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ON THE EDGEWORTH EXPANSION FOR THE SUM OF A FUNCTION OF UNIFORM SPACINGS

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An Edgeworth expansion for the sum of a fixed function g of normed uniform spacings is established under a natural moment assumption and a Cramér type condition. This condition is shown to hold under an easily verifiable and mild assumption on the function g.

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1. INTRODUCTION

Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0,1) random variables. For n=1,2,..., $U_{1:n} \le U_{2:n} \le \ldots \le U_{n:n}$ denote the ordered $U_1, U_2, \ldots, U_n$. Let $U_{0:n}=0$ and $U_{n+1:n}=1$. Uniform spacings are defined by

$$D_{jn} = U_{j:n} - U_{j-1:n}, \qquad j=1,2,\ldots,n+1. \tag{1.1}$$

Let $g:[0,\infty) \to \mathbb{R}$ be a fixed nonlinear measurable function and define statistics T_n by

$$T_n = \sum_{j=1}^{n+1} g((n+1)D_{jn}), \qquad n=1,2,\ldots . \tag{1.2}$$

Statistics of this form can be used for testing uniformity.

There has been considerable interest into the asymptotic distribution theory for statistics of type (1.2). An excellent survey of first order limit theory for statistics of the form (1.2) was given by Pyke (1972). We also refer to the paper of Koziol (1980). According to Pyke (1972) a study of the rate of convergence for sums of functions of uniform spacings is of interest.

We will use the following well-known characterization, which has been applied by Le Cam (1958) in order to prove first order limit theorems. Let Y_j, j=1,2,... be independent exponential random variables with expectation 1. Let, for n=1,2,...,

$$W_n = \sum_{j=1}^{n+1} g(Y_j) \tag{1.3}$$

and

$$S_n = \sum_{j=1}^{n+1} (Y_j - 1), \tag{1.4}$$

then

$$\mathcal{L}(T_n) = \mathcal{L}(W_n | S_n = 0), \tag{1.5}$$

i.e. T_n has the same distribution as a sum of independent random variables given another sum of independent random variables.

With the aid of (1.5) Does and Klaassen (1983, 1984) proved Berry-Esseen bounds of the order $n^{-\frac{1}{2}}$ for the normal approximation for statistics based on uniform spacings under natural moment assumptions. In Does and Helmers (1982) Edgeworth expansions were established for statistics of the form (1.2) under a natural moment assumption and an integrability condition. In the present paper it is shown that the latter integrability condition can be replaced by a much weaker and more natural Cramér type condition. This condition holds under an easily verifiable and mild assumption on the function g.

2. AN EDGEWORTH EXPANSION

Let Y be an exponential random variable with expectation 1 and let g be a fixed real-valued measurable function defined on $\mathbb{R}^+$. Introduce, whenever well-defined, a function $\tilde{g}$ by

$$\tilde{g}(y) = (g(y) - \mu - \tau(y-1))(\sigma^2 - \tau^2)^{-\frac{1}{2}}, \qquad y > 0, \tag{2.1}$$

where

$$\mu = Eg(Y), \qquad \sigma^2 = \operatorname{Var} g(Y) \tag{2.2}$$

and

$$\tau = \operatorname{Cov}(g(Y), Y). \tag{2.3}$$

Note that $\sigma^2 > \tau^2$ iff g is nonlinear.

We shall establish an asymptotic expansion with uniform remainder $o(n^{-1})$ for the distribution function

$$(2.4) \qquad F_n(x) = P\Big((n(\sigma^2-\tau^2))^{-\frac{1}{2}}(T_n-(n+1)\mu) \leq x\Big), \qquad x\in\mathbb{R}.$$

Let ρ denote the characteristic function of $(Y-1,g(Y))$, i.e.

$$(2.5) \qquad \rho(s,t) = Ee^{is(Y-1)+itg(Y)}, \qquad (s,t)\in\mathbb{R}^2.$$

Let Φ and ϕ denote the distribution function and density of the standard normal distribution and let $\|.\|$ denote the Euclidian norm in $\mathbb{R}^2$, i.e. $\|(s,t)\| = (s^2+t^2)^{\frac{1}{2}}$, for $(s,t)\in\mathbb{R}^2$.

THEOREM 2.1. *Let* $g:[0,\infty)\to\mathbb{R}$ *be a measurable function and let* F_n *be as in* (2.4) *(cf.* (1.2), (2.2) *and* (2.3)*). If the assumptions*

$$(2.6) \qquad Eg^4(Y) < \infty$$

and

$$(2.7) \qquad \limsup_{\|(s,t)\|\to\infty} |\rho(s,t)| < 1$$

are satisfied, then

$$(2.8) \qquad \lim_{n\to\infty} n \sup_{x\in\mathbb{R}} |F_n(x) - \tilde{F}_n(x)| = 0,$$

where

$$\tilde{F}_n(x) = \Phi(x) - \phi(x)\left\{n^{-\frac{1}{2}}\left(\frac{1}{6}\kappa_3(x^2-1) + a\right)\right.$$

(2.9)
$$+ n^{-1}\left(\frac{1}{24}\kappa_4(x^3-3x) + \frac{1}{72}\kappa_3^2(x^5-10x^3+15x)\right.$$
$$\left.\left. + \frac{1}{8}(-4a\kappa_3+b)x + \frac{1}{6}a\kappa_3 x^3\right)\right\}$$

with

$$\kappa_3 = E\tilde{g}^3(Y),$$

$$\kappa_4 = E\tilde{g}^4(Y) - 3 - 3\{E\tilde{g}^2(Y)(Y-1)\}^2,$$

(2.10)
$$a = -\tfrac{1}{2}E\tilde{g}(Y)(Y-1)^2,$$

$$b = 3\{E\tilde{g}(Y)(Y-1)^2\}^2 - 2E\tilde{g}^2(Y)(Y-1)^2 + 4E\tilde{g}^2(Y)(Y-1) + 6.$$

The assumptions of Theorem 2.1 seem to be natural ones. Condition (2.6) is obviously necessary for the expansion (2.9) to be well-defined. In Remark 3.1 it is indicated that condition (2.7) is an appropriate version of Cramér's condition for this case.

Does and Helmers (1982) use the integrability condition

(2.11)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\rho(s,t)|^p ds\, dt < \infty, \qquad \text{for some } p \geq 1,$$

instead of (2.7) to validate the expansion (2.9). An assumption equivalent to (2.11) is that there exists an integer k such that the k-th convolution of (Y-1, g(Y)) has a bounded density (cf. Bhattacharya and Rao (1976), Theorem 19.1). According to the Riemann-Lebesgue lemma (cf. Theorem 4.1 in Bhattacharya and Rao (1976)) this implies that $|\rho(s,t)|^k \to 0$ as $\|(s,t)\| \to \infty$, which is much stronger than (2.7). Integrability conditions like (2.11) are commonly encountered in problems of establishing asymptotic expansions for conditional distributions (see e.g. Michel (1979)).

If we standardize the statistic T_n (cf. (1.2)) exactly then it should be possible to verify that under the assumptions of Theorem

2.1 relation (2.8) holds, with F_n replaced by the distribution function of $(T_n - ET_n)(\mathrm{Var}T_n)^{-\frac{1}{2}}$ and $\tilde{F}_n$ by the right-hand side of (2.9) with a=b=0. One may prove this by a refinement of Lemma 3.4 from Does and Klaassen (1984).

We note that, although we have proved our results for a fixed function g it seems to be possible to generalize Theorem 2.1 to functions g_{jn}; i.e. functions depending on the j-th spacing and sample size n. A Berry-Esseen theorem for this more general case was proved in Does and Klaassen (1983).

If g is linear then T_n is degenerate and condition (2.7) can not hold. A sufficient condition for (2.7) is given in our second theorem.

THEOREM 2.2. *Let* $g:[0,\infty)\to\mathbb{R}$ *be a measurable function and let* $(c,d) \subset (0,\infty)$ *be a bounded open interval on which* g *is absolutely continuous with a bounded and almost everywhere continuous derivative* g'. *If* g' *is not essentially constant on* (c,d) *then* (2.7) *holds.*

In Section 3 of Pyke (1965) some examples of functions g are given. These functions are related to $f_1(x)=x^r$, $r > 0$, $r \neq 1$, $f_2(x)=(x-1)^2$, $f_3(x)=|x-1|$, $f_4(x)=\log x$ and $f_5(x)=x^{-1}$. Except for the function f_5, which does not satisfy (2.6) and moreover yields a nonnormal limit distribution, all these functions are clearly included in Theorem 2.2.

In Does, Helmers and Klaassen (1984) it is shown that for the well-known Greenwood statistic (f_1 with r=2 or f_2) a related Cornish-Fisher expansion provides a reasonable approximation to the percentage points.

3. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 depends heavily on the techniques of Does and Klaassen (1984) and the computations in Does and Helmers (1982).

Without loss of generality we may replace g by $\tilde{g}$ (cf. (2.1)), because this does nog affect F_n and the assumptions of the theorem. In other words we assume that $\mu=0$, $\sigma^2=1$ and $\tau=0$ (cf. (2.2) and (2.3)). Let χ_n denote the characteristic function of $n^{-\frac{1}{2}}T_n$; i.e.

$$(3.1)\qquad \chi_n(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x),$$

with F_n as in (2.4). By Esseen's smoothing lemma (see e.g. Feller (1971), Lemma XVI 3.2) it suffices to prove that

$$(3.2)\qquad \int_{|t|\leq n\log n} \frac{|\chi_n(t) - \tilde{\chi}_n(t)|}{|t|} dt = o(n^{-1}),$$

where $\tilde{\chi}_n$ is the Fourier-Stieltjes transform of $\tilde{F}_n$ (cf. (2.9)); i.e.

$$\tilde{\chi}_n(t) = \int_{-\infty}^{\infty} e^{itx} d\tilde{F}_n(x)$$

$$(3.3)\qquad = e^{-\frac{1}{2}t^2}\left\{1 + \frac{a}{n^{\frac{1}{2}}} it - \frac{\kappa_3}{6n^{\frac{1}{2}}} it^3 - \frac{b}{8n} t^2 + \frac{(\kappa_4 + 4a\kappa_3)}{24n} t^4 - \frac{\kappa_3^2}{72n} t^6\right\}.$$

Since

$$|t^{-1}(\chi_n(t)-1)| \leq E|n^{-\frac{1}{2}}T_n|$$

$$\leq n^{-\frac{1}{2}}(n+1)E|g((n+1)D_{1n})| = n^{\frac{1}{2}} \int_0^{n+1} |g(y)|\left(1 - \frac{y}{n+1}\right)^{n-1} dy$$

$$\leq n^{\frac{1}{2}} \int_0^{n+1} |g(y)|\exp\{- y + \frac{2y}{n+1}\} dy \leq e^2 n^{\frac{1}{2}} \int_0^{\infty} |g(y)|e^{-y}dy,$$

for any t, it is easily verified that

$$(3.4)\qquad \int_{|t|\leq n^{-2}} |t|^{-1}|\chi_n(t) - \tilde{\chi}_n(t)|dt = O(n^{-3/2}).$$

According to Lemma 3.1 of Does and Klaassen (1984) we can choose a regular version of the conditional distribution of $n^{-\frac{1}{2}}W_n$ given $n^{-\frac{1}{2}}S_n = x$ (cf. (1.3) - (1.5)), such that for this version

$$\chi_n(t) = Ee^{itn^{-\frac{1}{2}}T_n} = E(e^{itn^{-\frac{1}{2}}W_n} | n^{-\frac{1}{2}}S_n = 0). \tag{3.5}$$

Let ψ_n be the characteristic function of $(n^{-\frac{1}{2}}S_n, n^{-\frac{1}{2}}W_n)$; i.e.

$$\psi_n(s,t) = [\rho(sn^{-\frac{1}{2}}, tn^{-\frac{1}{2}})]^{n+1}, \tag{3.6}$$

with ρ as in (2.5).

With the aid of Plancherel's identity (see e.g. Theorem 4.1 of Bhattacharya and Rao (1976)) we check that for all t (cf. Does and Klaassen (1984), formulas (3.15) and (3.16))

$$\begin{aligned} \int_{-\infty}^{\infty} |\psi_n(s,t)| ds &= \int_{-\infty}^{\infty} |\rho(sn^{-\frac{1}{2}}, tn^{-\frac{1}{2}})|^{n+1} ds \\ &\leq n^{\frac{1}{2}} \int_{\infty}^{\infty} |\rho(s, tn^{-\frac{1}{2}})|^2 ds = 2\pi n^{\frac{1}{2}} \int_0^{\infty} e^{-2y} dy = \pi n^{\frac{1}{2}}. \end{aligned} \tag{3.7}$$

Let h_n be the density of $n^{-\frac{1}{2}}S_n$. In view of Lemma 3.1 of Does and Klaassen (1984), (3.5) and (3.7), Fourier inversion of

$$\psi_n(s,t) = \int_{-\infty}^{\infty} e^{isx}\{E(e^{itn^{-\frac{1}{2}}W_n} | n^{-\frac{1}{2}}S_n = x) h_n(x)\} dx \tag{3.8}$$

yields

$$\chi_n(t) = (2\pi h_n(0))^{-1} \int_{-\infty}^{\infty} \psi_n(s,t) ds. \tag{3.9}$$

The theory of asymptotic expansions for the density of a sum of independent and identically distributed random variables (see e.g. Feller (1971), Theorem XVI 2.2) implies

(3.10) $$h_n(0) = \frac{1}{(2\pi)^{\frac{1}{2}}}\left(1 - \frac{7}{12n}\right) + \mathcal{O}(n^{-3/2}).$$

With the definition (cf. Does and Helmers (1982), formula (3.5))

(3.11)
$$\begin{aligned}\tilde{\psi}_n(s,t) = e^{-\frac{1}{2}(s^2+t^2)}\Big\{1 &- \frac{i}{6n^{\frac{1}{2}}}(E\tilde{g}^3(Y)t^3 + 3E\tilde{g}^2(Y)(Y-1)t^2 s \\ &+ 3E\tilde{g}(Y)(Y-1)^2 ts^2 + 2s^3) + \frac{1}{24n}((E\tilde{g}^4(Y)-3)t^4 \\ &+ 4E\tilde{g}^3(Y)(Y-1)t^3 s + 6\{E\tilde{g}^2(Y)(Y-1)^2-1\}t^2 s^2 \\ &+ 4E\tilde{g}(Y)(Y-1)^3 ts^3 + 6s^4) \\ &- \frac{1}{72n}(E\tilde{g}^3(Y)t^3 + 3E\tilde{g}^2(Y)(Y-1)t^2 s \\ &+ 3E\tilde{g}(Y)(Y-1)^2 ts^2 + 2s^3)^2 - \frac{1}{2n}(t^2+s^2)\Big\}\end{aligned}$$

and (3.10) we find after some computations (cf. (3.3))

(3.12) $$\tilde{\chi}_n(t) = (2\pi h_n(0))^{-1}\int_{-\infty}^{\infty} \tilde{\psi}_n(s,t)ds + \mathcal{O}(n^{-3/2}(1+t^6)e^{-\frac{1}{2}t^2}),$$

uniformly in t.

Because of condition (2.7) an argument like in (3.7) shows that for any δ and t

(3.13) $$\int_{|s|\geq \delta n^{\frac{1}{2}}} |\psi_n(s,t)|ds \leq \pi n^{\frac{1}{2}} \sup_{|s|\geq\delta} |\rho(s,tn^{-\frac{1}{2}})|^{n-1}$$

is exponentially small. Hence, in view of (3.2), (3.4), (3.9) - (3.13) it suffices to show that for some $\delta > 0$

$$(3.14)\qquad \int_{n^{-2}\le|t|\le n\log n} |t|^{-1}\left|\int_{|s|\le\delta n^{\frac{1}{2}}} \{\psi_n(s,t) - \tilde{\psi}_n(s,t)\}ds\right|dt = o(n^{-1}).$$

Application of Theorem 9.12 of Bhattacharya and Rao (1976) (with V=I, s =4, k=2 and $\alpha=(0,0)$) yields

$$(3.15)\qquad \int_{1\le|t|\le\varepsilon n^{\frac{1}{2}}} |t|^{-1}\int_{|s|\le\delta n^{\frac{1}{2}}} |\psi_n(s,t) - \tilde{\psi}_n(s,t)|ds\ dt = o(n^{-1}),$$

for some $\varepsilon > 0$ and $\delta > 0$. By the same theorem (with $\alpha = (0,1)$) we obtain

$$(3.16)\qquad \int_{n^{-2}\le|t|\le 1}\left|\int_{|s|\le\delta n^{\frac{1}{2}}} \frac{1}{t}\int_0^t \frac{\partial}{\partial u}\,[\psi_n(s,u) - \tilde{\psi}_n(s,u)]du\ ds\right|dt = o(n^{-1}).$$

Moreover, the classical theory of Edgeworth expansions for sums of independent and identically distributed random variables yields

$$(3.17)\qquad \int_{n^{-2}\le|t|\le 1}\left|\frac{1}{t}\int_{|s|\le\delta n^{\frac{1}{2}}} \{\psi_n(s,0) - \tilde{\psi}_n(s,0)\}ds\right|dt = O(n^{-3/2}\log n).$$

Combining (3.14) - (3.17) we see that it remains to show that

$$(3.18)\qquad \int_{\varepsilon n^{\frac{1}{2}}\le|t|\le n\log n}\ \int_{|s|\le\delta n^{\frac{1}{2}}} \left\{\left|\frac{\psi_n(s,t)}{t}\right| + \left|\frac{\tilde{\psi}_n(s,t)}{t}\right|\right\}ds\ dt = o(n^{-1}).$$

But this is easily seen to be a simple consequence of formula (3.11) and condition (2.7) (cf. the argument leading to (3.13)). This completes the proof of Theorem 2.1. □

REMARK 3.1. As in the classical proof of Edgeworth expansions for sums of independent and identically distributed random variables one needs a condition to guarantee that for all $\varepsilon > 0$

$$(3.20) \qquad \int_{\varepsilon n^{\frac{1}{2}} \leq |t| \leq n \log n} \left| \frac{\chi_n(t)}{t} \right| dt = o(n^{-1}).$$

In view of (3.9), (3.10), (3.6) and an argument like (3.13), condition (3.20) is implied by and presumably almost equivalent to

$$(3.21) \qquad \int_{\varepsilon n^{\frac{1}{2}} \leq |t| \leq n \log n} |t|^{-1} \sup_{s \in \mathbb{R}} |\rho(sn^{-\frac{1}{2}}, tn^{-\frac{1}{2}})|^{n-1} dt = o(n^{-3/2}).$$

The natural Cramér type condition to ensure this reads as follows:

$$(3.22) \qquad \sup_{|t| \geq \varepsilon} \sup_{s \in \mathbb{R}} |\rho(s,t)| < 1, \qquad \text{for any } \varepsilon > 0.$$

This is easily checked to be equivalent to condition (2.7).

4. PROOF OF THEOREM 2.2

Let $\{s_m, t_m\}$ be a sequence with

$$(4.1) \qquad \lim_{m \to \infty} \| (s_m, t_m) \| = \infty$$

and

$$(4.2) \qquad \lim_{m \to \infty} |\rho(s_m, t_m)| = \limsup_{\|(s,t)\| \to \infty} |\rho(s,t)|.$$

Without loss of generality we assume that t_m is nonnegative and that there exists an $A \in [0, \infty]$ with

(4.3) $$\lim_{m\to\infty} |s_m t_m^{-1}| = A.$$

Suppose A = ∞. By partial integration we obtain

$$\left| \int_c^d \exp\{itg(y)+(is-1)y\}dy \right|$$

(4.4) $$= |is-1|^{-1} \left| [\exp\{itg(y)+(is-1)y\}]_{y=c}^{d} - \int_c^d itg'(y)\exp\{itg(y)+(is-1)y\}dy \right|$$

$$\leq (1+s^2)^{-\frac{1}{2}}\{e^{-c}+e^{-d}+|t| \int_c^d |g'(y)|e^{-y}dy\}.$$

Since g' is bounded, (4.4) yields

(4.5) $$\lim_{m\to\infty} |\rho(s_m,t_m)| \leq 1 - \int_c^d e^{-y}dy < 1,$$

which establishes (2.7) in case A = ∞.

If A is finite a more refined analysis is needed. By adding a linear function to g if necessary, we may and shall assume that g' is positive and bounded on the interval (c,d). This implies

(4.6) $$\sup_{c<y<d} g'(y) \{ \inf_{c<y<d} g'(y)\}^{-1} < \infty.$$

Let $\alpha > 0$ be fixed. Since g^{-1} is well-defined on (g(c),g(d)), the substitution $y=g^{-1}(g(z)-\alpha t^{-1})$ yields for sufficiently large t

$$\int_c^{g^{-1}(g(d)-\alpha t^{-1})} \exp\{itg(y)+(is-1)y\}dy$$

$$= \int_{g^{-1}(g(c)+\alpha t^{-1})}^{d} \exp\{itg(z)+(is-1)z-i\alpha+(is-1)[g^{-1}(g(z)-\alpha t^{-1})-z]\}$$

$$\cdot g'(z)[g'(g^{-1}(g(z)-\alpha t^{-1}))]^{-1}dz$$

$$= \int_{g^{-1}(g(c)+\alpha t^{-1})}^{d} \exp\{itg(z)+(is-1)z-i\alpha\}\exp\{-i\alpha A/g'(z)\}dz$$

$$+ \int_{g^{-1}(g(c)+\alpha t^{-1})}^{d} \exp\{itg(z)+(is-1)z-i\alpha\}[\exp\{is(g^{-1}(g(z)-\alpha t^{-1})-z)\}$$

(4.7)

$$- \exp\{-i\alpha A/g'(z)\}]dz$$

$$+ \int_{g^{-1}(g(c)+\alpha t^{-1})}^{d} \exp\{itg(z)+(is-1)z-i\alpha\}\exp\{is(g^{-1}(g(z)-\alpha t^{-1})-z)\}$$

$$\cdot[\exp\{z-g^{-1}(g(z)-\alpha t^{-1})\}-1]dz$$

$$+ \int_{g^{-1}(g(c)+\alpha t^{-1})}^{d} \exp\{itg(z)+(is-1)z-i\alpha\}\exp\{(is-1)(g^{-1}(g(z)-\alpha t^{-1})-z)\}$$

$$\cdot\left[\frac{g'(z)}{g'(g^{-1}(g(z)-\alpha t^{-1}))} -1\right]dz.$$

When in (4.7) the argument (s,t) is replaced by (s_m,t_m) satisfying (4.1) - (4.3), then dominated convergence implies, in view of the assumptions of the theorem and (4.6), that the last three integrals in the right-hand side of (4.7) converge to zero as $m \to \infty$. Hence we have

$$(4.8) \qquad \lim_{m\to\infty} \int_c^d \exp\{it_m g(y)+(is_m-1)y\}\left[1-\exp\{-i\alpha(1+A/g'(y))\}\right]dy=0$$

Since (4.8) also holds for $\alpha \leq 0$ we obtain for every density f on $\mathbb{R}$

$$(4.9) \qquad \lim_{m\to\infty} \int_{-\infty}^{\infty}\int_c^d \exp\{it_m g(y)+(is_m-1)y\}\left[1-\exp\{-i\alpha(1+A/g'(y))\}\right]dy \cdot f(\alpha)d\alpha = 0 .$$

Because g' is not essentially constant on (c,d) this implies

$$(4.10) \qquad \begin{aligned} &\lim_{m\to\infty} \left|\int_c^d \exp\{it_m g(y)+(is_m-1)y\}dy\right| \\ &\leq \int_c^d \left|\int_{-\infty}^{\infty} \exp\{-i(1+A/g'(y))\alpha\}f(\alpha)d\alpha\right| e^{-y}dy \\ &= \int_c^d \left|\zeta_f(1+A/g'(y))\right| e^{-y}dy < \int_c^d e^{-y}dy, \end{aligned}$$

where ζ_f denotes the characteristic function of f. Consequently we have

$$(4.11) \qquad \lim_{m\to\infty} \left|\rho(s_m,t_m)\right| < 1,$$

which in view of (4.2) establishes (2.7) in case $A < \infty$. □

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