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A LIMIT THEOREM FOR THE SUPERPOSITION OF RENEWAL PROCESSES

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The asymptotics of a superposition of renewal point processes is studied from the point of view of Palm theory.

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1. Introduction

In this section some known results from the theory of the superposition of point processes (p.p.) are given. In a generalized form they can be found in Matthes [1963].

Let N be a p.p. on the Borel line $(R^1, \mathfrak{B}^1)$. The translation T_t on the Borel line is the map $T_t: x \to x - t$. Define the translation $T_t N$ of N by $T_t N(A) = N(T_t A), A \in \mathfrak{B}^1$. The process N is stationary if for all t the p.p. N and $T_t N$ have the same distribution. We assume the intensity $\lambda = EN(0,1]$ is finite. If P is the distribution of N, we write P_0 for the Palm measure of P. The Palm measure can be seen as the distribution of N under the condition "a point occurs at 0". This condition has probability zero but there is a way to give the statement a proper meaning. If a p.p. N_0 has distribution P_0, with points in $\cdots < U_{-1} < U_0 = 0 < U_1 < \cdots$ then the p.p. $T_{U_n} N_0$ has distribution P_0 too.

Let $N_1,...,N_k$ be independent p.p. on the Borel line with distribution $P_1,...,P_k$. The distribution of the multivariate p.p. $N = (N_1,...,N_k)$ is denoted by $P_1 x ... x P_k$.

Now assume the p.p. N_j are stationary with intensity λ_j and do not have multiple points. The superposition of N is defined as $N^s = N_1 + ... + N_k$. The superposition has intensity $\lambda = \lambda_1 + ... + \lambda_k$. Let Q_0 be the Palm measure of the distribution $Q = P_1 x ... x P_k$ of N, that is the distribution of N conditioned to "a point of N^s occurs at 0". The Palm measure satisfies

$$Q_0 = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda} P^1 x ... x P^{i-1} x P_0^i x P^{i+1} x ... x P^k.$$

Let N_0 have distribution Q_0. The p.p. N_0^s has no multiple points. Denote the points of N_0^s by $\cdots < U_{-1} < U_0 < U_1 < \cdots$. The p.p. $T_{U_n} N_0$ has distribution Q_0 too.

Let $Q_n, n \geq 1$ and Q be distributions of a multivariate p.p. with k components. Say an interval I is Q-continuous if for both boundary points x the event $N^s\{x\} = 0$ has Q-measure 1. Say Q_n converges weakly to Q or

$$Q_n \Rightarrow Q$$

if for all Q-continuous intervals $J_1,...,J_l, l \geq 1$ the simultaneous distribution of the random vectors $N(J_1),...,N(J_l)$ under Q_n-measure converges weakly to the distribution of these vectors under Q-measure.

2. A limit theorem for the superposition of renewal processes

Let F be a distribution function on $(0,\infty)$ with finite mean $\mu > 0$. Assume F is not lattice, so F is not concentrated on any set $L_d = \{nd : n \in \mathbb{Z}\}$. F is called centered lattice if F is concentrated on a coset $\alpha + L_d$ of a lattice L_d. We define a p.p. called the stationary renewal process as follows. Assume $X_1, X_2, ..., Y_1, Y_2, ...$ and (X_0, Y_0) are independent. Let $X_1, X_2, ..., Y_1, Y_2, ...$ have distribution F. The simultaneous distribution of (X_0, Y_0) is given by

$$P(X_0 > x, Y_0 > y) = \int_{x+y}^{\infty} \frac{1 - F(t)}{\mu} dt, x, y \geq 0$$

The marginal distribution of X_0, and Y_0 is called the *survivor* distribution of F. Let the p.p. N have points

$$\{\sum_{i=0}^{n} X_i, n \geq 0\} \cup \{-\sum_{i=0}^{n} Y_i, n \leq 0\}.$$

It can be checked easily that N is stationary. A p.p. distributed as N is called a stationary renewal process. Its distribution will be denoted by P_F. The intensity of N is $\frac{1}{\mu}$. Let the p.p. N_0 have points

$$\{\sum_{i=1}^{n} X_i, n \geq 1\} \cup \{0\} \cup \{-\sum_{i=1}^{n} Y_i, n \leq 1\}.$$

It can be shown that the distribution of N_0 is the Palm measure $(P_F)_0$. The restriction of N_0 to $[0,\infty)$ describes the set of points visited by a random walk started in 0. This restriction is called the ordinary renewal process.

Let F_j be distribution functions on $(0,\infty)$ with mean μ_j and non lattice, $1 \leq j \leq k$. Let $\overline{M}_1, \ldots, \overline{M}_k$ be independent ordinary renewal processes with distribution $F_1,..,F_k$. The r.v. $X_1,...,X_k$ are assumed to be independent of $\overline{M}_1, \ldots, \overline{M}_k$. The modified (multivariate) renewal process with initial point $(X_1,...,X_k)$ is defined as

$$M = (T_{X_1}\overline{M}_1,...,T_{X_k}\overline{M}_k).$$

Let the superposition M^s have points

$$S_0 \leq S_1 \leq S_2 \leq \cdots$$

in which the equalities account for possible multiplicities of points of M^s. There are two important examples of modified renewal processes.

Example 1.

Let N have distribution $Q = P_{F_1}x...xP_{F_k}$. N is invariant under T_t. The restriction of the components of N to $[0,\infty)$ determines a modified renewal process M. The initial point $(X_1,...,X_k)$ consists of independent r.v., distributed as the survivor distribution of F_j.

Example 2.

Let N_0 have distribution Q_0, the Palm measure of Q. The points of N^s are denoted by

$$\cdots < U_{-1} < U_0 = 0 < U_1 < \cdots$$

N is invariant under T_{U_n}. The restriction of the components of N_0 to $[0,\infty)$ determines a modified renewal process. The initial point $(X_1,...,X_k)$ satisfies $P(X_j = 0) = \frac{\lambda_j}{\lambda} = \frac{1/\mu_j}{1/\mu_1 + ... + 1/\mu_k}$. Under the condition $X_j = 0$ the r.v. $X_i, i \neq j$ are independent; their distribution is the survivor distribution of F_i. Because of the invariance property the interval lengths $U_n - U_{n-1}$ in the superposed process have the same distribution, say G. As can be seen from the structure of Q_0

$$Q_0 = \sum_{j=1}^{k} \frac{\lambda_j}{\lambda} P_{F_1}x...x(P_{F_j})_0 x...xP_{F_k}$$

this distribution G is given by

$$G(x,\infty) = \sum_{j=1}^{k} \frac{\lambda_j}{\lambda}(1 - F_j(x)) \prod_{i \neq j} \int_x^{\infty} \frac{1 - F_i(t)}{\mu_i} dt.$$

Both examples 1 and 2 correspond to convergence theorems. Example 1 corresponds to:

Theorem 1. *If the distributions $F_j, 1 \leq j \leq k$ are non-lattice, the modified renewal process M satisfies*

$$P_{T_t M} \Rightarrow Q \quad \text{for} \quad t \rightarrow \infty.$$

Proof. For k = 1 the theorem is a well known consequence of the renewal theorem. Because of the independence of the component processes the case $k > 1$ is a consequence of $k = 1$. □

Example 2 corresponds to:

Theorem 2.

(i) If all distribution functions $F_j, 1 \leqslant j \leqslant k$, are not lattice and not centered lattice

or

(ii) if all distribution functions $F_j = F, 1 \leqslant j \leqslant k$ are identical and are not lattice then the modified renewal process M satisfies

$$P_{T_{S_n} M} \Rightarrow Q_0 \quad \text{for} \quad n \to \infty$$

Proof. The proof will be based on a coupling argument for random walks developed in Ornstein [1969]. Construct a probability space $(\Omega, \mathscr{Q}, P)$ with two independent processes $S_n^{(j)}$, $n \geqslant 0$, $1 \leqslant j \leqslant k$, and $T_n^{(j)}$, $-\infty < n < \infty$, $1 \leqslant j \leqslant k$, satisfying the following properties. The sets of points $\{S_n^{(j)}: n \geqslant 0\}$, $1 \leqslant j \leqslant k$, determine the points of a p.p., simultaneously distributed as the components of the given modified renewal process $M_j, 1 \leqslant j \leqslant k$. The set of points $\{T_n^{(j)}: -\infty < n < \infty\}$, $1 \leqslant j \leqslant k$, determine the component, of a p.p. N_0 with distribution Q_0. For all its components, $T_0^{(j)}$ is chosen to be its first non negative r.v. The increments of $S_n^{(j)}$ are denoted by $X_n^{(j)} = S_n^{(j)} - S_{n-1}^{(j)}$, $n \geqslant 1$ and of $T_n^{(j)}$ by $Y_n^{(j)} = T_n^{(j)} - T_{n-1}^{(j)}, n \geqslant 1$. All of these increments are independent and F_j-distributed.

Let $J_1, ..., J_l$ be Q_0-continuous intervals. Choose a number η such that

$$P(N_0^s(x - 2\eta, x + 2\eta) \geqslant 1 \text{ for any boundary point } x \text{ of } J_1, ..., J_l) < \epsilon$$

Extend the intervals J_i at both sides with 2η to get J_i^+ and let J_i shrink at both sides with 2η to get J_i^-. Choose an interval J of the form (a, ∞), $a \in \mathbb{R}^1$ that contains all intervals J_i^+. We shall give a coupling with coupling distance η. The proof of (ii) contains a problem that concerns the numbering of points of the coupled process. Therefore we have to deal with (i) and (ii) separately.

Case (i). Consider for $j = 1, ..., k$ the difference $S_n^{(j)} - T_n^{(j)} = S_0^{(j)} - T_0^{(j)} + \sum_{i=1}^{n} Z_i^{(j)}$ in which $Z_i^{(j)} = X_i^{(j)} - Y_i^{(j)}$. Because F_j is not centered lattice and not lattice distributed, the difference $Z_i^{(j)}$ is not lattice distributed, and has $EZ_i^{(j)} = 0$. The random walk $S_n^{(j)} - T_n^{(j)}$, $n \geqslant 1$ is recurrent. Take $\tau^{(j)}$ the first entrance time into the neighbourhood $(-\eta, \eta)$ of zero. Because of the Markov property, given $\tau^{(j)}$ the increments $(X_n^{(j)})_{n > \tau^{(j)}}$ and $(Y_n^{(j)})_{n > \tau^{(j)}}$ are independent of the process $(S_n^{(j)}, T_n^{(j)})$ for $n \leqslant \tau^{(j)}$, and also independent of the other components. Given $\tau^{(j)}$ the increments are independent and F_j-distributed. Now exchange in $S_n^{(j)}$ the increments $X_n^{(j)}$ by $Y_n^{(j)}$ for $n > \tau^{(j)}$. We obtain the process $\tilde{S}_n^{(j)}$, $n \geqslant 0$, distributed as $S_n^{(j)}, n \geqslant 0$. This procedure has to be performed for all indexes $1 \leqslant j \leqslant k$. The result is a p.p. $\tilde{M}$, distributed as M, with points $\{\tilde{S}_n^{(j)}: n \geqslant 0\}$, $1 \leqslant j \leqslant k$, such that $\tilde{S}_n^{(j)}$ and $T_n^{(j)}$ differ at most η for all n exceeding a random time $\tau = \max \tau^{(j)}$.

Let the superposition $\tilde{M}^s$ have points $\tilde{S}_0 \leqslant \tilde{S}_1 \leqslant \cdots$, and the superposition N_0^s points $T_0 < T_1 < T_2 < \cdots$. Consider the interval $\tilde{S}_n + J$. Choose n so large that on $A \subset \Omega$ with probability $P(A) > 1 - \epsilon$, the points of the components of $\tilde{M}$ and N_0 are coupled at distance at most η on this interval $\tilde{S}_n + J$. Then also for the superposed processes $|\tilde{S}_n - T_n| < \eta$ on A. So the points of the translated p.p. $T_{\tilde{S}_n} \tilde{M}$ and $T_{T_n} N_0$ are coupled with distance at most 2η on J. On the set A

$$T_{T_n} N_0(J_i^-) \leqslant T_{\tilde{S}_n} \tilde{M}(J_i) \leqslant T_{T_n} N_0(J_i^+).$$

Because of the invariance of N_0 under T_{T_n} and the choice of η

$$P(T_{T_n} N_0(J_i^-) \neq T_{\tilde{S}_n} \tilde{M}(J_i)) \text{ for some i }) < \epsilon.$$

Therefore

$$P(T_{T_n} N_0(J_i) \neq T_{\tilde{S}_n} \tilde{M}(J_i)) \leqslant P(A^c) + \epsilon.$$

Because $T_{T_n} N_0$ has distribution Q_0 the asserted convergence follows from this inequality.

Case (ii). We consider the case of k identical distributions $F_j = F$. Because of what is proved in case (i) we only have to consider a non-lattice, centered-lattice distributed F. Let L_d be the minimal lattice for which F is concentrated on a coset of L_d. Assume F is concentrated on $\alpha + L_d$. The ratio α / d cannot be rational, and so the ratio $k\alpha / d$ is rational. Choose numbers $p_j, 1 \leqslant j \leqslant k$, depending on $S_0^{(j)} - T_0^{(j)}$ such that $p_j \cdot k\alpha + S_0^{(j)} - T_0^{(j)} = \delta_j$ (mod d) with $|\delta_j| < \eta$. Let $p = p_1 + ... + p_k$; then the sum of $\tilde{p}_1 = kp_1 - p, ..., \tilde{p}_k = kp_k - p$, equals zero. Now choose positive numbers $m_1, ..., m_k$ with $\tilde{p}_1 = m_1 - m_2, \ldots, \tilde{p}_{k-1} = m_{k-1} - m_k$. We remark that $m_k - m_1 = (m_k - m_{k-1}) + ... + (m_2 - m_1) = -\tilde{p}_{k-1} - ... - \tilde{p}_1 = \tilde{p}_k$. Because F is concentrated on $\alpha + L_d$ it follows that

$$S_{m_1}^{(1)} - T_{m_2}^{(1)} = kp_1\alpha - p\alpha + S_0^{(1)} - T_0^{(1)} (mod\ d) = (\delta_1 - p\alpha)(mod\ d)$$

....

$$S_{m_{k-1}}^{(k-1)} - T_{m_k}^{(k-1)} = \cdots = (\delta_{k-1} - p\alpha)(mod\ d)$$

$$S_{m_k}^{(k)} - T_{m_1}^{(k)} = ... = (\delta_k - p\alpha)(mod\ d)$$

The numbers $v_j = \delta_j - p\alpha$, $1 \leqslant j \leqslant k$ and $v = p\alpha$ satisfy $|v_j - v| < \eta$. We have reached that (mod d) the S-process shifted over a distance v and the T-process have distance at most η at the indicated times. Consider $S_{m_1+n}^{(1)} - T_{m_2+n}^{(1)}, n \geqslant 0$. The numbers m_j depend on $S_0^{(j)}$ and $T_0^{(j)}$, $1 \leqslant j \leqslant k$ only, so $S_{m_1+n}^{(1)} - T_{m_2+n}^{(1)}$ is a random walk with increments $Z_n^{(1)} = X_{m_1+n}^{(1)} - Y_{m_2+n}^{(2)}$. Because of the choice of L_d the r.v. $Z_n^{(1)}$ have a distribution with minimal lattice L^d, and expectation zero. So the random walk $S_{m_1+n}^{(1)} - T_{m_2+n}^{(1)}, n \geqslant 0$ on $v_1 + L_d$ is recurrent. Take τ_1 the first entrance time of the random walk into v_1. Now exchange in $S_n^{(j)}$ the increments $(X_{n+m_1}^{(1)})_{n > \tau_1}$ with $(Y_{n+m_2}^{(1)})_{n > \tau_1}$ to obtain the process $\tilde{S}_n^{(j)}$. Using the Markov property, it follows again that the replacement does not change the distribution. This coupling has to be performed for all components. The result is a process $\tilde{S}$ such that

$$|v + \tilde{S}_{m_1+n}^{(1)} - T_{m_2+n}^{(1)}| < \eta$$

....

$$|v + \tilde{S}_{m_{k-1}+n}^{(1)} - T_{m_k+n}^{(k-1)}| < \eta$$

$$|v + \tilde{S}_{m_k+n}^{(k)} - T_{m_1+n}^{(k)}| < \eta$$

for all n exceeding some random time τ. Denote by $\tilde{M}$ the p.p. determined by the $\tilde{S}$-process.

About the numbering of points we remark the following. Take from each component of the $\tilde{S}_n^{(j)}$-process a "coupled" point, say with index $m_j + n_j$, $1 \leqslant j \leqslant k$. The index sum is $\sum_{j=1}^{k} (m_j + n_j)$. Consider the set of points of the $T_n^{(j)}$- process that coincide by the coupling with the selected points of the $\tilde{S}_n^{(j)}$-process. Remark that the index sum of these points is $\sum_{j=1}^{k} (m_j + n_j)$. So each selection of points of the components of $\tilde{M}$ coincides by the coupling with points of N_0 in such way that both index sums are equal. The points of the superposed process of $\tilde{M}$ are given by $\tilde{S}_n = \max_{n_1 + ... + n_k = n} (\min_j \tilde{S}_{n_j}^{(j)})$ and for N_0 by $T_n = \max_{n_1 + ... + n_k = n} (\min_j T_{n_j}^{(j)})$. So if n is large enough the superposition points satisfy $|v + \tilde{S}_n - T_n| < \eta$. Consider again the interval $\tilde{S}_n + J$. Choose n so large that on $A \subset \Omega$ with $P(A) > 1 - \epsilon$ the components of $\tilde{M}$ and N_0 only have coupled points. So on that interval the points of the $\tilde{S}_n^{(j)}$- process shifted over a distance v coincide with the points of the $T_n^{(j)}$-process with error at most η. But because $|v + \tilde{S}_n - T_n| < \eta$ the points of the translated p.p. $T_{\tilde{S}_n} \tilde{M}$ on J lie separated from the points of $T_{T_n} N_0$ with distance at most 2η. The rest of the proof is the same as in case (i). □

Corollary 3. *Let G be the distribution of the interval length of the Palm process. The convergence to the Palm measure entrails the convergence of the interval length $S_{n+1} - S_n$ to G.*

Proof. Choose Q_0-continuous intervals $J_+ = (\eta, x)$ and $J_- = (-\eta, x)$ such that $P(N_0(-\eta, \eta) > 1) < \epsilon$. Let $n \to \infty$ in the inequality

$$P(T_{S_n} M(J_+)=0) \leqslant P(S_{n+1}-S_n > x) \leqslant P(T_{S_n} M(J_-) \geqslant 1)$$

According to theorem 2 and the choice of ϵ right- and left-hand limits differ at most ϵ from $P(N_0(0,x)=0) = 1-G(x)$. □

If the distributions $F_1,...,F_k$ are non lattice, but centered lattice the convergence to the Palm measure stated in theorem 2 cannot be proved in general.

Counterexample. We give a counterexample for corollary 3. Take non lattice distributions F_1 and F_2 with F_1 concentrated on $\alpha+L_d$ and F_2 concentrated on $-\alpha+L_d$. Assume the initial point is (0,0). Let M denote the modified renewal process, with superposition points $S_0 \leqslant S_1 \leqslant \cdots$. If both S_n and S_{n+1} are points of the first component of M, then $S_{n+1}-S_n \in \alpha+L_d$. If S_n is a point of the first component (say the κ th point) and S_{n+1} is a point of the second component (the $(n+1-\kappa)$th point) then $S_{n+1}-S_n \in (n+1)\alpha+L_d$. A consideration of all possibilities gives that $S_{n+1}-S_n \in \{\alpha\} \cup \{-\alpha\} \cup \{(n+1)\alpha\} \cup \{-(n+1)\alpha\}+L_d$, a.s. However the structure of Q_0 shows G has an absolute component. Therefore the distribution of $S_{n+1}-S_n$ does not converge to G.

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