM A.3. (Paley-Wiener). *Let* F *be an entire function which is uniformly bounded in the closed right half plane. Then* F *is of exponential* type τ *and* L^2*-integrable along the imaginary axis if and only if*

$$F(z) = \int_0^\tau e^{-zt}\phi(t)dt$$

where $\phi \in L^2(0,\tau)$ *and* ϕ *does not vanish a.e. in any neighborhood of* τ.

We shall call the entire functions described by the above theorem *Paley-Wiener functions*.

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