Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

D.M. Chibisov, W.R. van Zwet

On the Edgeworth expansion for the logarithm of the likelihood ratio, II

Department of Mathematical Statistics Report MS-R8401 January

ON THE EDGEWORTH EXPANSION FOR THE LOGARITHM OF THE LIKELIHOOD RATIO, II

D.M. CHIBISOV
Steklov Mathematical Institute, Moscow, U.S.S.R.

W.R. VAN ZWET
University of Leyden, The Netherlands

In this paper we discuss conditions under which the distribution function of the logarithm of the likelihood ratio possesses an Edgeworth expansion. The underlying model is that of independent but not necessarily identically distributed random variables and the two sequences of product distributions are assumed to be contiguous. First we deal with the problem in full generality and obtain a result in the spirit of Oosterhoff and Van Zwet (1979), who characterized contiguity and asymptotic normality of the logarithm of the likelihood ratio. Next we show how this result may be simplified for differentiable likelihoods. In a companion paper, Chibisov and Van Zwet (1984), we discuss the special case of independent and identically distributed random variables and differentiable likelihoods in considerably more detail.

1980 MATHEMATICS SUBJECT CLASSIFICATION: 62E20.
KEY WORDS & PHRASES: Edgeworth expansions, contiguity.
NOTE: This paper will appear in *Proceedings of the Third Prague Symposium on Asymptotic Statistics*. Research of D.M. Chibisov supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

Report MS-R8401
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1. INTRODUCTION

For $n = 1,2,\ldots,$ let $(\mathcal{X}_{n1},\mathcal{A}_{n1}),\ldots,(\mathcal{X}_{nn},\mathcal{A}_{nn})$ be arbitrary measurable spaces. Let P_{nj} and Q_{nj} be probability measures defined on $(\mathcal{X}_{nj},\mathcal{A}_{nj})$ with densities p_{nj} and q_{nj} with respect to a σ - finite measure μ_{nj} , $j = 1,\ldots,n$, and define product probability measures $P_n = \Pi_{j=1}^n P_{nj}$ and $Q_n = \Pi_{j=1}^n Q_{nj}$. For each n and j , X_{nj} will denote the identity map from $\mathcal{X}_{nj}$ onto itself. Thus P_n and Q_n represent the two possible distributions of the random vector $(X_{n1},\ldots,X_{nn})$ as well as the probability measures on the underlying probability space. Obviously, $X_{n1},\ldots,X_{nn}$ are independent under both P_n and Q_n , with P_{nj} or Q_{nj} as marginal distributions of X_{nj} . Expectations and variances under these models will be indicated by E_P, E_Q, σ_P^2 and σ_Q^2 . Define the logarithm of the likelihood ratio for the individual experiments by

$$(1.1)\qquad \Lambda_{nj} = \log(q_{nj}(X_{nj})/p_{nj}(X_{nj})) \ , \quad j = 1,\ldots, n \ ,$$

and for the combined experiment by

$$(1.2)\qquad \Lambda_n = \sum_{j=1}^n \Lambda_{nj} \ , \qquad n = 1,2,\ldots \ .$$

It was shown in Oosterhoff and Van Zwet (1979) that the sequences $\{P_n\}$ and $\{Q_n\}$ are mutually contiguous if and only if, as $n \to \infty$,

$$(1.3)\qquad \sum_{j=1}^n \int \{q_{nj}^{\frac{1}{2}} - p_{nj}^{\frac{1}{2}}\}^2 \, d\mu_{nj} = \mathcal{O}(1)$$

and whenever $c_n \to \infty$, then

$$(1.4)\qquad \sum_{j=1}^n Q_{nj}(q_{nj}(X_{nj})/p_{nj}(X_{nj}) \geq c_n) = o(1) \ ,$$

$$(1.5)\qquad \sum_{j=1}^n P_{nj}(p_{nj}(X_{nj})/q_{nj}(X_{nj}) \geq c_n) = o(1) \ .$$

A property which is slightly stronger than mutual contiguity is that under P_n , Λ_n is asymptotically normal with mean $-\frac{1}{2}\sigma^2$ and variance $\sigma^2 \in (0,\infty)$, i.e.

$$(1.6)\qquad \sup_x \left| P_n(\Lambda_n \leq x) - \Phi\left(\frac{x+\frac{1}{2}\sigma^2}{\sigma}\right)\right| = o(1) \ ,$$

where Φ denotes the standard normal distribution function. It is well-known that (1.6) is equivalent to

$$(1.7)\qquad \sup_x |Q_n(\Lambda_n \le x) - \Phi\left(\frac{x-\frac{1}{2}\sigma^2}{\sigma}\right)| = o(1)$$

and according to Oosterhoff and Van Zwet (1979), both (1.6) and (1.7) are implied by

$$(1.8)\qquad \lim_{n\to\infty} \sum_{j=1}^{n} \int \{q_{nj}^{\frac{1}{2}} - p_{nj}^{\frac{1}{2}}\}^2 \, d\mu_{nj} = \frac{\sigma^2}{4} \in (0,\infty) \ ,$$

$$(1.9)\qquad \sum_{j=1}^{n} Q_{nj}(q_{nj}(X_{nj})/p_{nj}(X_{nj}) \ge 1+\varepsilon) = o(1) \ ,$$

$$(1.10)\qquad \sum_{j=1}^{n} P_{nj}(p_{nj}(X_{nj})/q_{nj}(X_{nj}) \ge 1+\varepsilon) = o(1) \ ,$$

for every $\varepsilon > 0$. These sufficient conditions (1.8) - (1.10) are also necessary, provided that the summands Λ_{nj} are uniformly asymptotically negligible under P_n , i.e.

$$\max_{1\le j\le n} P_{nj}(|\Lambda_{nj}| \ge \varepsilon) = o(1) \qquad \text{for every } \varepsilon > 0 \ .$$

Since Λ_n is the test statistic of the most powerful test of P_n against Q_n , (1.6) and (1.7) provide the limiting envelope power for this testing problem and this may serve as a yardstick for expressing the limiting performance of other tests for the same problem in terms of asymptotic relative efficiencies. In recent years, however, much better approximations of the power of many tests have become available, for which the error is $o(n^{-1})$ - or more generally $o(n^{-s})$ - rather than $o(1)$ as before. This has led to a more refined asymptotic comparison of tests, as expressed in terms of deficiencies or higher order efficiencies. Of course, the standard for comparison is still the most powerful test, so for any higher order efficiency calculation one needs, first of all, approximations of $P_n(\Lambda_n \le x)$ and $Q_n(\Lambda_n \le x)$ to the required order $o(n^{-s})$. Typically, such approximations are provided by Edgeworth expansions.

For an integer $r \ge 2$, we shall say that the distribution function (d.f.) of Λ_n under P_n admits an Edgeworth expansion F_{nr} with remainder $o(n^{-\frac{1}{2}r+1})$, if

$$(1.11)\qquad \sup_x |P_n(\Lambda_n \le x) - F_{nr}(x)| = o(n^{-\frac{1}{2}r+1})$$

as $n \to \infty$, where F_{nr} is obtained by inversion of a formal expansion of the characteristic function of Λ_n under P_n (cf. Petrov (1975)). Without going into the precise structure of F_{nr}, we note that it depends only on $\sum_{j=1}^{n} \kappa_k(\Lambda_{nj})$ for $k = 1,\ldots,r$, where $\kappa_k(\Lambda_{nj})$ denotes the k-th cumulant of Λ_{nj} under P_n, provided these cumulants are finite; if they are not, then (1.11) may still hold if one replaces the cumulants of Λ_{nj} in F_{nr} by pseudo-cumulants, which are cumulants of suitable approximations of Λ_{nj}. Similarly, we shall say that the d.f. of Λ_n under Q_n admits an Edgeworth expansion G_{nr} with remainder $o(n^{-\frac{1}{2}r+1})$, if

$$(1.12)\quad \sup_x |Q_n(\Lambda_n \le x) - G_{nr}(x)| = o(n^{-\frac{1}{2}r+1})$$

as $n \to \infty$, where G_{nr} depends only on the first r (pseudo)-cumulants of the Λ_{nj} under Q_n.

The main result of this paper is

THEOREM 1.1.

Let $r \ge 2$ *be an integer and let* c *and* η_n *be positive numbers with* $\lim \eta_n = \infty$. *Suppose that for every positive* ε *and* δ, *the following assumptions hold*

$$(1.13)\quad \sum_{j=1}^{n} P_{nj}(p_{nj}(X_{nj})/q_{nj}(X_{nj}) \ge 1+\varepsilon) = o(n^{-\frac{1}{2}r+1}),$$

$$(1.14)\quad \sum_{j=1}^{n} Q_{nj}(q_{nj}(X_{nj})/p_{nj}(X_{nj}) \ge 1+\varepsilon) = o(n^{-\frac{1}{2}r+1}),$$

$$(1.15)\quad \liminf_n \sum_{j=1}^{n} \int (q_{nj}^{\frac{1}{2}} - p_{nj}^{\frac{1}{2}})^2 \, d\mu_{nj} \ge c,$$

$$(1.16)\quad \sum_{j=1}^{n} \int |q_{nj}^{1/r} - p_{nj}^{1/r}|^r \, d\mu_{nj} = o(n^{-\frac{1}{2}r+1}),$$

$$(1.17)\quad \liminf_n \inf_{\delta \le |t| \le \eta_n n^{\frac{1}{2}(r-1)}} \sum_{j=1}^{n} \left[\frac{1 - |E_P \exp\{itn^{\frac{1}{2}}\Lambda_{nj}\}|^2}{\log n}\right] > r - 2.$$

Then both under P_n *and* Q_n, *the d.f. of* Λ_n *admits an Edgeworth expansion with remainder* $o(n^{-\frac{1}{2}r+1})$. *For* $r = 2$, *assumption* (1.17) *may be omitted.*

For $r = 2$, the conditions of the theorem reduce to (1.8) - (1.10) except for the trivial modification that we don't require convergence in (1.8), but only that all possible limit points are in $(0,\infty)$. Since the leading terms in the Edgeworth expansions turn out to be normal d.f.'s with the right parameters, theorem 1.1 contains the result on the asymptotic normality of Λ_n in Oosterhoff and Van Zwet (1979).

For $r \geq 3$, theorem 1.1 is concerned with higher order asymptotics as opposed to limit theory. As will be seen from the proof of the theorem, assumptions (1.13) - (1.17) are equivalent to the best available conditions for the existence of Edgeworth expansions for sums of independent random variables and in this sense the assumptions are optimal. Assumptions (1.13) and (1.14) allow the necessary truncation of the Λ_{nj}, (1.15) prevents Λ_n from degenerating and (1.16) controls the order of magnitude of the pseudo-cumulants involved. Finally, (1.17) is an assumption of Cramér-type which ensures the necessary smoothness of the d.f. of Λ_n.

A number of authors have dealt with this problem before and the reader is referred to Chibisov and Van Zwet (1984) for an account of the present state of affairs. We note that all previous results concern the parametric setup with differentiable likelihoods: for a parametric family of densities p_θ satisfying certain differentiability conditions with respect to θ, it is assumed that $p_{nj} = p_0$ and $q_{nj} = p_{\theta_n}$ with $\theta_n = tn^{-\frac{1}{2}}$.

In section 2 we provide a proof of theorem 1.1 and indicate how one may compute the pseudo-cumulants that determine the Edgeworth expansions. Since the verification of assumptions (1.13) - (1.16) and the computation of the pseudo-cumulants may be laborious, we simplify the situation in section 3, by requiring the likelihood ratio to satisfy a differentiability assumption in the mean. The resulting theorem 3.2 comes very close to the results in Chibisov and Van Zwet (1984). We do not discuss assumption (1.17) any further in the present paper, but refer the reader once more to Chibisov and Van Zwet (1984).

2. PROOF OF THEOREM 1.1

Let 1_B denote the indicator function of a set B. For $\varepsilon > 0$, define

$$\Lambda_{nj}(\varepsilon) = \Lambda_{nj}\, 1_{\{|\Lambda_{nj}|\leq\varepsilon\}} .$$

LEMMA 2.1

Let $r \geq 2$ *be an integer and let* c *and* η_n *be positive numbers with* $\lim \eta_n = \infty$. *Suppose that for every positive* ε *and* δ *the following assumptions hold*

$$(2.1)\qquad \sum_{j=1}^{n} P_{nj}(|\Lambda_{nj}| > \varepsilon) = o(n^{-\frac{1}{2}r+1}) ,$$

$$(2.2)\qquad \liminf_{n} \sum_{j=1}^{n} \sigma_P^2(\Lambda_{nj}(\varepsilon)) \geq c ,$$

$$(2.3)\qquad \sum_{j=1}^{n} E_P|\Lambda_{nj}(\varepsilon)|^r = O(n^{-\frac{1}{2}r+1}) ,$$

$$(2.4)\qquad \sup_{\delta \leq |t| \leq \eta_n n^{\frac{1}{2}(r-1)}} \prod_{j=1}^{n} |E_P \exp\{it\, n^{\frac{1}{2}}\Lambda_{nj}(\varepsilon)\}| = o(n^{-\frac{1}{2}r+1}/\log n) .$$

Then under P_n *the d.f. of* Λ_n *admits an Edgeworth expansion with remainder* $o(n^{-\frac{1}{2}r+1})$. *The pseudo-cumulants in the expansion are the cumulants of* $\Lambda_{nj}(\varepsilon)$ *(for any* $\varepsilon > 0$ *) under* P_n .

PROOF. Apply a slight extension of theorem 7 in chapter VI of Petrov (1975) (cf. also Bhattacharya and Ranga Rao (1976), theorem 20.6) to the truncated variables $\Lambda_{nj}(\varepsilon)$. In view of (2.1) the lemma follows. □

To prove theorem 1.1 we have to show that (1.13) - (1.17) imply not only (2.1) - (2.4) but also (2.1*) - (2.4*), which are derived from (2.1) - (2.4) by replacing P by Q . Let us write $r_{nj} = q_{nj}/p_{nj}$ and $R_{nj} = q_{nj}(X_{nj})/p_{nj}(X_{nj})$. Clearly (1.13) and (1.14) are equivalent to (2.1) and (2.1*) since

$$Q_{nj}(R_{nj}^{-1} \geq 1+\varepsilon) \leq \frac{1}{1+\varepsilon} P_{nj}(R_{nj}^{-1} \geq 1+\varepsilon) ,$$

$$P_{nj}(R_{nj} \geq 1+\varepsilon) \leq \frac{1}{1+\varepsilon} Q_{nj}(R_{nj} \geq 1+\varepsilon) .$$

If (1.13) and (1.14) hold, then (1.16) is equivalent to (2.3) and (2.3*) because

$$(2.5)\qquad \int_{|\log r_{nj}| \geq \varepsilon} |q_{nj}^{1/r} - p_{nj}^{1/r}|^r \, d\mu_{nj} \leq P_{nj}(R_{nj}^{-1} \geq e^{\varepsilon}) + Q_{nj}(R_{nj} \geq e^{\varepsilon}) ,$$

$$\log r_{nj} = r \log(1 + p_{nj}^{-1/r}(q_{nj}^{1/r} - p_{nj}^{1/r})) = -r \log(1 + q_{nj}^{-1/r}(p_{nj}^{1/r} - q_{nj}^{1/r})) .$$

If (1.13), (1.14) and (1.16) hold, then (1.15) is equivalent to (2.2) and (2.2*) - with a different choice of c - by an argument similar to the one leading to (3.13) and (3.14) in Oosterhoff and Van Zwet (1979). It follows that (1.13)-(1.16) are equivalent to (2.1) - (2.3) and (2.1*) - (2.3*).

It remains to show that (1.13) - (1.17) imply (2.4) and (2.4*). We have

$$\begin{aligned}(2.6)\qquad &\prod_{j=1}^{n} |E_P \exp\{it\, n^{\frac{1}{2}}\Lambda_{nj}(\varepsilon)\}|^2 \le \exp\{-\sum_{j=1}^{n}[1-|E_P \exp\{it\, n^{\frac{1}{2}}\Lambda_{nj}(\varepsilon)\}|^2]\} \le \\ &\le \exp\{-\sum_{j=1}^{n}[1-|E_P \exp\{it\, n^{\frac{1}{2}}\Lambda_{nj}\}|^2] + \sum_{j=1}^{n} P_{nj}(|\Lambda_{nj}| > \varepsilon)\} ,\end{aligned}$$

and (2.4) follows from (1.17) and (2.1). Define an auxiliary random variable $\tilde{\Lambda}_{nj}$ such that Λ_{nj} and $\tilde{\Lambda}_{nj}$ are independent and identically distributed under both P_{nj} and Q_{nj}. Then, for every $\eta > 0$,

$$\begin{aligned}1 - |E_Q \exp\{it\, n^{\frac{1}{2}}\Lambda_{nj}\}|^2 &= E_{Q\times Q}[1-\cos(t\, n^{\frac{1}{2}}(\Lambda_{nj}-\tilde{\Lambda}_{nj}))] \ge \\ &\ge (1+\eta)^{-2} E_{P\times P}[1-\cos(t\, n^{\frac{1}{2}}(\Lambda_{nj}-\tilde{\Lambda}_{nj}))] - 4(1+\eta)^{-2} P_{nj}(R_{nj}^{-1} \ge 1+\eta) = \\ &= (1+\eta)^{-2}[1-|E_P \exp\{it\, n^{\frac{1}{2}}\Lambda_{nj}\}|^2 - 4\, P_{nj}(R_{nj}^{-1} \ge 1+\eta)] .\end{aligned}$$

Taking η sufficiently small and combining this with (2.6) with P replaced by Q, we obtain (2.4*) and the theorem. □

What we have achieved in passing from lemma 2.1 to theorem 1.1 is to remove the truncated random variables $\Lambda_{nj}(\varepsilon)$ from the assumptions and replace them by integrable functions. According to lemma 2.1, however, the pseudo-cumulants of the Λ_{nj} occurring in the expansions are the cumulants of $\Lambda_{nj}(\varepsilon)$ under both P_n and Q_n. To remove the $\Lambda_{nj}(\varepsilon)$ at this point also, we define

$$(2.7)\qquad Z_{nj} = p_{nj}^{-1/r}(X_{nj})\{q_{nj}^{1/r}(X_{nj}) - p_{nj}^{1/r}(X_{nj})\}$$

and note that $\Lambda_{nj} = r\log(1+Z_{nj})$. It follows that $\Lambda_{nj}(\varepsilon) = r\log(1+Z_{nj}(\varepsilon))$, where

$$Z_{nj}(\varepsilon) = \begin{cases} Z_{nj} & \text{if } -1+\exp\{-\varepsilon/r\} \le Z_{nj} \le \exp\{\varepsilon/r\}-1 , \\ 0 & \text{otherwise.}\end{cases}$$

For $k = 1,\ldots,r$, consider the Taylor expansion

$$(2.8)\qquad \{r\log(1+z)\}^k = \sum_{\nu=k}^{r} a_{k\nu} z^\nu + o(|z|^r)$$

which, for any $\varepsilon > 0$, is valid uniformly for $-1+\exp\{-\varepsilon/r\} \le z \le \exp\{\varepsilon/r\}-1$. Define pseudo-moments

$$(2.9)\qquad \mu_{nj}(k) = \sum_{\nu=k}^{r} a_{k\nu} E_P Z_{nj}^\nu , \qquad k = 1,\ldots,r .$$

By (1.16), (2.1), (2.5) and Hölder's inequality for $k < r$, we have

$$\sum_{j=1}^{n} E_P|Z_{nj}(\varepsilon)|^r \le \sum_{j=1}^{n} E_P|Z_{nj}|^r = O(n^{-\frac{1}{2}r+1}) ,$$

$$\sum_{j=1}^{n} |E_P Z_{nj}^k - E_P Z_{nj}^k(\varepsilon)| \le \sum_{j=1}^{n} E_P|Z_{nj}|^k 1_{\{|\Lambda_{nj}|>\varepsilon\}} = o(n^{-\frac{1}{2}r+1})$$

for $k = 1,\ldots,r$. In view of (2.8) this yields

$$(2.10)\qquad \sum_{j=1}^{n} |E_P \Lambda_{nj}^k(\varepsilon) - \mu_{nj}(k)| = o(n^{-\frac{1}{2}r+1}) , \qquad k = 1,\ldots,r .$$

Let $\kappa_k(\Lambda_{nj}(\varepsilon))$ denote the k-th cumulant of $\Lambda_{nj}(\varepsilon)$ under P_n and define the pseudo-cumulants $\kappa_{nj}(k)$ corresponding to the pseudo-moments $\mu_{nj}(k)$ by

$$(2.11)\qquad \kappa_{nj}(k) = \frac{d^k}{dt^k} \log\left(\sum_{m=0}^{k} \mu_{nj}(m) \frac{t^m}{m!}\right)\Bigg|_{t=0} , \qquad k = 1,\ldots,r ,$$

where we may omit terms containing $\prod_i E_P Z_{nj}^{\nu_i}$ with $\sum \nu_i > r$, if we wish. From the results obtained above we find by standard inequalities that

$$\sum_{j=1}^{n} |\kappa_k(\Lambda_{nj}(\varepsilon)) - \kappa_{nj}(k)| = o(n^{-\frac{1}{2}r+1}) , \qquad k = 1,\ldots,r ,$$

and in view of the structure of Edgeworth expansions, this implies that we may replace the $\kappa_k(\Lambda_{nj}(\varepsilon))$ by the pseudo-cumulants $\kappa_{nj}(k)$ in the Edgeworth expansion under P_n without impairing its validity. Hence, in order to obtain the expansion, it suffices to compute

$$(2.12)\qquad E_P Z_{nj}^\nu = \int (q_{nj}^{1/r} - p_{nj}^{1/r})^\nu \, p_{nj}^{(r-\nu)/r} d\mu_{nj} , \qquad \nu = 1,\ldots,r .$$

For the expansion under Q_n we clearly need the integrals obtained from (2.12) by interchanging p and q. At the present level of generality this seems to be all that one can say about the problem.

3. DIFFERENTIABLE LIKELIHOODS

THEOREM 3.1

Let $r \geq 2$ *be an integer let* c *and* η_n *be positive numbers with* $\lim \eta_n = \infty$ *and let* ϕ_{nj} *be measurable functions on* (X_{nj}, A_{nj}) . *Suppose that for every positive* ε *and* δ , (1.17) *is satisfied and*

$$(3.1) \qquad \sum_{j=1}^{n} \int |q_{nj}^{1/r} - p_{nj}^{1/r}(1 + n^{-\frac{1}{2}}\phi_{nj})|^r \, d\mu_{nj} = o(n^{-\frac{1}{2}r+1}) ,$$

$$(3.2) \qquad \sum_{j=1}^{n} P_{nj}(\phi_{nj}(X_{nj}) \leq -\varepsilon n^{\frac{1}{2}}) = o(n^{-\frac{1}{2}r+1}) ,$$

$$(3.3) \qquad \sum_{j=1}^{n} Q_{nj}(\phi_{nj}(X_{nj}) \geq \varepsilon n^{\frac{1}{2}}) = o(n^{-\frac{1}{2}r+1}) ,$$

$$(3.4) \qquad \liminf_{n} \frac{1}{n} \sum_{j=1}^{n} \int \phi_{nj}^2 \, p_{nj} \, d\mu_{nj} \geq c ,$$

$$(3.5) \qquad \frac{1}{n} \sum_{j=1}^{n} \int |\phi_{nj}|^r \, p_{nj} \, d\mu_{nj} = O(1) .$$

Then the conclusion of theorem 1.1 *holds.*

PROOF. Take $0 < \varepsilon < \frac{1}{2}$ and write $R_{nj} = q_{nj}(X_{nj})/p_{nj}(X_{nj})$ as before. Then

$$\int |q_{nj}^{1/r} - p_{nj}^{1/r}(1 + n^{-\frac{1}{2}}\phi_{nj})|^r d\mu_{nj} \geq$$

$$\geq \varepsilon^r \{Q_{nj}(R_{nj} \geq (1-2\varepsilon)^{-r}) - Q_{nj}(\phi_{nj}(X_{nj}) \geq \varepsilon n^{\frac{1}{2}})\}$$

and (1.14) follows from (3.1) and (3.3). Similarly, (3.1) and (3.2) yield (1.13). Since $|x^{1/r}-1| \leq |x^{\frac{1}{2}}-1|$ for $x \geq 0$, (1.15) is a consequence of (3.1), (3.4) and (3.5). Finally, (1.16) follows from (3.1) and (3.5). □

If we take $\phi_{nj} = n^{\frac{1}{2}}(q_{nj}^{1/r} - p_{nj}^{1/r})/p_{nj}^{1/r}$, then theorem 3.1 is equivalent to theorem 1.1, but of course this is not what we have in mind. We are thinking of the situation where we have a parametric family of densities p_θ and where e.g. $p_{nj} = p_0$ and $q_{nj} = p_{\theta nj}$, with $\sum \theta_{nj}^2 = 1$ and $\sum|\theta_{nj}|^r = O(n^{-\frac{1}{2}r+1})$. Let $\phi_{nj} = (\theta_{nj} n^{\frac{1}{2}} \dot{p}_0)/(r\ p_0)$ on the set where $\dot{p}_0 = \partial p_\theta/\partial_\theta|_{\theta=0}$ exists. In this case assumption (3.1) concerns differentiability in the r-th mean and (3.4) asserts that the Fisher information of the model p_θ at $\theta = 0$ is positive.

In theorem 3.1 the assumptions are given in terms of the derivatives ϕ_{nj}, but we'd also like to express the pseudo-cumulants occurring in the Edgeworth expansions in a similar way. If we would have a formal expansion

$$(q_{nj}^{1/r} - p_{nj}^{1/r})/p_{nj}^{1/r} = \sum_{m=1}^{r} n^{-\frac{1}{2}m} \phi_{nj}^{(m)} + \ldots$$

this would yield a formal expansion

$$E_P Z_{nj}^\nu = E_P \left\{ \sum_{m=1}^{r} n^{-\frac{1}{2}m}\ \phi_{nj}^{(m)}(X_{nj}) \right\}^\nu + \ldots = \sum_{m=\nu}^{r} b_{nj\nu m}\ n^{-\frac{1}{2}m} + \ldots ,$$

with Z_{nj} as in (2.7). Substituting this in (2.9) and (2.9) in (2.11), at each step omitting all terms containing a factor n^{-s} with $s > \frac{1}{2}r$, we would obtain a formal expansion for the pseudo-cumulants occurring in the Edgeworth expansion under P. Of course the same result would be obtained if we would start with an expansion for $(q_{nj}-p_{nj})/p_{nj}$ or for $\log(q_{nj}/p_{nj})$ and use this to obtain a formal expansion for the pseudo-cumulants of Λ_{nj} under P directly instead of via the Z_{nj}. In the same way one may formally obtain expansions for the pseudo-cumulants occurring in the Edgeworth expansion under Q.

Of course the question is whether such formal computations are legitimate and theorem 3.2 provides conditions under which they are.

THEOREM 3.2

Let $r \geq 2$ *be an integer, let* c *and* η_n *be positive numbers with* $\lim \eta_n = \infty$ *and let* $\psi_{nj}^{(m)}$ *be measurable functions on* (X_{nj}, A_{nj}). *Suppose that for every positive* ε *and* δ, (1.17) *is satisfied and*

$$(3.6)\qquad \sum_{j=1}^{n} \int \left|q_{nj} - p_{nj}\left(1 + \sum_{m=1}^{r} n^{-\frac{1}{2}m}\ \psi_{nj}^{(m)}\right)\right| d\mu_{nj} = o(n^{-\frac{1}{2}r+1}) ,$$

$$(3.7)\qquad \liminf_{n} \frac{1}{n} \sum_{j=1}^{n} \int \{\psi_{nj}^{(1)}\}^2\ p_{nj}\ d\mu_{nj} \geq c ,$$

$$(3.8)\quad \frac{1}{n}\sum_{j=1}^{n}\int |\psi_{nj}^{(m)}|^{r/m}\, p_{nj}\, d\mu_{nj} = O(1), \quad \text{for} \quad m = 1,\ldots,r\ ,$$

$$(3.9)\quad \frac{1}{n}\sum_{j=1}^{n}\int_{|\psi_{nj}^{(m)}|\geq \varepsilon n^{\frac{1}{2}m}} |\psi_{nj}^{(m)}|^{r/m}\, p_{nj}\, d\mu_{nj} = o(1)\ .$$

Then the conclusion of theorem 1.1 *holds and the pseudo-cumulants occurring in the Edgeworth expansions under both* P *and* Q *may be computed formally from the expansions (cf.* (3.6))

$$(3.10)\quad q_{nj} = p_{nj}(1 + \sum_{m=1}^{r} n^{-\frac{1}{2}m}\, \psi_{nj}^{(m)}) + \ldots\ .$$

PROOF. Define

$$\psi_{nj\varepsilon}^{(m)} = \begin{cases} \psi_{nj}^{(m)} & \text{if } |\psi_{nj}^{(m)}| \leq \varepsilon n^{\frac{1}{2}m}\ , \\ 0 & \text{otherwise.} \end{cases}$$

Assumption (3.9) implies that

$$\sum_{j=1}^{n}\int n^{-\frac{1}{2}m}|\psi_{nj}^{(m)} - \psi_{nj\varepsilon}^{(m)}|p_{nj}\, d\mu_{nj} = o(n^{-\frac{1}{2}r+1})\ .$$

For any $\varepsilon > 0$, we may therefore replace $\psi_{nj}^{(m)}$ by $\psi_{nj\varepsilon}^{(m)}$ in (3.6) - (3.9) without affecting the validity of these assumptions. Then (1.13) and (1.14) follow. As a result of this we may formally derive an expansion

$$(3.11)\quad q_{nj}^{1/r} = p_{nj}^{1/r}(1 + \sum_{m=1}^{r} n^{-\frac{1}{2}m}\, \phi_{nj}^{(m)}) + \ldots$$

from (3.10) and assert that

$$(3.12)\quad \sum_{j=1}^{n}\int |(q_{nj}^{1/r}-p_{nj}^{1/r})/p_{nj}^{1/r} - \sum_{m=1}^{r} n^{-\frac{1}{2}m}\, \phi_{nj}^{(m)}|p_{nj}\, d\mu_{nj} = o(n^{-\frac{1}{2}r+1})$$

and that (3.7) - (3.9) hold with $\psi_{nj}^{(m)}$ replaced by $\phi_{nj}^{(m)}$. Again we may, of course, replace the $\phi_{nj}^{(m)}$ by truncated versions $\phi_{nj\varepsilon}^{(m)}$ with $|\phi_{nj\varepsilon}^{(m)}| \leq \varepsilon\, n^{\frac{1}{2}m}$ with impunity. But this implies by standard inequalities that for $\nu = 2,\ldots,r$,

$$\sum_{j=1}^{n} \int \{(q_{nj}^{1/r}-p_{nj}^{1/r})/p_{nj}^{1/r}\}^{\nu} \, p_{nj} \, d\mu_{nj} =$$

$$= \sum_{j=1}^{n} \int \{\sum_{m=1}^{r} n^{-\frac{1}{2}m} \phi_{nj\varepsilon}^{(m)}\}^{\nu} p_{nj} \, d\mu_{nj} + o(n^{-\frac{1}{2}r+1}) =$$

(3.13)
$$= \sum_{j=1}^{n} \int \sum_{\substack{m_1+\ldots+m_\nu \le r \\ m_t \ge 1}} \cdots \sum \prod_{t=1}^{\nu} n^{-\frac{1}{2}m_t} \phi_{nj\varepsilon}^{(m_t)} \, p_{nj} \, d\mu_{nj} + n^{-\frac{1}{2}r+1}(o(1) + O(\varepsilon)) =$$

$$= \sum_{j=1}^{n} \int \sum_{\substack{m_1+\ldots+m_\nu \le r \\ m_t \ge 1}} \cdots \sum \prod_{t=1}^{\nu} n^{-\frac{1}{2}m_t} \phi_{nj}^{(m_t)} \, p_{nj} \, d\mu_{nj} + o(n^{-\frac{1}{2}r+1}) ,$$

since $\varepsilon > 0$ is arbitrary.

First of all, (3.13) together with (3.7) and (3.8) for the $\phi_{nj}^{(m)}$, yields (1.15) and (1.16) so that the conclusion of theorem 1.1 follows. Secondly, (3.12) and (3.13) imply the validity of the formal expansion of $E_P Z_{nj}^{\nu}$ in (2.9) and therefore in the computation of the pseudo-cumulants under P. The proof that formal expansion is legitimate under Q also is similar. □

REFERENCES

[1] BHATTACHARYA, R.N. and RANGA RAO, R., *Normal Approximation and Asymptotic Expansions* (Wiley, New York, 1976).

[2] CHIBISOV, D.M. and VAN ZWET, W.R., On the Edgeworth expansion for the logarithm of the likelihood ratio, I. To appear in *Teor. Verojatnost. i Primenen.* (1984).

[3] OOSTERHOFF, J. and VAN ZWET, W.R., A note on contiguity and Hellinger distance, in: Jurečková, J. (ed.), *Contributions to Statistics, Jaroslav Hájek Memorial Volume* (Academia, Prague, 1979), 157-166.

[4] PETROV, V.V., *Sums of Independent Random Variables* (Springer, Berlin, 1975).