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A case of a not so strange strange attractor

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A CASE OF A NOT SO STRANGE STRANGE ATTRACTOR

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An example is given of an iterative two-dimensional map of the horseshoe type in which everything can be expressed in simple trigonometric functions. The strange attractor is an analytic curve with a fractal dimension.

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1. INTRODUCTION

In this note a simple example is given of an iterative horeseshoe map in which almost everything can be expressed in explicit analytic expressions involving only elementary trigonometry. In this way a nice illustration is obtained of an unstable manifold which is also a strange attractor with a Cantor-like transsection having a simple fractal dimension. In our example a particular transsection yields a set of homoclinic and of heteroclinic points.

The map (cf. fig. 2.1, 2.2) is defined by

(1.1) $$\begin{cases} x \to bx(1-2y) + y, \\ y \to 4y(1-y), \quad 0 < b < 1. \end{cases}$$

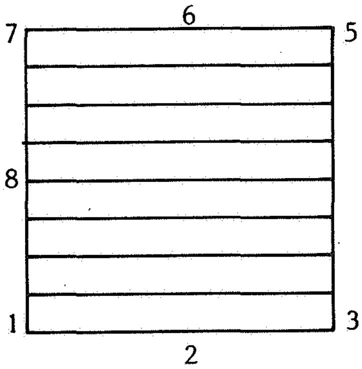

Fig. 2.1

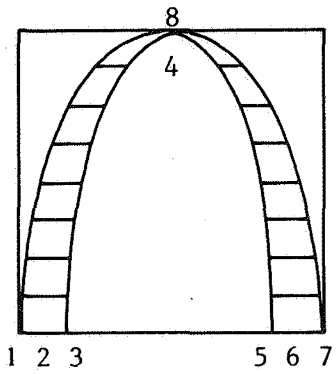

Fig. 2.2

By this map the unit square is mapped into itself as an internal horseshoe. There are two fixed points both hyperbolic, the origin and a fixed point at the level y = 3/4. The lines y = 0 and y = 3/4 are their stable manifolds. We show a.o. that the unstable invariant curve J_0 of the origin is an analytic curve determined by

(1.2) $$\begin{cases} x = \frac{1}{2} - \frac{1}{2}(\frac{1}{b} - 1) \sum_{k=1}^{\infty} b^k \phi_k(t), \\ y = \sin^2(t/2), \end{cases}$$

where

$$(1.3) \qquad \phi_k(t) = \frac{\sin t}{2^k \sin(2^{-k}t)} .$$

We note that $|\phi_k| \leq 1$ for all k and t. J_0 is like a sine curve folded up an infinite number of times so that it fits inside the unit square. The expression (1.2) can be used to show that J_0 is also a strange attractor. For the unstable manifold J_1 of the second fixed point a similar representation is obtained. However, after addition of their limit sets J_0 and J_1 become identical. It is as if J_1 starts at the "end" of J_0. Any point of the strange attractor can be obtained from a double-infinite binary expansion of $t/(2\pi)$

$$(1.4) \qquad \frac{t}{2\pi} = \ldots\ b_3b_2b_1b_0.b_{-1}b_{-2}b_{-3}\ \ldots = n + \theta$$

where n is the integer part and θ the fractional part. Substitution of (1.4) into (1.2) and (1.3) gives at first $y = \sin^2\pi\theta$ so that a constant θ gives the transsection of J_0 with a horizontal line. The value $\theta = 1/3$ gives a set of heteroclinic points. For constant θ and arbitrary n (or β) we may replace (1.3) by

$$(1.5) \qquad \phi_k(\beta) = \frac{2^{-k}\sin 2\pi\theta}{\sin((.b_{k-1}b_{k-2}\ldots b_1b_0b_{-1}b_{-2}\ldots)2\pi)} .$$

Substitution in (1.2) gives for each fraction β a point $x(\beta)$ of the strange attractor. The Lyapunov numbers of T are $\lambda_1 = 2$ and $\lambda_2 = b/2$. From this the Lyapunov dimension of the strange attractor is obtained as

$$(1.6) \qquad 1 + \frac{\log 2}{\log 2/b}, \qquad 0 < b < 1.$$

Details will be given in the next few sections. In the last section a generalisation is indicated. The general idea is to lift a one-dimensional map $y \to f(y)$ of which $y \to 4y(1-y)$ is perhaps the simplest non-trivial case for which there exists a complete parametrisation. According to Poincaré [1] any map

$$(1.7) \qquad y_{n+1} = f(y_n)$$

with

$$f(0) = 0, \quad |f'(0)| > 1,$$

for which f(y) is holomorphic, can be parametrised by an analytic function F(z) satisfying the functional equation

$$F(az) = f(F(z)), \tag{1.8}$$

where $a = f'(0)$, with the initial condition

$$F(0) = 0, \quad F'(0) = 1.$$

If f(y) is a polynomial or an entire function then also F(z) is an entire function. In a similar way as in the special case (1.1) the unstable manifold of the fixed point (0,0) can be parametrised as

$$x = E(t), \quad y = F(t), \tag{1.9}$$

where E(t) is another analytic function. In section 4 this is worked out a little for the slightly more general logistic map $y \to ay(1-y)$. We may safely conjecture that for a value of the parameter a for which the logistic map is chaotic the general situation is roughly as in the special case (1.1). A few more details can be found in a previous publication [4].

2. UNSTABLE INVARIANT CURVES

In the map (cf. fig. 1.1, 1.2)

$$\begin{cases} x \to bx(1-2y) + y, \\ y \to 4y(1-y), \end{cases} \tag{2.1}$$

with $0 < b \le 1$ the vertical line $x = \xi$, $0 \le y \le 1$ is transformed into a parabolic arc

$$y = 1 - \left(\frac{1-2x}{1-2b\xi}\right)^2.$$

A horizontal line $y = \eta$, $0 \le x \le 1$ is transformed into a similar line at the level $4\eta(1-\eta)$. We observe at once that the lines $y = 0$ and $y = 3/4$ are invariant. They can be interpreted as the stable manifolds of the fixed points $(0,0)$ and $(\frac{3}{4+2b}, \frac{3}{4})$. By the map (2.1) the unit square becomes a horseshoe which is overlapping itself for $\frac{1}{2} \le b \le 1$.
The case $b = 1$ is somewhat special and will be considered separately. If $0 < b < 1$ we may perform the substitution

$$(2.2) \qquad \begin{cases} 2bx = b - (1-b)u, \\ 2y = 1 - v, \end{cases}$$

which changes (2.1) into

$$(2.3) \qquad \begin{cases} u_{n+1} = b(1+u_n)v_n, \\ v_{n+1} = 2v_n^2 - 1, \end{cases}$$

written as an iterative process. A simple calculation shows that

$$(2.4) \qquad u_n = bv_{n-1} + b^2 v_{n-1}v_{n-2} + b^3 v_{n-1}v_{n-2}v_{n-3} + \dots + b^n v_{n-1}v_{n-2}v_{n-3} \dots v_0(1+u_0).$$

The sequence v_n can be parametrised by

$$(2.5) \qquad v_n = \cos(2^n z).$$

We note that

$$(2.6) \qquad v_0 v_1 v_2 \dots v_{m-1} = \frac{2^{-m}\sin(2^m z)}{\sin z} .$$

Thus (2.4), (2.5) can be written as

$$(2.7) \qquad \begin{cases} u_n = \sum_{k=1}^{n} \frac{(b/2)^k \sin(2^n z)}{\sin(2^{n-k} z)} + \frac{(b/2)^n \sin(2^n z)}{\sin z} u_0, \\ v_n = \cos(2^n z). \end{cases}$$

We see that for $n \to \infty$ the effect of the initial value u_0 vanishes

exponentially like $(b/2)^n$. On the other hand v_n is extremely sensitive to small changes in the initial value v_0.
For $n \to \infty$ the expression (2.7) becomes meaningless but if at the same time the parameter z is rescaled by writing $z = 2^{-n}t$ we obtain in the limit

$$(2.8) \qquad \begin{cases} u(t) = \sum_{k=1}^{\infty} \frac{(b/2)^k \sin t}{\sin(2^{-k}t)}, \\ v(t) = \cos t, \end{cases}$$

which can be interpreted as a parametrisation of the unstable manifold J_0. The iteration (2.3) shows that the effect of a single iteration of points on J_0 is equivalent to doubling the parameter t

$$(2.9) \qquad t \to 2t.$$

In fact, for $v = \cos t$ the second relation of (2.3) becomes the well-known duplication formula $\cos 2t = 2\cos^2 t - 1$.
In terms of the original variables x,y (2.8) may be written as

$$(2.10) \qquad \begin{cases} 2x = 1 - (1/b-1) \sum_{k=1}^{\infty} \phi_k(t) b^k, \\ 2y = 1 - \cos t, \end{cases}$$

where

$$(2.11) \qquad \phi_k(t) = \frac{\sin t}{2^k \sin(2^{-k}t)} = \cos\frac{t}{2} \cos\frac{t}{2^2} \cdots \cos\frac{t}{2^k}.$$

It may be of interest to determine the shape of J_0 near the origin by developing x and y into powers of t. Writing (2.10) shortly as

$$(2.12) \qquad x = E(t), \quad y = F(t),$$

it is obvious that

$$(2.13) \qquad F(t) = \tfrac{1}{2}(1-\cos t) = \frac{1}{4}t^2 - \frac{1}{48}t^4 + \ldots .$$

The expansion of E(t) can be derived from the identity

(2.14) $$E(2t) = bE(t)(1-2F(t)) + F(t),$$

which follows at once from (2.1) and (2.9). In the special case $b = 1/3$ we find

(2.15) $$E(t) = \frac{3}{44}t^2 - \frac{17}{16\times 11\times 47}t^4 + \dots .$$

Any branch of J_0 can be represented in the form $x = \phi(y)$ by substituting $t = 2\arcsin\sqrt{y}$ in the power series $x = E(t)$ with a corresponding meaning of the multivalued function arcsin.
In the special case $b = 1/3$ we find for the initial branch

(2.16) $$x = \frac{3}{11}y + \frac{30}{11\times 47}y^2 + \dots .$$

The turning points of J_0 are obtained for the parameter value $t_n = 2^n\pi$, $n = 1,2,3,\dots$. They give

(2.17) $$x_n = (1-\frac{b}{2})\,b^{n-1}, \quad y_n = 0.$$

These are all points of J_0 on $y = 0$. In fact, for $t = 2m\pi$ where m is an odd natural number we have always $\phi_1 = -1$ and $\phi_2 = \phi_3 = \dots = 0$ so that for all those parameters values we obtain the same point $x_1 = 1 - \frac{b}{2}$. If $t = 2m\pi$ where m is an even number we obtain iterates (2.17) of x_1.
This shows that each point (2.17) is a turning point of an infinity of folds of the unstable manifold. They are points of homoclinic tangency of J_0 with the stable manifold $y = 0$. There is also an infinity of heteroclinic points as the intersections of J_0 with the stable manifold $y = 3/4$ of the second fixed point. On the invariant line $y = 3/4$ the iterative map reduces to a linear recurrent relation

(2.18) $$x_{n+1} = -\frac{b}{2}x_n + \frac{3}{4}.$$

It has the particular solution $3/(4+2b)$ so its general solution is

(2.19) $$x_n = \frac{3}{4+2b} + C(-\frac{1}{8})^n.$$

This shows already that the successive loops of J_0 come arbitrarily close to the second fixed point. The heteroclinic points are all given by the parameter values $t = \pm \frac{2\pi}{3} + 2m\pi$.

The unstable manifold J_1 of the second fixed point can be parametrised in a similar way.
The fixed point is $(\frac{3}{4+2b}, \frac{3}{4})$ and it has the multipliers $-b/2$ and -2. Instead of (2.5) we take the parametrisation

(2.20) $$v_n = \cos(\frac{2\pi}{3} + (-2)^n z).$$

We note that

(2.21) $$v_0 v_1 v_2 \cdots v_{m-1} = \frac{(-2)^{-m}\sin(\frac{2\pi}{3} + (-2)^m z)}{\sin(\frac{2\pi}{3} + z)}.$$

Eventually we obtain the result

(2.22) $$\begin{cases} x = \frac{1}{2} - \frac{1}{2}(\frac{1}{b} - 1) \sum_{k=1}^{\infty} \psi_k(t)(b/2)^k, \\ y = \sin^2(\frac{\pi}{3} + \frac{t}{2}), \end{cases}$$

where

(2.23) $$\psi_k(t) = \frac{\sin(\frac{2\pi}{3} + t)}{(-2)^k \sin(\frac{2\pi}{3} + \frac{t}{(-2)^k})}.$$

Again $|\psi_k(t)| \leq 1$. For $t = 0$ we obtain the fixed point $x = \frac{3}{4+2b}$, $y = 3/4$. In this case we have two branches corresponding to $t > 0$ and to $t < 0$.

In the special case $b = 1$ the substitution (2.2) cannot be used. However, a simple observation shows that $x = \frac{1}{2}$ is an invariant line. The unstable manifold J_0 is described here by

(2.24) $$\begin{cases} x = \frac{1}{2}(1 - \frac{\sin t}{t}), \\ y = \frac{1}{2}(1 - \cos t). \end{cases}$$

This shows that the line $x = \frac{1}{2}$, $0 \leq y \leq 1$ is the corresponding attractor, no longer strange.

3. THE STRANGE ATTRACTORS

In the preceding section we have seen that for $0 < b < 1$ the unstable manifold J_0 of $(0,0)$ is determined by

$$(3.1)\qquad \begin{cases} x = \frac{1}{2} - \frac{1}{2}(\frac{1}{b} - 1)\sum_{k=1}^{\infty} \phi_k(t)b^k, \\ y = \sin^2 t/2, \end{cases}$$

where $\phi_k(t)$ is given by (2.10). The limit set, i.e. the strange attractor, is obtained by taking sequences $t_n \to \infty$ for $n \to \infty$. Let us consider the intersection of the strange attractor with a horizontal line. This means that we take subsequences from

$$(3.2)\qquad t = (n+\theta)2\pi, \quad 0 < \theta < 1.$$

Let

$$(3.3)\qquad \theta = b_{-1}b_{-2}b_{-3}b_{-4}\ldots,$$

be the binary expansion of θ. We also introduce the real number β with a similar expansion

$$(3.4)\qquad \beta = .b_0b_1b_2b_3\ \ldots, \quad 0 < \beta < 1.$$

We use β to define a subsequence of n-values

$$(3.5)\qquad b_0,\ b_1b_0,\ b_2b_1b_0\ \ldots,$$

again in binary notation.

Assuming for a while that this sequence never stops we obtain a limit point of J in the form

$$(3.6)\qquad \begin{cases} x = \frac{1}{2} - \frac{1}{2}(\frac{1}{b} - 1)\sum_{k=1}^{\infty} \dfrac{(b/2)^k \sin 2\pi\theta}{\sin((.b_{k-1}b_{k-2}\ldots b_0b_{-1}b_{-2}\ldots)2\pi)}, \\ y = \sin^2 \pi\theta. \end{cases}$$

The effect of the binary unit b_m upon the position of the corresponding element of the strange attractor is of the order b^m. This shows that also finite sequences correspond to points of the strange attractor since every rational β is approximated by an irrational β. This means that all points of the unstable manifold J_0 are accumulation points of the strange attractor and that in fact the closure of the invariant curve J_0 is the

strange attractor.
Clearly the intersection points determined by (3.6) for fixed θ and all β $(0 \leq \beta < 1)$ form an uncountable perfect set. The dimension of the strange attractor can be determined through the intermediary of the Lyapunov numbers of T. The special form of the Jacobian of T shows that the first Lyapunov number λ_1 is that of the one-dimensional map $y \to 4y(1-y)$ which is 2. In a similar way the second Lyapunov number equals $\lambda_2 = b/2$. The corresponding Lyapunov exponents are

$$(3.7) \qquad \sigma_1 = \log 2, \quad \sigma_2 = \log b/2.$$

Thus the strange attractor has the Lyapunov dimension (cf. Farmer et al [2])

$$(3.8) \qquad d_L = 1 + \frac{\log 2}{\log 2/b}.$$

It is a safe conjecture that also the Hausdorff dimension and the capacity have the same fractal value.

4. GENERALISATIONS

If the special map $y \to 4y(1-y)$ is replaced by the general logistic map $y \to ay(1-y)$ where $3 < a \leq 4$ we have the same overall picture provided the case is chaotic. According to Poincaré [1] the iterative sequence y_n can be parametrised as

$$(4.1) \qquad y_n = F(a^n z)$$

where $F(z)$ is an entire analytic function satisfying the functional equation

$$(4.2) \qquad F(az) = aF(z)(1-F(z))$$

with

$$(4.3) \qquad F(0) = 0, \quad F'(0) = 1.$$

It can be shown that $F(z)$ is entire when $a > 1$ and that its exponential order is log2/loga [3], [4]. The only elementary cases are

$$a = 4 \qquad F(z) = \sin^2\sqrt{z},$$
$$a = 2 \qquad F(z) = \tfrac{1}{2}(1-e^{-2z}).$$

Explicitly

$$F(z) = z - \frac{z^2}{a-1} + \frac{2z^3}{(a-1)(a^2-1)} - \dots . \tag{4.4}$$

Repeated use of (4.2) and (4.4) enables us to compute F(z) for arbitrary large values of z. Here we are interested only in the behaviour of F(t) for positive real values of t. F(t) looks like the infinite iteration of the map $y \to ay(1-y)$ on a suitable horizontal scale. For $t \to \infty$ the function F(t) is almost periodic with

$$\tfrac{1}{4}a^2(1-\tfrac{1}{4}a) \leq F \leq \tfrac{1}{4}a. \tag{4.5}$$

Proceeding as in section 2 we consider the map

$$\begin{cases} x \to bx(1-2y) + y, \\ y \to ay(1-y). \end{cases} \tag{4.6}$$

A parametrisation of the unstable manifold J_0 of the origin can be obtained in the form (2.11) where F(t) is the entire Poincaré function defined by (4.2),(4.3). The action of T along J_0 is now

$$t \to at. \tag{4.7}$$

This gives for E(t) a similar functional equation as (2.13)

$$E(at) = bE(t)(1-2F(t)) + F(t). \tag{4.8}$$

Again repeated use of this relation and a few terms of the power series expansion of E(t) makes it possible to compute points of the unstable manifold even for very large values of t.

In a slightly different approach we may also derive an expansion of the kind (2.9) or (2.22).

In terms of u and v defined by (2.2) the map is written as

(4.9) $$\begin{cases} u \to b(1+u)v, \\ v \to \frac{1}{2}av^2 - (\frac{1}{2}a-1). \end{cases}$$

For v we now use the parametrisation

(4.10) $$v = G(t) \overset{\text{def}}{=} 1 - 2F(\frac{1}{a}t^{\frac{\log a}{\log 2}}),$$

which makes G(t) almost periodic within the interval $1 - a/2$, $1 - a^2/2+a^3/8$ and for which a single iteration is equivalent to $t \to 2t$.
Writing

(4.11) $$u = \sum_{k=1}^{\infty} \phi_k(t)b^k,$$

we obtain for $\phi_k(t)$ the duplication rule

(4.12) $$\phi_k(2t) = \phi_{k-1}(t)G(t), \quad \phi_0(t) = 1,$$

so that

(4.13) $$\phi_k(t) = G(\frac{t}{2})G(\frac{t}{2^2})\ldots G(\frac{t}{2^k}).$$

As a check we observe that for a = 4

(4.14) $$F(t) = \sin^2\sqrt{t}, \quad G(t) = \cos t$$

so that indeed (4.13) coincides with (2.10).

Similar interesting maps may be derived from the scheme

(4.15) $$T\begin{cases} x \to b(1+x)f(y) \\ y \to g(y) \end{cases}$$

where $0 < b < 1$, $|f(y)| \le 1$, $|g(y)| \le 1$ for which the origin is a hyperbolic fixed point with an analytic invariant unstable curve J.

(4.16) $$\begin{cases} x = \sum_{k=1}^{\infty} b^k\phi_k(t), \\ y = P(t). \end{cases}$$

The restriction of T to J_0 is required to be

(4.17) $t \to 2t$.

The simplest assumption is that P(t) is periodic with period 1. We should also have

(4.18) $P(2t) = g(P(t))$

and

(4.19) $\phi_k(2t) = b\phi_{k-1}(t)f(P(t))$

with $\phi_0(t) \equiv 1$.

5. ILLUSTRATIONS

In fig. 5.1 a plot is given of the map T with $b = 1/3$ in the scale -0.2, 1.2, -0.1, 1.1. Shown is an orbit of some 500 points attracted by the strange attractor.
In fig. 5.2 the first few arcs of the unstable manifold J_0 of the map of the previous illustration are shown.
In fig. 5.3 the first few arcs of the unstable manifold J_1 with $t > 0$ are shown for the same case. In table 5.4 the first 64 heteroclinic points are given as the intersections of J_0 and $y = 3/4$ for $b = 1/4$.

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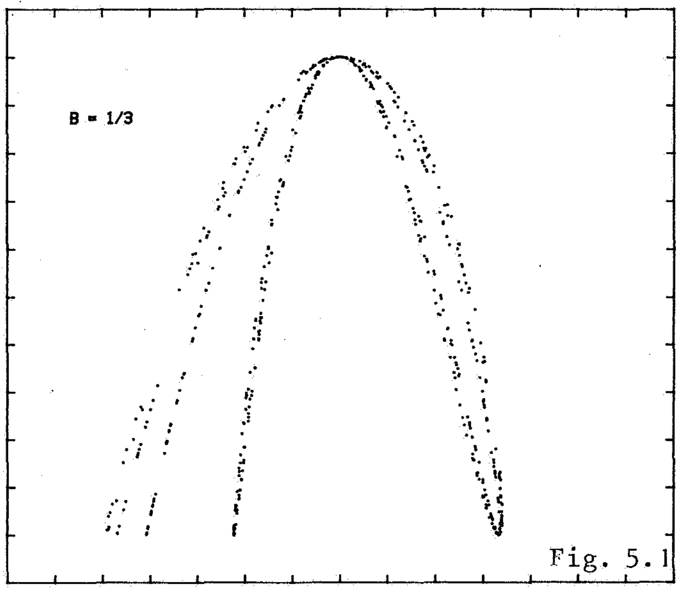

Fig. 5.1

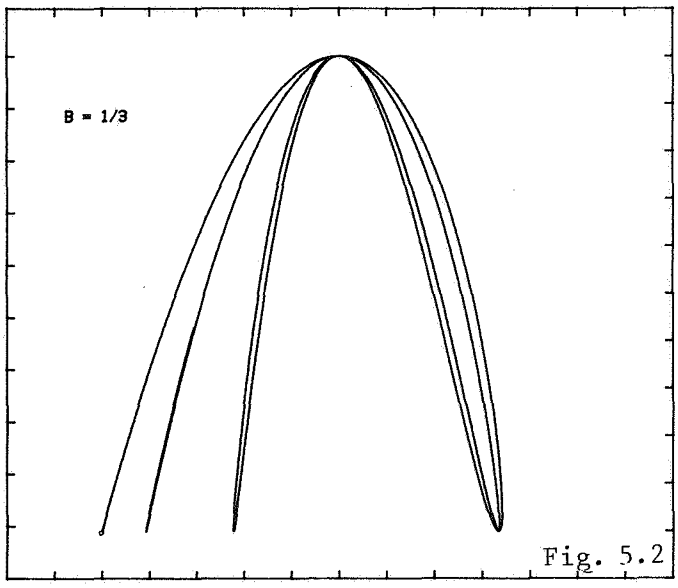

Fig. 5.2

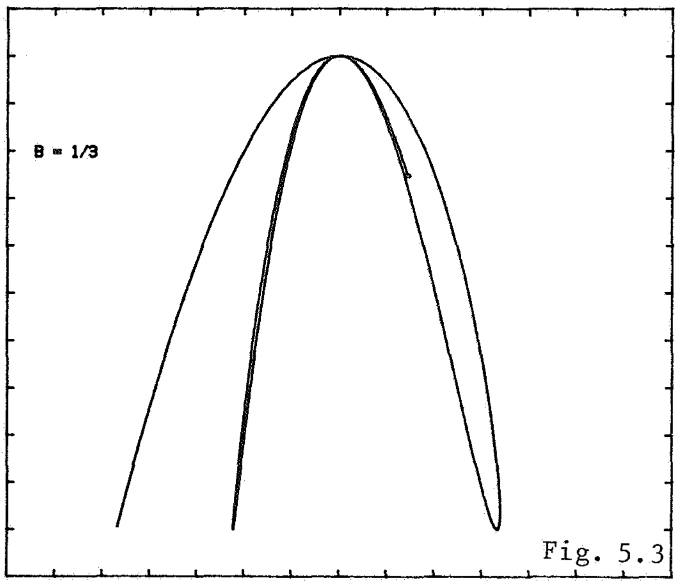

Fig. 5.3

1	.258864297	33	.260272426
2	.717641963	34	.716939586
3	.660294755	35	.660645617
4	.356406456	36	.356125347
5	.349949820	37	.350162016
6	.667463156	38	.667285254
7	.705449193	39	.705593699
8	.282106671	40	.281974013
9	.281319458	41	.281434437
10	.706256273	42	.706147964
11	.666567106	43	.666665111
12	.350924597	44	.350830555
13	.355147590	45	.355235531
14	.661818851	46	.661733197
15	.714736666	47	.714819028
16	.264581113	48	.264499821
17	.264483316	49	.264563531
18	.714835068	50	.714754873
19	.661717966	51	.661799181
20	.355250410	52	.355168120
21	.350816288	53	.350902016
22	.666679112	54	.666590919
23	.706134425	55	.706229434
24	.281447777	56	.281348174
25	.281961015	57	.282073139
26	.705606551	58	.705485876
27	.667272644	59	.667417760
28	.350164528	60	.350001465
29	.356112990	61	.356334738
30	.660657917	62	.660384080
31	.716927361	63	.717464421
32	.260284632	64	.259218446

Table 5.4. Heteroclinic points;
fixed point at 2/3 for b=1/4